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VECTORS OF GEVREY CLASSES AND APPLICATIONS

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Introduction.

In several problems in partial differential equations one is led to study the space of functions u defined in a domain Ω of R^n with smooth boundary Γ and which satisfy conditions of the following type (we take here the simplest possible case):

(1) $\left(\int_\Omega |\Delta^k u|^2 \, dx\right)^{1/2} \leq cL^k M_k \qquad \forall\, k,$

(2) $\Delta^k u = 0 \qquad \text{on} \qquad \Gamma \qquad \forall\, k,$

where c and L are suitable constants (which depend on u) and M_k is a given sequence — For example, if

(3) $M_k = (2k)!$

then (1) (2) imply that u is analytic in $\bar{\Omega} = \Omega \cup \Gamma$ (assuming Γ to be real-analytic). A much more general result of this type will be reported in Section 4 below.

Once one is led to study classes of functions satisfying conditions of type (1) (2), it is natural to put this question in a more general framework and to replace in (1) (2) Δ by an unbounded operator A in a Banach space E, condition (2) being then replaced by

($\tilde{2}$) $u \in$ domain of A, $Au \in$ domain of A, and so on, and condition (1) being replaced by

($\tilde{1}$) $||A^k u|| \leq cL^k M_k \qquad \forall\, k,$

(where $||\quad||$ denotes the norm in).

In Sections 1,2 we give some (simple) remarks on the spaces defined by

[1] Expositery lecture. All details and other results are contained in the book [4] by E. Magenes and the A.

$\tilde{(1)}$ $\tilde{(2)}$ (the so — called "vectors of Gevrey class" when $\{M_k\}$ is a Gevrey sequence) when $(-A)$ is the infinitesimal generator of a semi-group. [This contains (1) (2) by taking $E = L^2(\Omega)$, $A = -\Delta$, the domain of A consisting of those functions u which are zero on Γ].

The plan is as follows:

1. Domains $D(A^\infty; M_k)$.
2. A criterion of non triviality.
3. The semi group on $D(A^\infty; M_k)$.
4. The case when A is an elliptic operator.
5. Transposition.
6. Cauchy problem.
7. Some examples.

Bibliography

1. Domains $D(A^\infty; M_k)$.

Let E be a reflexive Banach space, norm $\| \quad \|$; let A be an unbounded operator given in E; we assume (for semi-group theory we refer to [2], [10]):
(1.1) $(-A)$ is the infinitesimal generator of a continuous semi-group $G(t)$ in E. Let $D(A)$ be the domain of A. We set

$$D(A^\infty) = \{u \mid A^k u \in D(A) \qquad \forall\ k\};$$

it is well known [2], [10] that $D(A^\infty)$ is dense in E.

Let now $\{M_k\}$ be a given sequence of positive numbers.

We define

$$(1.2) \quad \begin{cases} D(A^\infty; M_k) = \{u \mid u \in D(A^\infty); \text{ there exist constants } c \text{ and } L \text{ (de-} \\ \text{pending on } u) \text{ such that } \|A^k u\| \le c L^k M_k \ \forall\ k\}. \end{cases}$$

Example 1.1.

If $M_k = (k!)^\alpha$, $\alpha > 1$, the coresponding $D(A^\infty; M_k)$ space is called: the space of vectors of Gevrey class α.

Example 1.2.

If $M_k = k!$, the coresponding $D(A^\infty; M_k)$ is the space of analytic vectors. (See [8])

Remark 1.1

Definition 1.2 is purely algebraic. There is a "natural" locally convex topology on $D(A^\infty; M_k)$: firstly, fix L in (1.2) (but not C) and call $D^L(A^\infty; M_k)$

the coresponding space; provided with the norm $\sup_{k\geq 0} \frac{1}{L^k M_k} ||A^k u||$, it is a Banach space; then $D(A^\infty; M_k)$ = inductive limit of $D^{L_n}(A^\infty; M_k)$, $L_n \to +\infty$. For details see [4].

Remark 1.2.

Hypothesis (1.1) is perfectly useless in Definition (1.2). But it will be useful in the proofs below.

The "natural questions" are now:

(i) when is $D(A^\infty; M_k) \neq \{O\}$?

(ii) what is the "abstract" interest of $D(A^\infty; M_k)$?

(iii) how can one characterice, in "concrete" situations, the spaces $D(A^\infty; M_k)$ in "concrete" terms?

