Roberto Conti

Problems in linear control theory

In: Valter Šeda (ed.): Differential Equations and Their Applications, Proceedings of the Conference held in Bratislava in September 1966. Slovenské pedagogické nakladateľstvo, Bratislava, 1967. Acta Facultatis Rerum Naturalium Universitatis Comenianae. Mathematica, XVII. pp. 73--80.

Persistent URL: http://dml.cz/dmlcz/700200

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XVII — 1967

PROBLEMS IN LINEAR CONTROL THEORY

R. CONTI, Firenze

1.

Given a Banach space X and a real $T > 0$ let $A: t \to A(t)$ be a function of $t \in [0, T]$ with values in the space of linear (possibly unbounded) operators in X.

We shall assume the existence of the Green's function (evolution operator) associated with A. By this we mean a function $G : t, s \to G(t, s)$ defined for $0 \leq s \leq t \leq T$, with values in the space $\mathscr{L}(X, X)$ of linear bounded operators in X, strongly continuous in the two variables jointly and satisfying the conditions:

$$G(t, s)\, G(s, r) = G(t, r), \qquad 0 \leq r \leq s \leq t \leq T,$$
$$G(s, s) = \mathbf{1} \qquad \text{(the identity in } X\text{)}$$

$$\frac{\partial G(t, s)\, x}{\partial t} = A(t)\, G(t, s)\, x, \qquad x \in D(A(s))$$

$$\frac{\partial G(t, s)\, x}{\partial s} = -G(t, s)\, A(s)\, x, \qquad x \in D(A(s))$$

where $\partial/\partial t$, $\partial/\partial s$ denote strong derivatives and $D(A(s)) \subset X$ is the domain of $A(s)$.

There are various known sufficient conditions for the existence of Green's function (T. KATO [9], J. KISYNSKI [10], E. T. POULSEN [14]).

Let $1 \leq p \leq \infty$. Given a Banach space E we denote by $L^p(0, T; E)$ the Banach space of all E-valued, strongly measurable functions f defined in $[0, T]$, such that

$$|f|_p = \left(\int_0^T |f(t)|_E^p \mathrm{d}t\right)^{1/p} < \infty \qquad \text{if } p < \infty$$

$$|f|_\infty = \operatorname{ess\,sup} \{|f(t)|_E : 0 \leq t \leq T\} < \infty, \qquad \text{if } p = \infty.$$

If $c : t \to c(t)$ belongs to $L^1(0, T; X)$ then

$$\int_0^t G(t, s)\, c(s)\, \mathrm{d}s \in X, \qquad 0 \leq t \leq T$$

the integral understood in the sense of Bochner.

Beside X we shall, consider another Banach space U and the space $\mathscr{L}(U,X)$, of linear bounded operators from U into X.

Let $B : t \to B(t)$ belong to $L^{p'}(0, T; \mathscr{L}(U, X))$ with $p' = p(p-1)^{-1}$ for $1 < p < \infty$, $p' = 1$ for $p = \infty$, $p' = \infty$ for $p = 1$.

If $u : t \to u(t)$ belongs to $L^p(0, T; U)$ then $t \to B(t)\, u(t)$ will belong to $L^1(0, T; X)$ and

$$\int_0^t G(t, s)\, B(s)\, u(s)\, \mathrm{d}s \in X, \qquad 0 \leq t \leq T.$$

Summing up, if G exists and if $v \in X$, $u \in L^p(0, T; U)$, $B \in L^{p'}(0, T; \mathscr{L}(U,X))$, $c \in L^1(0, T; X)$, we may define

$$(1.1)\quad x(t, u, v) = G(t, 0)\, v + \int_0^t G(t, s)\, B(s)\, u(s)\, \mathrm{d}s + \int_0^t G(t, s)\, c(s)\, \mathrm{d}s, \quad 0 \leq t \leq T.$$

We shall denote by V, W, and $\mathscr{U}$ three convex, bounded, closed subsets of X, X and $L^p(0, T; U)$ respectively and consider the following:

Problem *P. Given X, U, p, T, A (or rather G), B, c, V, W, $\mathscr{U}$, determine whether there are $v \in V$, $u \in \mathscr{U}$ such that $x(T, u, v) \in W$.*

A few comments before we go further.

