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SOME REGULARITY RESULTS FOR QUASILINEAR PARABOLIC SYSTEMS

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We present three results on the regularity of the weak solutions of quasilinear parabolic systems of second order. Two of them concern the relation between the regularity of the system and the Liouville property. The third one is the example of the system for which there exists a weak solution of the boundary value problem in the cylinder Q with Lipschitz continuous boundary data which develops the singularity in the interior of Q.

Denote $z = [t,x] \in R \times R^n$. Let us consider the system of m equations for m unknown functions $u = [u^1,\dots,u^m]$ of the form

$$u_t^i - D_\alpha(A_{\alpha\beta}^{ij}(z,u)D_\beta u^j) = -f^i + D_\alpha g_\alpha^i, \quad i = 1,\dots,m,$$

where we sum over j from 1 to m and over α, β from 1 to n. For the sake of brevity we rewrite it at the form

$$(1) \quad u_t - \operatorname{div}(A(z,u)Du) = -f + \operatorname{div} g.$$

1. Interior regularity. Let Q be a domain in $R \times R^n$. Suppose that the following assertions on A, f and g are satisfied:

(i) $A = A(z,u)$ is continuous on $Q \times R^m$.

(ii) $(A(z,u)\xi,\xi) > 0$ for all $[z,u] \in Q \times R^m$, $\xi \neq 0$.

(iii) $f \in L_{s,loc}(Q)$ with some $s > n/2 + 1$, $g \in L_{r,loc}(Q)$ with $r > n + 1$.

Denote by $W_{2,loc}^{0,1}(Q)$ the set of all functions belonging to the $L_{2,loc}(Q)$ together with their spatial derivatives. Recall that the function $u \in W_{2,loc}^{0,1}(Q)$ is a *weak solution of (1) in* Q if for all $\varphi \in \mathcal{D}(Q)$

$$\int_Q [(u,\varphi_t) - (ADu,D\varphi)]\,dz = \int_Q [(f,\varphi) + (g,D\varphi)]\,dz .$$

Definition 1. The system (1) is said to be *regular* if every bounded weak solution of (1) is locally Hölder contiunuous in the domain Q.

Definition 2. The system (1) has *the interior Liouville property in* Q if for each $z_0 \in Q$ every bounded weak solution of the system

$w_t - \operatorname{div}(A(z_0,w)Dw) = 0$ in all $R \times R^n$ is constant.

Theorem 1. The system (1) is regular iff it has interior Liouville property in Q.

Sketch of the proof. To have the regularity of the weak solution u in Q, it is sufficient to prove that for each $z_0 \in Q$

$$\liminf_{R \to 0+} [R^{-n-2} \int_{Q(z_0,R)} |u(z) - u_{z_0,R}|^2 dz] = 0 ,$$

where $Q(z_0,R) = (t_0 - R^2, t_0) \times \{x; |x - x_0| < R\}$ and $u_{z_0,R}$ is an integral mean value of u over $Q(z_0,R)$.

Using the blowing-up technique, we obtain this form the interior Liouville property. (For the details see [1], [2].)

Remark. (The elliptic case.) J. Daněček from Brno proved in his disertation that Theorem 1 (modified for elliptic systems) remains to be true if we change the request of boundedness of the weak solution (in both the definitions 1 and 2) by the assumption that the weak solutions in question belong to the space BMO. Further, he described the nontrivial class of the systems which satisfy BMO-interior Liouville property.

2. Regularity of the Cauchy problem. Let Ω be a domain in R^n and $T > 0$. Denote $Q^+ = (0,T) \times \Omega$, $\Gamma = \{[0,x];\ x \in \Omega\}$ and $Q = Q^+ \cup \Gamma$. To the conditions (i) - (iii) we add the assumption concerning the initial function u_0, namely:

(iv) $u_0 \in W^1_{q,loc}(\Omega) \cap L_\infty(\Omega)$, $q > n$.

The function $u \in W^{0,1}_{2,loc}(Q)$ is a *weak solution of Cauchy problem for the system (1) in Q with initial function* u_0 if for each $\varphi \in C^\infty(\overline{Q})$ with supp $\varphi \subset Q$

$$\int_Q [(u,\varphi_t) - (ADu,D\varphi)]\,dz = \int_Q [(f,\varphi) + (g,D\varphi)]\,dz - \int_\Omega u_0\varphi(0,x) .$$

Definition 3. Cauchy problem is *regular* if its each bounded solution is locally Hölder continuous on Q.

