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3. APPLICATIONS AND NUMERICAL METHODS

ON SOME FUNCTIONS WHICH VERIFY DIFFERENTIAL INEQUALITIES

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1. In the very first chapters of the mathematical analysis we find differential inequalities which characterize important classes of functions with a given comportement.

Thus it is well-known that the differential inequality $f'(x) \geqq 0$ characterizes the derivable and nondecreasing functions, i.e. the functions whose divided difference $[x_1, x_2; f] = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ remains $\geqq 0$ on every set of two distinct points x_1, x_2 of the definition set of the function.

2. For the consideration of a more general case we introduce the notion of convex function of higher order.

In the first place the *divided difference* $[x_1, x_2, \ldots, x_{n+1}; f]$ (of order n) of the function f on the *knots* $x_1, x_2, \ldots, x_{n+1}$ may be defined by the recurrence relation

$$[x_1, x_2, \ldots, x_{n+1}; f] = \frac{[x_2, x_3, \ldots, x_{n+1}; f] - [x_1, x_2, \ldots, x_n; f]}{x_{n+1} - x_1},$$

$$[x_1; f] = f(x_1)$$

or in another way, as the coefficient of x^n of the polynomial of interpolation (of degree n) which takes the same values as the function f on the knots x_α, $\alpha = 1, 2, \ldots, n + 1$. Another definition is with the quotient of the form (7) of two determinants (for the functions (11) in this case).

For the sake of simplification and in the first stage ours considerations we may suppose that the knots x_α are distinct. The case of knots which are not distinct is obtained by a convenient limite process.

A function f is called *nonconcave of order* n ($n \geqq -1$) if we have

$$[x_1, x_2, \ldots, x_{n+2}; f] \geqq 0 \tag{1}$$

on any set of $n + 2$ distinct points $x_1, x_2, \ldots, x_{n+2}$ of the set of definition of the function. If the strict inequality (with the sign $>$) is valid in (1), the function is called *convex of order n*. In a similar way we can introduce the nonconvex functions respective the concave ones of order n, on condition that the divided difference in the first member of the formula (1) should remain $\leqq$ respective < 0.

3. If we suppose that function f is definite and has a derivative of order $n + 1$ ($n \geqq 0$) on the interval $[a, b]$, the condition

$$f^{(n+1)}(x) \geqq 0 \qquad (\text{on } [a, b]) \tag{2}$$

is necessary and sufficient for the nonconcavity of order n, and the condition $f^{(n+1)}(x) > 0$ (on $[a, b]$) is sufficient for the convexity of order n of the function. An analogous property exists for the nonconvexes and the concave functions of order n, respectively.

The demonstration of these properties is based on the mean-value formula ($n \geqq 0$)

$$[x_1, x_2, \ldots, x_{n+2}; f] = \frac{f^{(n+1)}(\xi)}{(n+1)!} \tag{3}$$

where ξ is within the smallest interval which contains all the knots x_α. The formula (3) is due to CAUCHY [2] but it can be perfected and admits various generalizations [11].

4. The *global* characterization through the inequality (1) of the nonconcave functions of order n, corresponds to the *local* characterization (2). There is also a more general, local characterization which does not require existence of the derivative of order $n + 1$ of the function. A function is nonconcave of order n on the point x if there is a neighbourhood of x, on which it is nonconcave of order n. If a function is nonconcave of order n on any point of the finite and closed interval $[a, b]$, it is nonconcave of order n on $[a, b]$. The demonstration of this property results from the general mean-value formula which we give below and also from the application of the well-known Borel—Lebesgue's covering lemma.

The general mean-value formula referred to is

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_{n+2}}; f] = \sum_{\alpha=1}^{m-n-1} A_\alpha [x_\alpha, x_{\alpha+1}, \ldots, x_{\alpha+n+1}; f] \tag{4}$$

where $x_1 < x_2 < \ldots < x_m$, $m \geqq n + 2$, $1 = i_1 < i_2 < \ldots < i_{n+1} < i_{n+2} = m$ and the A_α, $\alpha = 1, 2, \ldots, m - n - 1$ are coefficients independent of the function f and $A_\alpha \geqq 0$, $\alpha = 1, 2, \ldots, m - n - 1$, $A_1 > 0$, $A_{m-n-1} > 0$, $\sum_{\alpha=1}^{m-n-1} A_\alpha = 1$.