Equation (1.1) can be considered as the Bochner integral version of the linear differential equation

$$(1.2)\qquad \mathrm{d}x/\mathrm{d}t - A(t)\, x = B(t)\, u(t) + c(t)$$

with initial condition

$$(1.3)\qquad x(0, u, v) = v.$$

Sufficient conditions in order that (1.1) yield (1.2) are known (T. Kato [9], J. Kisynski [10], E. T. Poulsen [14]).

The problem we are dealing with is a typical one in linear control theory where x represents the state of some physical system, u, v are controls, permanent and initial, respectively, and it is required to determine such controls from given sets $\mathscr{U}$, V which transfer x from V into W in a given time interval $[0, T]$ along a trajectory of (1.2).

If $\dim X < \infty$ then (1.1) is in fact equivalent to the ordinary differential equation (1.2) and $G(t, s) = \Phi(t)\, \Phi^{-1}(s)$ where $\Phi(t)$ is any fundamental matrix associated with A. However control problems involving partial differential equations (distributed parameter controls) require that also infinite dimensional spaces X be considered (A. G. Butkovskii [3], P. K. C. Wang [16]).

2.

The linear operator

$$\Gamma_T : x \to G(T, 0)\, x$$

from X into itself is bounded, therefore the image $\Gamma_T V$ of V is a bounded convex subset of X.

Also the linear operator

$$\Lambda_T : u \to \int_0^T G(T, s)\, B(s)\, u(s)\, ds$$

from $L^p(0, T; U)$ into X is bounded and the image $\Lambda_T \mathcal{U}$ of $\mathcal{U}$ is a bounded convex subset of X.

Therefore $W - \Gamma_T V - \Lambda_T \mathcal{U}$ is a bounded convex subset of X.

By virtue of (1.1) Problem P reduces then to establish whether

$$-\int_0^T G(T, s)\, c(s)\, ds \in -W + \Gamma_T V + \Lambda_T \mathcal{U}. \tag{2.1}$$

Let us first consider the weaker relation

$$-\int_0^T G(T, s)\, c(s)\, ds \in \overline{-W + \Gamma_T V + \Lambda_T \mathcal{U}}, \tag{2.2}$$

the closure of $-W + \Gamma_T V + \Lambda_T U$.

Recall that for any bounded subset $C \subset X$ a supporting function $h_C(x')$ is defined in the dual space X' by

$$h_C(x') = \sup_{x \in C} \langle x, x' \rangle$$

We need the following lemmas.

Lemma 1.

$$h_{\overline{C}}(x') = h_C(x'), \qquad x' \in X' \tag{2.3}$$

Proof. Since $C \subset \overline{C}$ it follows $h_C(x') \leq h_{\overline{C}}(x')$ by definition. Conversely, for a fixed $x' \in X'$ let $x_k \in \overline{C}$ be such that $\lim_k \langle x_k, x' \rangle = \sup_{x \in \overline{C}} \langle x, x' \rangle = h_{\overline{C}}(x')$. Now choose $\chi_k \in C$, $|\chi_k - x_k|_X < k^{-1}$.

Then $\langle x_k, x' \rangle = \langle \chi_k, x' \rangle + \langle x_k - \chi_k, x' \rangle \leq h_C(x') + k^{-1}|x'|_{X'}$, and letting $k \to \infty$ we have $h_{\overline{C}}(x') \leq h_C(x')$.

Lemma 2. *If C is a bounded convex set $\subset X$, then*

$$\langle \chi, x' \rangle \leq h_C(x'), \qquad x' \in X' \Leftrightarrow \chi \in \overline{C}. \tag{2.4}$$

Proof. $\chi \in \overline{C}$ means $\langle \chi, x' \rangle \leq \sup_{x \in \overline{C}} \langle x, x' \rangle = h_{\overline{C}}(x') = h_C(x')$ by lemma 1. Let $\chi \notin \overline{C}$, i.e. let $\{\chi\} \cap \overline{C}$ be void. Since $\{\chi\}$, $\overline{C}$ are convex, closed sets and $\{\chi\}$ is compact the "strict separation" theorem holds, i.e. there are two real

numbers $\varepsilon > 0$, c and some $\chi' \in X'$ such that $\langle x, \chi' \rangle \leq c - \varepsilon < c \leq \langle \chi, \chi' \rangle$ $x \in \overline{C}$, hence $h_{\overline{C}}(\chi') \leq \langle \chi, \chi' \rangle$ and $h_C(\chi') < \langle \chi, \chi' \rangle$ by lemma **1.**

By applying (2.4) to (2.2) we have

Theorem 1. *The inequality*

$$(2.5)\quad \langle -\int_0^T G(T, s)\, c(s)\, ds, x' \rangle \leq h_{-W+\Gamma_T V+\Lambda_T U}(x'), \qquad x' \in X'$$

is equivalent to (2.2), *therefore it is equivalent to* (2.1) *iff the set* $-W + \Gamma_T V + \Lambda_T \mathcal{U}$ *is closed.*

3.