Partial answers to these questions are respectively given in Sections 2, 3, 4 below — some applications being given in Sections 5, 6, 7.

2. A criterion of non triviality.

Theorem 2.1. *Let $\{M_k\}$ be a non quasi-analytic sequence*[1] [1] [7]. *Then $D(A^\infty; M_k)$ is dense in E.*

Proof. 1) If $\{M_k\}$ is non quasi-analytic, one can find a sequence ϱ_n of functions with the following properties [7] [9]

$$(2.1) \begin{cases} \varrho_n \in D_{M_k},\ \varrho_n(t) = 0 \text{ if } t \leq 0 \text{ or if } t \geq \varepsilon_n,\ \varepsilon_n \to 0 \text{ if } n \to \infty, \\ \varrho_n \geq 0,\ \int_0^\infty \varrho_n(t)\, dt = 1. \end{cases}$$

2) Define next $G(\varrho_n) \in L(E;E)$ by

$$(2.2)\quad G(\varrho_n)\, e = \int_0^\infty G(t)\, e \,.\, \varrho_n(t)\, dt,\ e \in E$$

One checks easily that $G(\varrho_n)\, e \in D(A^\infty)$ and that

(2.3) $A^k G(\varrho_n)\, e = G(\varrho_n^{(k)})\, e\ \forall\ k$.

Thanks to the fact that $\varrho_n \in D_{M_k}$ it follows that $G(\varrho_n)\, e \in D(A^\infty; M_k)$.

3) Let now e be arbitrarely given in E; by (2.1) $G(\varrho_n)\, e \to e$ in E, and by 2), $G(\varrho_n)\, e \in D(A^\infty; M_k)$, hence the result follows.

Remark 2.1. It can happen that $D(A^\infty; M_k)$ is dense in E even with $M_k = 1\ \forall\ k$ example: assume that A has a complete set in E of eigenvectors ω_n then $A\omega_n = \lambda_n \omega_n$ hence $||A^k \omega_n|| \leq ||\omega_n||\, \lambda_n^k$, i.e. belongs to $D(A^\infty; 1)$.

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(2) This means: let D_{M_k} be the space of C^∞ scalar functions φ on R with compact support and satisfying $|\ldots\ | \leq |\varphi^{(k)}(t)| \leq c L^k M_k\ \forall k$ then $D_{M_k} \neq \{O\}$.

But in can happen that $D(A^\infty; M_k) = \{O\}$ if M_k is quasi — analytic; example: $E = L^p(0, \infty)$, $A = \frac{d}{dx}$, $D(A) = \left\{f \mid f, \frac{df}{dx} \in L^p(0, \infty), f(0) = 0\right\}$.

3. The semi-group on $D(A^\infty; M_k)$.

Theorem 3.1. *The necessary and sufficient condition for $u \in E$ to be in $D(A^\infty; M_k)$ is that the function*

(3.1) $G(.)\, u = \text{"}t \to G(t)\, u\text{"}$

is of class M_k with values in E, i.e.:

(3.2) $\begin{cases} \text{for every finite } T \text{ there exist constants } C_1 \text{ and } L_1 \text{ (depending on } T \text{ and } u) \text{ such that} \\ \left\| \frac{d^k}{dt^k} G(t)\, u \right\| \leq C_1 L_1^k M_k \;\; \forall\, k, \;\; t \in [0, T]. \end{cases}$

Remark 3.1. This property justifies the terminology introduced in Examples 1.1 and 1.2.

Proof of Theorem 3.1.

1) (3.2) implies (1.2) (with $C = C_1$, $L = L_1$). Obvious, take $t = 0$ in (3.2) and use $\frac{d^k G(t)}{dt^k} \cdot u \,|_0 = (-1)^k A^k u$.

2) (1.2) implies (3.2). Obvious too. Indeed $\frac{d^k}{dt^k} G(t)\, u = (-1)^k G(t)\, A^k u$ hence, for $t \in [0, T]$

$$\left\| \frac{d^k G(t)}{dt^k} u \right\| \leq \sup_{t \in [0, T]} \|G(t)\|_{L(E;E)} \|A^k u\|,$$

hence 3.2 follows.

It follows easily from Theorem 3.1 that (see [4] for details).