Thus the divided difference in the first member of the formula (4) is comprised between the smallest and the greatest of the divided differences $[x_\alpha, x_{\alpha+1}, \ldots, x_{\alpha+n+1}; f]$, $\alpha = 1, 2, \ldots, m - n - 1$. The mean-value formula (3) is deduced from a limit case of the general formula (4).

5. We mention by the way that the global characterization through the inequality (1) of the nonconcave (convexe, etc.) functions of order n may be replaced by a weaker characterization if we suppose that the function is continuous.

A necessary and sufficient condition for function f which is defined and continuous on the interval $[a, b]$ to be nonconcave of order n, is that the inequality (1) should be valid whenever the knots $x_1, x_2, \ldots, x_{n+2}$ are *equally spaced* [7].

It seems that this last condition may be replaced by another one which implies that the mutual ratios of the distances between the knots should have certain given values, so that the inequality (1) may be replaced by

$$[x + \lambda_1 h, x + \lambda_2 h, \ldots, x + \lambda_{n+2} h; f] \geqq 0$$

where λ_α, $\alpha = 1, 2, \ldots, n + 2$ are distinct given numbers and $x + \lambda_\alpha h \in \in [a, b]$, $\alpha = 1, 2, \ldots, n + 2$. This is certainly true if $n = 1$, $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = p =$ given natural number > 1. The cas of a given p comes to that of $p - 1$, as we have shown for $p = 3$ [12].

6. We come to a generalization of the differential inequality (2) through a convenient generalization of the notion of non-concave function of order n. Let us write

$$V\begin{pmatrix} g_1, g_2, \ldots, g_m \\ x_1, x_2, \ldots, x_m \end{pmatrix} = |g_\beta(x_\alpha)|_{\alpha,\beta} = 1, 2, \ldots, m \tag{5}$$

the determinant of the values of the functions $g_1, g_2, \ldots, g_m$ on the points x_α, $\alpha = 1, 2, \ldots, m$.

Let us consider now a sequence of $n + 2$ functions

$$f_0, f_1, \ldots, f_{n+1} \tag{6}$$

checking certain conditions of regularity [13], with which we shall deal below.

The quotient

$$V\begin{pmatrix} f_0, f_1, \ldots, f_n, f \\ x_1, x_2, \ldots, x_{n+2} \end{pmatrix} : V\begin{pmatrix} f_0, f_1, \ldots, f_n, f_{n+1} \\ x_1, x_2, \ldots, x_{n+2} \end{pmatrix}, \tag{7}$$

which we write $[x_1, x_2, \ldots, x_{n+2}; f]$, is the divided difference of the function f on the knots x_α, $\alpha = 1, 2, \ldots, n + 2$ with respect to the sequence of (given) functions (6).

We can generally consider the knots distinct. The divided difference may be defined also on knots which are not distinct, by limite process. By this limiting process we can get the divided difference on all knots coinciding with the same point x which can be written $[\underbrace{x, x, \ldots, x}_{n+2}; f]$ and which has the forme

$$(8)\quad \frac{[\underbrace{x, x, \ldots, x}; f]}{n+2} = \varphi_0(x) f^{(n+1)}(x) + \varphi_1(x) f^{(n)}(x) + \ldots + \varphi_{n+1}(x) f(x) = L[f]$$

where by virtue of the admitted conditions of regularity, the function $\varphi_0(x)$ does not vanish and the function f admits derivatives required by the existence of the formula.

The conditions which are fulfilled by the functions (6) are of such a kind that the quotient (7) should have sense (the denominator is $\neq 0$) and the divided difference taken into consideration should exist.

7. The definition of the nonconcave functions with respect to the sequence of functions (6) is given also by the inequality (1), when the divided difference in the first member has the value (7). The convexity, concavity and non-convexity with respect to the given sequence of functions may be similarly defined.

In the condition of regularity mentioned the corresponding mean-value formula (3) has the following form

$$[x_1, x_2, \ldots, x_{n+2}; f] = \frac{[\underbrace{\xi, \xi, \ldots, \xi}; f]}{n+2}$$

and the general mean-value formula (4) subsists.

It come out that the inequality

$$(9)\qquad L[f] \geqq 0$$

which on the basis of (8) is a differential inequality, characterizes the non-concave functions with respect to the sequence (6) and if they admit a $n + 1$ order derivative.

The differential equation

$$(10)\qquad L[f] = 0$$

in the already mentioned conditions of regularity has just the functions $f_0, f_1, \ldots, f_n$, as a system of liniarly independent solutions.