We are now going to indicate some criteria for the validity of

$$(3.1)\quad -W + \Gamma_T V + \Lambda_T \mathcal{U} = \overline{-W + \Gamma_T V + \Lambda_T \mathcal{U}}.$$

This can be insured by

$$(3.2)\quad W = \overline{W}, \qquad \Gamma_T V = \overline{\Gamma_T V}, \qquad \Lambda_T \mathcal{U} = \overline{\Lambda_T \mathcal{U}},$$

plus an additional assumption namely that

(3.3) *X is a reflexive Banach space.*

We recall in fact that in a Banach space X: *i*) all bounded weakly closed subset are weakly compact iff X is reflexive; *ii*) convex sets are weakly closed iff they are closed; *iii*) any finite sum of weakly compact sets is weakly closed. The implication (3.2) + (3.3) $\Rightarrow$ (3.1) then follows from the fact that all sets involved are convex and bounded.

Now $W = \overline{W}$ by assumption. Also $\Gamma_T V = \overline{\Gamma_T V}$ since Γ_T, as a linear operator continuous in the norm topology of X is also weakly continuous and V is, by assumption, weakly compact. On the contrary the validity of $\Lambda_T \mathcal{U} = \overline{\Lambda_T \mathcal{U}}$ requires some further assumption on $\mathcal{U}$. In particular the case $p = 1$ has to be put aside since there are examples of $\Lambda_T \mathcal{U} \neq \overline{\Lambda_T \mathcal{U}}$ in $L^1(0, T; U)$ even for $U = R$, the real number system.

Therefore we shall consider, from now on, only the case $1 < p \leq \infty$ and make a further assumption, namely

$$U = \varrho \mathcal{U}_1$$

with given $\varrho > 0$ and $\mathcal{U}_1 = \{u : |u|_p \leq 1\}$, the unit ball of $L^p(0, T; U)$. What we have to show is then that $\Lambda_T \mathcal{U}_1$ is (weakly) closed, or, equivalently, weakly compact.

Since Λ_T is continuous (in the norm hence) in the weak topologies of $L^p(0, T; U)$, X, we have weak compactness of $\Lambda_T \mathcal{U}_1$ when also $\mathcal{U}_1$ is weakly compact, which is equivalent to the assumption that

(3.4) *$L^p(0, T; U)$ is a reflexive Banach space*[1].
We thus have

Theorem 2. *Let X be a reflexive Banach space and let V, W be convex, bounded, closed subsets of X.*

Then Problem P has solutions if, $\mathscr{U} = \varrho\mathscr{U}_1$, $\varrho > 0$, $\mathscr{U}_1$ the unit ball of $L^p(0, T; U)$, $1 < p < \infty$ and U is such that $L^p(0, T; U)$ be reflexive.

Let us now turn to the case $p = \infty$.

We have (P. L. FALB [6]).

Lemma 3. *If U is such that $L^p(0, T; U)$ is reflexive, $1 < p < \infty$, then the unit ball $\mathscr{U}_1$ of $L^\infty(0, T; U)$ is a weakly compact subset of $L^p(0, T; U)$.*

Proof. Clearly $\mathscr{U}_1$ is a bounded subset of $L^p(0, T; U)$. Further if a sequence $u_k \in \mathscr{U}_1$ converges in $L^p(0, T; U)$ towards some $v \in L^p(0, T; U)$ then $v \in \mathscr{U}_1$, i.e. $\mathscr{U}_1$ is a closed subset of $L^p(0, T; U)$. In fact $u_k \to v$ in measure, hence $u_{k_n} \to v$ a.e. in $[0, T]$ for some subsequence u_{k_n}. Since $|u|_U \leq 1$ is closed, $|v(t)|_U \leq 1$ a.e. in $[0, T]$, i.e. $v \in \mathscr{U}_1$. Since $\mathscr{U}_1$ is also convex it is also weakly closed in $L^p(0, T; U)$, hence is weakly compact in $L^p(0, T; U)$ as $L^p(0, T; U)$ is reflexive.