Theorem 3.2. *For every t, $G(t)$ is a continuous linear mapping from $D(A^\infty; M_k)$ into itself; the semi group $G(t)$ in $D(A^\infty; M_k)$ is C^∞ (and of infinitesinal generator $- A$).*

One can also show [4] that if for a suitable constant d

(3.3) $M_{k+j} \leq d^{k+j} M_k M_j \;\; \forall\, k, j$

then for every $u \in D(A^\infty; M_k)$ the function $t \to G(t)\, u$ is of class M_k in $t \geq 0$ with values in $D(A^\infty; M_k)$ (i.e., for every finite T, there exists a bounded set B in $D(A^\infty; M_k)$ and a constant L such that $\frac{1}{L^k M_k} \frac{d^k}{dt^k} G(t)\, u \in B \;\forall\, k$, $t \in [0, T]$).

4. The case when A is an elliptic operator.

Let us recall first a classical definition: a complex-valued function φ defined on a compact set of R^n is said of Gevrey order $\beta > 1$ (resp. real analytic) if for suitable constants c and L one has

$$|D^p\varphi(x)| \leq cL^{p_1+\dots+p_n}(p_1!\, p_2! \dots p_n!)^\beta \quad (\text{resp. } \beta = 1)$$

$\forall\, p = \{p_1, \dots, p_n\}$, $\forall\, x \in$ compact set of definition of φ.

Let Ω be a bounded open set of R^n, of boundary Γ; we assume

(4.1) Γ is a $(n - 1)$ dimensional variety, of Gevrey order β (resp. real analytic)

Let A be a differential operator in Ω; we assume that

(4.2) A is an elliptic operator of order $2m$ (and properly elliptic if $n = 2$) and that

(4.3) the coefficients of A are of Gevrey order β (resp. real analytic) in $\overline{\Omega}$.

We are going to characterize $D(A^\infty; M_k)$, taking

(4.4) $E = L^2(\Omega)$.

(4.5) $D(A) = \{u \mid u \in H^{2m}(\Omega) \cap H_0^m(\Omega)\}$ (that is: $D^p u \in L^2(\Omega)\ \forall\, p$, $|p| \leq 2m$, $D^p u = 0$ on $\Gamma\ \forall$, $|p| \leq m - 1$),

and when we choose

(4.6) $M_k = [(2km)!]^\beta$.

One can prove (see [5], [6], [4]):

Theorem 4.1. *We assume the hypotheses* (4.1), (4.2), (4.3) *to hold choosing* $D(A)$ *and* M_k *by* (4.5) (4.6) *one has*

(4.7) $D(A^\infty; M_k) \equiv$ *functions of Gevrey order* β *in* $\overline{\Omega}$ *(resp. real analytic) which satisfy the boundary conditions* "$A^k u \in H_0^m(\Omega)\ \forall\, k$".

Remark 4.1. Under the hypothesis (4.2), $-A$ is the infinitesimal generator of a semi-group in E and even of an analytical semi-group. [2], [10].

One can replace $E = L^2(\Omega)$ by $L^p(\Omega)$, $1 < p < \infty$, $p \neq 2$, without changing $D(A^\infty; M_k)$.

Remark 4.2. The same result holds true for other boundary conditions than the Dirichlet boundary conditions considered above. — See [4].

Remark 4.3. If u satisfies $||A^k u|| \leq cL^k((2km)!)\ \forall\, k$ and no boundary conditions, then one can conclude that u is real analytic on every compact subset of Ω; see [3]; this result in contained in Theorem 4.1.

Remark 4.4. A more general result is proved in [4] when we also consider "non-zero boundary conditions".

5. Transposition

Since E is assumed to be a reflexive Banach Space (actually "reflexive" is used here for the first time — and in a non essential manner!) all what we said in Sections 1, 2, 3 is valid after replacing

E by E' = dual of E

$G(t)$ by $G^*(t)$ = adjoint of $G(t)$

A by A^*, A^* being the adjoint of A in the sense of unbounded operators in E or the (opposite to the) infinitesimal generator of the adjoint semi-group $G^*(t)$.

Consequently:

(5.1) $G^*(t)$ is a semi-group in $D(A^{*\infty}; M_k)'$.