The connection between the differential equation (10) and the differential inequality (9) is thus made by means of the divided difference (8) and of the notion of non-concave function with respect to a system of liniarly independent solutions of the equation (10) which checks certain supplementary regularity conditions.

The above mentioned remarks are to be found generally in the case $n = 1$, in a remarcable note of H. Poincaré [5]. Various authors have completed the results previously achieved. In this respect one may consult one of G. Pólya works [6] and some bibliographical indications given in my monograph on convex functions [9]. Some properties of continuity and derivability of the convex (nonconcave, etc.) functions with respect to a sequence (6) were studied in a previous work of mine [8].

The case of inequality (2) with the usual divided difference is the „polynomial" case when

(11) $$f_\alpha = x^\alpha, \quad \alpha = 0, 1, \ldots, n+1.$$

In this case all the conditions of regularity, of the existence of the divided differences, etc. are satisfied, whatever the definition interval of the considered functions may be. Another important, peculiar case will be examinad below.

8. The convexity notion with respect to a given sequence of functions may be applied to the study of the structure of the remainder of some linear approximation formulae of the analysis.

Consider a linear functional (additive and homogeneous) $R[f]$ definite on a vectorial space S formed by continuous functions on the interval E. Suppose the functions (6) belong to S and check the regularity conditions required. In this case we say that the linear functional $R[f]$ is of *simple form* if for any $f \in S$ it is of the form

(12) $$R[f] = K \, . \, [\xi_1, \xi_2, \ldots, \xi_{n+2}; f]$$

where $K \neq 0$ is a constant independent of function f, and ξ_α , $\alpha = 1, 2, \ldots,$ $n+2$ are $n+2$ distinct points from E and generally dependent on function f.

Supposing that the linear functional $R[f]$ vanishes in the first $n+1$ functions (6), a necessary and sufficient condition for it to be of simple form is that we should have $R[f] \neq 0$ on any convex (not non-concave!) $f \in S$ with respect to functions (6). Then the universal constant K from the formula (12) may be easily calculated and is equal to $R[f_{n+1}]$.

9. The previous result may be applied to the remainder of various numerical derivation and numerical integration formulae. [10, 13, 14, 15]. The conclusion may be applied to the remainder of the well known quadrature formula

(13) $$\int_0^{2\pi} f(x)\, dx = \frac{2\pi}{m+1} \sum_{\alpha=0}^{m} f\left(\frac{2\alpha\pi}{m+1}\right) + R[f]$$

where m is a natural number and f a continuous function on $[a, b]$.

We have here a „trigonometric" case in which for functions (6) we can take ($n = 2m$)

$$f_0 = 1, \quad f_{2m+1} = x, \quad f_{2\alpha-1} = \cos \alpha x, \quad f_{2\alpha} = \sin \alpha x, \quad \alpha = 1, 2, \ldots, m.$$

In this case the conditions of regularity required are satisfied on the interval $[0, 2\pi)$ (closed on the left and open on the right) [13].

The remainder of the formula (13) is of simple form [13] and in this case we have

$$R[f] = \frac{2\pi^2}{m+1} \, [\xi_1, \xi_2, \ldots, \xi_{2m+2}; f].$$

We find J. RADON's formula if a derivative of order $2m + 1$ exists [16],

$$R[f] = \frac{2\pi^2}{(m+1)(m!)^2} \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1^2\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2^2\right) \cdots \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + m^2\right) f\right]_{x=\xi},$$

$$\xi \in (0, 2\pi).$$

In the demonstration we can initialy suppose that the function has a continuous $(2m+1)^{th}$ derivative on the interval $[0, 2\pi]$. Later on we can omit this supposition.

10. The conditions of regularity to which we referred and which are cheeked by sequence (6), or by the subsequence of its first $n + 1$ terms, is the same with interpolator property of the set of all linear combinations of these functions.

The sequence $g_1, g_2, \ldots, g_m$ is an interpolator sequence (system), or a system of TSCHEBYSCHEFF, if the determinant (5) is $\neq 0$ for any system of m distinct points x_α, $\alpha = 1, 2, \ldots, m$ of the definition set of the functions g_α, $\alpha = 1, 2, \ldots, m$.

The regularity conditions of the sequence (6) also contain limit properties which ensure the existence of divided differences wich knots not all different.

We can study much more general interpolator systems.