Definition 4. The system (1) has *boundary Liouville property on* Γ if for each $z_0 \in \Gamma$ every bounded weak solution of Cauchy problem for the system $w_t - \operatorname{div}(A(z_0,w)Dw) = 0$ in the set $\{[t,x];\ t \ge 0,\ x \in R^n\}$ with initial function $u_0 \equiv 0$ is equal zero identically.

Theorem 2. Cauchy problem for (1) in Q with initial function u_0 is regular iff the system (1) has interior Liouville property in Q^+ and boundary Liouville property on Γ.

Sketch of the proof. We extend the coefficients and the right hand side function of (1) to the cylinder $G = Q \cup (-Q^+)$. The weak solution of the Cauchy problem for (1) in Q can be shifted and prolonged in a suitable manner to the weak solution of the extended system on the whole G. Using now the interior regularity result in G we obtain the assertion of Theorem 2 immediately. (For the details see [2].)

3. Example. Let $m = n = 3$, $Q = (0,\infty) \times B$ (B is the unit ball in R^3). We obtain the example of the system (1) for which the boundary value problem with Lipschitzian boundary data on $\Gamma = [\{0\} \times B] \cup [(0,\infty) \times \partial B]$ has a solution u which develops the singularity for some $t_0 > 0$ in two steps: a) We choose the suitable u and b) to this u we construct the system for which u is the weak solution of the boudary value problem.

In the choice of the solution we were inspired by M. Struwe [3] who constructed the example for the system $u^i_t - \Delta u^i = f^i(t,x,u,Du)$, $i = 1,\dots,m$, with f^i growing quadratically in $|Du|$. We set

$$(2)\quad u^i(t,x) = \begin{cases} \dfrac{x_i}{|x|} & \\ \Phi * \dfrac{x_i}{|x|} & \text{if } t < 1, \end{cases} \qquad i = 1,\dots,3\ ,$$

where Φ is fundamental solution of the equation $w_t + \Delta w = 0$. It is easy to see that u is locally Lipschitz continuous on $R \times R^n$ with except of the half-line $p = \{[t,0]\,;\ t \geq 1\}$, where it ceases to be continuous.

To construct the system, we modify the procedure due to M. Giaquinta and J. Souček. At first we seek the system with bounded and measurable coefficients in the form

$$(3)\quad w_t - \operatorname{div}(A(z)Dw) = 0,$$

$$A^{ij}_{\alpha\beta}(z) = \delta_{\alpha\beta}\delta_{ij} - \frac{\tilde d_{\alpha i}\tilde d_{\beta j}}{(\tilde d, Du)},\ \text{where } \tilde d_{\alpha i} = D_\alpha u^i - b_{\alpha i}\ .$$

Substituting u for w into this system we obtain the conditions on $b_{\alpha i}$ under which u is a solution of the system (3):

$$(4)\quad u^i_t = D_\alpha b_{\alpha i},\quad i = 1,\dots,3\ .$$

Choosing reasonably the form of $b_{\alpha i}$ we get after tedious calculations an explicit form of coefficients A:

$$(5)\quad A^{ij}_{\alpha\beta}(z) = \delta_{\alpha\beta}\delta_{ij} + d_{\alpha i}d_{\beta j}\ ,$$

where

$$d_{\alpha i} = \frac{1}{\sqrt{4(a-2)}} \{ \delta_{\alpha i}(a-2) - \frac{x_i x_\alpha}{|x|^2}(6+a)\} \text{ if } t \geq 1,$$

$$d_{\alpha i} = \frac{1}{\sqrt{4(a-2)+(6+a)q(4-3q)}} \{-\delta_{\alpha i}[a-2+(6+a)\frac{q}{2}] - $$
$$- \frac{x_i x_\alpha}{|x|^2}(6+a)(1-\frac{3q}{2})\} \qquad \text{if } \quad t < 1 .$$

Here $a > 2$ is a real parameter, $\xi = \frac{|x|}{2\sqrt{1-t}}$ and $q = q(\xi) = \xi^{-2} -$ $- \xi^{-1}e^{-\xi^2}(\int\limits_0^\xi e^{-\tau^2}d\tau)^{-1}$.

Theorem 3. The function u given by (2) is a weak solution of the boundary value problem for (3) with the coefficients given by (5) and the boundary function u_0= Trace u on Γ. This solution is Lipschitz continuous with except of the half-line $p = \{[t,0]; t \geq 1\}$ where it ceases to be continuous.

Remark. Rewriting $x_1 x_\alpha |x|^{-2}$ in the coefficients we can pass to the desired quasilinear system of the type (1). For the details see [4].

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