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ON A CERTAIN BOUNDARY VALUE PROBLEM OF THE THIRD ORDER

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1. The boundary value problem of the form

$$y''' + [f(x) + \lambda g(x)]y' + \lambda h(x)y = 0 \tag{1}$$

$$y(-a,\lambda) = y(a,\lambda) = 0, \quad a > 0 \tag{2}$$

$$\lambda \int_{-a}^{a} [r(t)-k]\{g(t)y(t,\lambda) + \int_{-a}^{t} [h(\tau) - g'(\tau)]y(\tau,\lambda)d\tau\}dt = \tag{3}$$

$$= \int_{-a}^{a} [r(t)-k]\{f(t)y(t,\lambda) + \int_{-a}^{t} f(\tau)y'(\tau,\lambda)d\tau\}dt ,$$

where $f'(x)$, $g'(x)$, $h(x)$, $r''(x)$ are continuous functions on the interval $\langle -a,a\rangle$ and k is a constant, will be studied.

The boundary condition (3) is in the integral form. For the first time, such a condition was formulated in [1] and the problem (1), (2), (3) is a natural generalization of the problem discussed in [1].

It will be shown that under certain conditions on the function $r(x)$, the problem (1), (2), (3), is equivalent to the boundary problem (1), (4), where

$$y(-a,\lambda) = y''(-a,\lambda) = 0, \quad y(a,\lambda) = 0 \quad a > 0 . \tag{4}$$

In order to solve the problem (1), (4), the theory of the third order linear differential equation [2] can be applied. Moreover some special results will be formulated.

2. Consider the problem (1), (2), (3). Let the functions f, g, h, r fulfil the conditions formulated in Section 1. Then the following theorem is true.

THEOREM 1. *The problem (1), (2), (3) is equivalent to the problem (1), (4) if the function $r = r(t)$ solves the problem*

$$r'' + f(t)r = kf(t) \tag{5}$$

$$r(-a) = k, \quad r(a) = k . \tag{6}$$

Proof. Integrating the differential equation (1), written in the form

$$y''' + \{[f(x) + \lambda g(x)]y\}' + \{-f'(x) + \lambda[h(x) - g'(x)]\}y = 0 ,$$

term by term from $-a$ to x, $x \le a$, and considering (2), we get

$$y'' + f(x)y + \lambda g(x)y + \int_{-a}^{x} \{-f'(\tau) + \lambda[h(\tau) - g'(\tau)]\}y(\tau,\lambda)d\tau =$$

$$= y''(-a,\lambda) .$$

Now suppose that $y''(-a,\lambda) = 0$, multiply the last equality by $r(x)-k$, where k is a constant, and integrate it from $-a$ to a. We come to the equality

$$\text{(7)} \qquad \int_{-a}^{a} [r(t)-k][y''(t,\lambda) + f(t)y(t,\lambda)]dt =$$

$$= \int_{-a}^{a} [r(t)-k]\{g(t)y(t,\lambda) +$$

$$+ \int_{-a}^{t} [h(\tau)-g'(\tau)]y(\tau,\lambda)d\tau\}dt -$$

$$- \int_{-a}^{a} [r(t)-k]\{-f(t)y(t,\lambda) + \int_{-a}^{t} f(\tau)y'(\tau,\lambda)d\tau\}dt .$$

The right-hand side of (7) contains the expression which stands in condition (3). Therefore it is necessary to prove that the integral on the left-hand side of (7) is equal to zero.

Calculate this integral. Under the conditions (2), it follows that

$$\int_{-a}^{a} [r(t)-k][y''(t,\lambda) + f(t)y(t,\lambda)]dt = y'(a,\lambda)[r(a)-k] -$$

$$-y'(-a,\lambda)[r(-a)-k] + \int_{-a}^{a} [r''(t) + f(t)r(t)-k\, f(t)]y(t,\lambda)dt .$$

This implies that the boundary condition (3) will be fulfilled if $y''(-a,\lambda) = 0$ and if the function $r(t)$ solves the boúndary problem (5), (6). Thus the theorem is proved.