If we make the hypothesis (see Theorem 1.1):

(5.2) $D(A^{*\infty}; M_k)$ is dense in E'

then we can identify E to a sub-space of the dual $D(A^{*\infty}; M_k)'$ of $D(A^{*\infty}; M_k)$; summing up, we have

(5.3) $D(A^{\infty}; M_k) \subset E \subset D(A^{*\infty}; M_k)!$

Taking the adjoint of (5.1) we obtain:

(5.4) $[G^*(t)]^*$ is a semi-group in $D(A^{*\infty}; M_k)'$.

But one easily checks that $(G^*(t))^*$ is an extension of $G(t)$, that we can still denote by $G(t)$. Therefore:

(5.5) $\begin{cases} G(t) \text{ is a semi-group in } D(A^{*\infty}; M_k)', \text{ which is } C^\infty \text{ and whose infinite-} \\ \text{simal generators is } -A. \end{cases}$

For more details, see [4].

Remark 5.1. In the applications, $D(A^{*\infty}; M_k)'$ is not a space of distributions but a space of functionals (analytic functionals of Gervey's functionals). Structure theorems for the elements of $D(A^{*\infty}; M_k)'$ are given in [4].

6. Cauchy problem.

If $-A$ is the infinitesimal generator of a semi-group $G(t)$, then the unique solution of the Cauchy problem

(6.1) $Au + u' = 0 \qquad \left(u' = \dfrac{du}{dt}\right)$,

(6.2) $\begin{cases} u(t) \in D(A), \\ u(0) = u_0 \end{cases}$

is given by

(6.3) $u(t) = G(t)\, u_0$.

See [2], [10].

Thanks to Theorem 3.2 and its "transposed" version (5.5) we obtain:

Theorem 6.1. *We assume that* (5.2) *holds true — For* u_0 *given in* $D(A^\infty; M_k)$ *(resp. in* $D(A^{*\infty}; M_k)'$*) the Cauchy problem* (6.1), (6.2) *admits a unique solution, given by* (6.3), *which is* C^∞ *from* $t \geq 0 \to D(A^\infty; M_k)$ *(resp.* $D(A^{*\infty}; M_k)'$*). Moreover, in case* (3.3) *holds true, the solution* $u(t)$ *is of class* M_k.

Remark 6.1. In case $G(t)$ is analytic (see Remark 4.1) then, even starting with $u_0 \in D(A^{*\infty}; M_k)'$ (i.e. with an extremely general Cauchy data), one has

$$u(t) \in D(A^\infty; M_k) \ \forall\, t > 0.$$

See [4].

7. Some examples.

We take the two as simple as possible cases.

7.1. *Heat equation.*

Combining results of Sections 4 and 6 we obtain the following result: let u_0 be given in Ω, satisfying

$$(7.1)\quad \begin{cases} u_0 \text{ is of Gevrey order } \beta \text{ (resp. real analytic) in } \overline{\Omega}, \text{ and } \Delta^k u_0 = 0 \text{ on} \\ \Gamma\ \forall\, k. \end{cases}$$

Then the solution of

$$(7.2)\quad -\Delta u + \frac{\partial u}{\partial t} = 0 \text{ in } \Omega \times]0, \infty[,$$

$$(7.3)\quad u(x, t) = 0 \text{ if } x \in \Gamma,\ t > 0,$$

$$(7.4)\quad u(x, 0) = u_0(x),\ x \in \Omega$$

is of Gevrey order β in x (resp. real analytic if $\beta = 1$) and of Gevrey order 2β in t.

We have just to take: $M_k = [(2k)!]^\beta$ in the general theory.

Moreover in this case Remark 6.1 applies —

7.2. *Wave equation.*

We consider now

$$(7.5)\quad -\Delta u + \frac{\partial^2 u}{\partial t^2} = 0 \text{ in } \Omega \times]0, \infty[,$$

$$(7.6)\quad u(x, t) = 0 \text{ if } x \in \Gamma,\ t > 0,$$

$$(7.7)\quad \begin{cases} u(x, 0) = u_{00}(x),\ x \in \Omega, \\ \dfrac{\partial u}{\partial t}(x, 0) = u_{01}(x),\ x \in \Omega. \end{cases}$$

Writing (7.5) as a first order system in t one can apply semi-group theory. One obtains:

$$(7.8) \begin{cases} \text{if } u_{01} \text{ and } u_0, \text{ satisfy conditions analogous to (7.1) for } u_0, \text{ then } u(x, t) \\ \text{is of Gevrey order } \beta \text{ in } x \text{ and in } t. \end{cases}$$

See [4] Vol 3 for technical details.

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