From this follows

Theorem 2'. *Let X, V, W be as in Theorem 2.*

Then Problem P has solutions if $\mathscr{U} = \varrho\mathscr{U}_1$, $\varrho > 0$, $\mathscr{U}_1$ the unit ball of $L^\infty(0, T; U)$, provided that $L^p(0, T; U)$, $1 < p < \infty$ be reflexive, and

(3.5) *$B \in L^{1+\alpha}(0, T; \mathscr{L}(U, X))$, for some $\alpha > 0$.*

Proof. In fact (3.5) allows to consider Λ_T as a mapping of $L^{1+1/\alpha}(0, T; U)$ into X, continuous (in the norm, hence) in the weak topologies and by lemma 3 ($p = 1 + 1/\alpha$) it follows, again, that $\Lambda_T\mathscr{U}_1$ is a weakly compact subset of X.

Assumption (3.5) is actually stronger than $B \in L^1(0, T; \mathscr{L}(U, X))$ which would be the natural one in the case $u \in L^\infty(0, T; U)$. It can be avoided, however, at the expense of heavier assumptions on U, X, by using a particular case of the well-known Alaoglu's theorem, namely

Lemma 4. *If $L^\infty(0, T; U) = (L^1(0, T; U'))'$, then the unit ball $\mathscr{U}_1$ of $L^\infty(0, T; U)$ is weakly * compact.*

Let u_k be any sequence in $\mathscr{U}_1$. We may assume that u_k converges weakly * towards some $u \in \mathscr{U}_1$, i.e.

$$(3.6) \qquad \int_0^T \langle v, u_k \rangle \, dt \to \int_0^T \langle v, u \rangle \, dt \qquad \text{for all } v \in L^1(0, T; U').$$

This will imply $\Lambda_T u_k \to \Lambda_T u$ strongly in X in some cases, for instance when

---

[1] Recall that the reflexivity of $L^p(0, T; U)$ depends on U, but not on p, $1 < p < \infty$.

U, X are both finite dimensional: $\dim U = m$, $\dim X = n$. In fact $\Lambda_T u_k$, $\Lambda_T u$ are n-vectors with components, respectively

$$\int_0^T \langle v_j, u_k \rangle \, dt, \quad \int_0^T \langle v_j, u \rangle \, dt, \qquad j = 1, 2, \ldots, n$$

where v_j denotes the j. th row of the n by m matrix $G(T, t) B(t)$.

We thus have (H. A. ANTOSIEWICZ [1]).

Theorem 2''. *Let V, W, be convex, bounded, closed subsets of X, $\dim X = n$. Then Problem P has solutions if $\mathcal{U} = \varrho \mathcal{U}_1$, $\varrho > 0$, $\mathcal{U}_1$ the unit ball of $L^\infty(0, T; U)$, $\dim U = m$.*

4.

We shall now write the right hand side of (2.5) under the assumption $\mathcal{U} = \varrho \mathcal{U}_1$ in a more explicit form. We have

$$h_{-W+\Gamma_T V+\varrho \Lambda_T \mathcal{U}_1}(x') = h_{-W}(x') + h_{\Gamma_T V}(x') + \varrho h_{\Lambda_T U_1}(x')$$

with

$$h_{\Gamma_T V}(x') = \sup_{v \in \Gamma} \langle v, x' G(T, 0) \rangle$$

and

$$h_{\Lambda_T U_1}(x') = \left(\int_0^T |x' G(T, s) B(s)|_{U'}^{p'} ds \right)^{1/p'}.$$

Therefore (2.5) becomes

$$(4.1) \quad \langle -\int_0^T G(T, s) c(s) \, ds, x' \rangle \leq \sup_{w \in -W} \langle w, x' \rangle + \sup_{v \in \Gamma} \langle v, x' G(T, 0) \rangle +$$

$$+ \varrho \left(\int_0^T |x' G(T, s) B(s)|_{U'}^{p'} ds \right)^{1/p'}, \quad x' \in X'.$$

This inequality already appeared in the literature in many particular instances, both finite (H. A. ANTOSIEWICZ [1], R. CONTI [4], R. GABASOV—F. M. KIRILLOVA [8], W. T. REID [15]) and infinite dimensional (W. MIRANKER [11], G. MOCHI [12]).

5.