3. In [3] the problem

$$\text{(8)} \qquad y''' + \{[1 + \lambda g(x)]y\}' = 0$$

$$\text{(9)} \qquad y(-a) = y(a) = 0, \qquad \int_{-a}^{a} (\cos t - \cos a)g(t)y(t)dt = 0$$

and its generalization

$$(10)\qquad y''' + [1 + \lambda g(x)]y' + \lambda h(x)y = 0$$

$$(11)\qquad y(-a) = y(a) = 0, \qquad \int_{-a}^{a} (\cos t - \cos a)\{g(t)y(t) + $$

$$+ \int_{-a}^{x} [h(\tau)-g'(\tau)]y(\tau)d\tau\}dt = 0$$

where $a > 0$, $g'(x)$, $h(x)$ are continuous functions on $\langle -a,a\rangle$, were discussed.

REMARK 1. The problems (8), (9), and (10), (11) are special cases of the problem (1), (2), (3).

Clearly, if we suppose $f(x) = 1$, $h(x) = g'(x)$, $k = \cos a$, we get that (8), (9) is a special case of (1), (2), (3) and from Theorem 1 it follows hat $r(x) = \cos x$. Similarly, if $f(x) = 1$ and $k = \cos a$ we get that (10), (11) is a special case of (1), (2), (3) if $r(x) = \cos x$ and $k = \cos a$. But $r(x) = \cos x$ solves the problem (5), (6), where $k = \cos a$ and $f(x) = 1$.

In [3] it has been proved that under the condition $a = \pi/2$ the problems (8), (9) and (10), (11), respectively are equivalent to the problems (8), (4) and (10), (4) respectively.

Now we prove the following theorem (the formulation will be only for the equation (8), in the case of the equation (10) the equation is similar).

THEOREM 2. *Let* $g(x)$ *be continuous on* $\langle -a,a\rangle$ *and let* $0 < a < \pi/2$. *Then the problem* (8), (13), *where*

$$(13)\qquad y(-a) = y(a) = 0, \qquad \int_{-a}^{a} [r(t)-1] \quad g(t)y(t)dt = 0$$

is equivalent to the problem (8), (4) *if*

$$(14)\qquad r(x) = \int_{-a}^{a} G(x,\xi)d\xi + \varphi_1(x) + \varphi_2(x) ,$$

where $G(x,\xi)$ *is the Green function of the problem*

$r'' + r = 0$, $r(-a) = r(a) = 0$, $0 < a < \frac{\pi}{2}$,

$\varphi_1(x)$, *and* $\varphi_2(x)$, *respectively, are the solutions of the problem* $r'' + r = 0$, $r(-a) = 1$, $r(a) = 0$, *and of the problem* $r'' + r = 0$, $r(-a) = 0$, $r(a) = 1$ *respectively.*

The proof of Theorem 2 is similar to that of Theorem 1. Integrating (8) term by term from $-a$ to $x \le a$ and considering (2), we get $y'' + y + \lambda g(x)y = y''(-a,\lambda)$.
Let $y''(-a,\lambda) = 0$, multiply this equation by $r(x)-1$ and integrate it from $-a$ to a. We obtain

$$(15) \qquad - \int_{-a}^{a} [y''(t) + y(t)][r(t)-1]dt = \lambda \int_{-a}^{a} [r(t)-1]g(t)y(t)dt \ .$$

It is necessary to find such an $r(x)$ that the integral on the left-hand side of (15) be equal to zero.
Calculating it we get

$$\int_{-a}^{a} [y''(t) + y(t)][r(t)-1]dt = y'(a)[r(a)-1] - y'(-a)[r(-a)-1] + \\ + \int_{-a}^{a} y(t)[r''(t) + r(t)-1]dt \ .$$

From this equality it follows that $r(x)$ must solve the problem

$$r'' + r = 1$$
$$r(-a) = 1 , \quad r(a) = 1 \ .$$

Thus the theorem is proved.

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