In order to be more precise, let's consider a set F of functions, definite and continuous on the finite and closed interval $[a, b]$. We shall say that the set F is *interpolator of order* $n + 1$ $(n \geqq 0)$ if for any system of $n + 1$ distinct point x_α, $\alpha = 1, 2, \ldots, n+1$ of $[a, b]$ and for any continuous function f, definite on the interval $[a, b]$, there is an element and only one $\varphi \in F$ which checks the equalities $\varphi(x_\alpha) = f(x_\alpha)$, $\alpha = 1, 2, \ldots, n + 1$. We can write $\varphi(x) = L(x_1, x_2, \ldots, x_{n+1}; f|x)$ this element of F. In this definition the continuity hypothesis of f is not essential. Still we take into consideration this hypothesis because in what we are going to present all the functions will be supposed to be continuous.

The difference $(n \geqq 0)$

$$[x_1, x_2, \ldots, x_{n+2}; f]_D = f(x_{n+2}) - L(x_1, x_2, \ldots, x_{n+1}; f|x_{n+2}) \tag{14}$$

where $x_1 < x_2 < \ldots < x_{n+2}$, plays the part of divided difference (7) in the linear case.

We can extend the mean-value theorems of the divided difference [3, 4]. Thus we get the following property: if f is a continuous function on $[a, b]$ and $x_1 < x_2 < \ldots < x_{n+2}$ are $n + 2$ points on this interval, there is a point $\xi \in (x_1, x_{n+2})$ (open interval) so that in all neighbouhood of ξ should exist the points $\xi_1 < \xi_2 < \ldots < \xi_{n+2}$ for wich we have

$$\mathrm{sg}\,[x_1, x_2, \ldots, x_{n+2}; f]_D = \mathrm{sg}\,[\xi_1, \xi_2, \ldots, \xi_{n+2}; f]_D$$

There are various other mean-value formulae which are closely connected with the given formula and which all come to mean-value formulae of the divided difference when F is the set of linear combinations of the first $n + 1$ terms of the sequence (6) checking the respective interpolator conditions.

A function f definite on $[a, b]$ is called nonconcave (convex, concave, nonconvex) with respect to the interpolator set F, if we have $[x_1, x_2, \ldots, x_{n+2}; f]_D \geqq$ $\geqq (>, <, \leqq) \, 0$, for any system of $n + 2$ points $x_1 < x_2 < \ldots < x_{n+2}$ of $[a, b]$.

The continuous convex (concave) functions have an equivalent definition because they cannot coincide in more than $n + 1$ points with a certain element of F.

The mean-value formulae already mentioned or such properties of intersection allon us to study the properties of the nonconcave (convex, concave, nonconvex) functions with respect to the interpolator set F [4, 17].

11. By a convenient particularizing of our researches we come to differential inequalities more general than the inequality (9) studied above.

Consider the differential equation ($n \geqq 0$)

$$y^{(n+1)} - G(x, y, y', \ldots, y^{(n)}) = 0 \tag{15}$$

where the function $G(z_1, z_2, \ldots, z_{n+2})$ is continuous for $z_1 \in [a, b]$, $z_\alpha \in (-\infty, +\infty)$, $\alpha = 2, 3, \ldots, n + 2$.

Suppose that the set of the solutions of the equation (15) is interpolator of order $n + 1$ and that Cauchy's problem has always a unique solution on any point of $[a, b]$.

If f is a function which has a continuous derivative of order $n + 1$ ($n \geqq 0$) on $[a, b]$ and if it coincide with a solution of the equation (15) on the points $x_1 < x_2 < \ldots < x_{n+2}$ of $[a, b]$ there is a point $\xi \in (x_1, x_{n+2})$ so that

$$f^{(n+1)}(\xi) - G(\xi, f(\xi), f'(\xi), \ldots, f^{(n)}(\xi)) = 0$$

The differential inequality

$$f^{(n+1)}(x) - G(x, f(x), f'(x), \ldots, f^{(n)}(x)) \geqq 0 \text{ (respectively } >) \ 0$$

for $x \in [a, b]$ characterizes the functions f which have a continuous derivative of order $n + 1$ and which are nonconcave (convex) with respect to the set of solutions of the equation (15), [3].

As regards the hypotheses according to which the set of solutions of a differential equation (15) is interpolator of order $n + 1$, on a given interval, they constitute the so-called plurilocal problem of the differential equations. This problem was studied by many authors and has been for many years the preoccupation of the researchers of the Institut of Calculus from, CLUJ, of the Academy of the Socialist Republic of Romania [1].

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