Some existence theorems for certain typical optimum control problems can be drawn from (4.1) along the lines followed by H. A. ANTOSIEWICZ [1] in the finite dimensional case.

a) Let ϱ_0 be the infimum of ϱ's such that (4.1) holds and let $\varrho_k \downarrow \varrho_0$ be a sequence of such ϱ's. Then (4.1) must hold also with $\varrho = \varrho_0$ and we have

Theorem 3. *Under the assumptions of Theorems 2,2', 2'' if Problem P has a solution, then it also has a solution v, u with minimum $|u|_p$.*

Sometimes $|u|_p$ is called the "effort" associated with the control system and Theorem 3 states that under the assumptions of Theorems 2, 2', 2'' there is a solution of the minimum effort control problem (W. A. PORTER—J. P. WILLIAMS [13]) as soon as the corresponding control problem has solutions.

b) Another typical problem in optimum control theory is the so-called "final value" problem (A. V. BALAKRISHNAN [2]). For instance it is required to minimize $|x(T, u, v) - w^0|_X$ for a given $w^0 \in X$. To this purpose we may assume the set W to be a closed ball of radius $\varepsilon > 0$ with center at w^0, i.e. $W = \{w^0\} + \varepsilon X_1$, X_1 the unit ball of X. Then $-W = \{-w^0\} + \varepsilon X_1$, and $h_{-W}(x') = -\langle w^0, x'\rangle + \varepsilon|x'|_{X'}$. Substituting into (4.1), the same argument we used for ϱ, applied to the infimum of ε's for which (4.1) holds leads to

Theorem 4. *Under the assumptions of Theorems 2.2', 2'' if Problem P with* $W = \{w^0\} + \varepsilon X_1$ *has a solution, then it also has a solution* v, u *such that* $|x(T, u, v) - w^0|_X$ *is minimum.*

c) In a similar way we could consider an "initial value" problem by taking $V = \{v^0\} + \sigma X_1$, $\sigma > 0$. Then $h_{\Gamma_T V}(x') = \langle v^0, x'G(T, 0)\rangle + \sigma|x'G(T, 0)|_{X'}$, etc.

d) The best known problem in optimum control theory is perhaps the "minimum time" problem: to find solutions yielding the minimum time T of transfer from V to W.

Since both sides of (4.1) are continuous functions of T, denoting by T_0 the infimum of T's for which (4.1) holds and by $T_k \downarrow T$ a sequence of such T's we obtain

Theorem 5. *Under the assumptions of Theorems 2,2', 2'' if Problem P has a solution, then it also has a solution such that T is minimum.*

For an infinite dimensional X particular cases of this Theorem were obtained by Y. V. EGOROV [5], H. O. FATTORINI [7], A. V. BALAKRISHNAN [2].

REFERENCES

[1] H. A. ANTOSIEWICZ, *Archive Rat. Mech. Anal., 12 (1963), 313.*

[2] A. V. BALAKRISHNAN, *Jour. SIAM Control, 3 (1965), 152.*

[3] A. G. BUTKOVSKII, *Teoriya optimal'nogo upravleniya sistemami s raspredelennymi parametrami, Izdat. Nauka, Moskva, 1965.*

[4] R. CONTI, *Jour. Diff. Equations, 1 (1965), 427.*

[5] Y. V. EGOROV, *Matem. Sbornik, 64 (106) (1964), 79.*

[6] P. L. FALB, *Jour. Math. Anal. Appl., 9 (1964), 12.*

[7] H. O. FATTORINI, *Jour. SIAM Control, 2 (1964), 54.*

[8] R. GABASOV—F. M. KIRILLOVA, *Avtom. i Telem., 25 (1964), 1058.*

[9] T. KATO, *A. M. S. Symposia in Appl. Math., vol. 17 (1965), 50.*

[10] J. KISYNSKI, *Studia Math., 23 (1964), 285.*
[11] W. MIRANKER, *Jour. Math. Anal. Appl., 10 (1965), 378.*
[12] G. MOCHI, *Boll. Un. Mat. Ital., (3) 21 (1966), 35.*
[13] W. A. PORTER—J. P. WILLIAMS, *Jour. Math. Anal. Appl., 13 (1966), 251.*
[14] E. T. POULSEN, *Math. Zeitsch., 90 (1965), 286.*
[15] W. T. REID, *Duke Math. Jour., 29 (1962), 591.*
[16] P. K. C. WANG, *Advances in Control Theory, edited by C. T. Leondes, vol. 1, Academic Press, 1964, 75.*