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FREE BOUNDARY PROBLEMS IN FLUID DYNAMICS

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The velocity potential of a 2-dimensional ideal incompressible and irrotational fluid satisfies $\Delta\phi = 0$; further, Bernoulli's law $|\nabla\phi|^2 + 2p = \text{const.}$ yields $|\nabla\phi| = \text{const.}$ on the (free) boundary of the fluid in contact with air. Since $\nabla\phi$ is tangential to the free boundary, the stream function u (i.e., the harmonic conjugate of ϕ) satisfies:

$$\begin{aligned} &\Delta u = 0 \quad \text{in the fluid} \\ &u = c,\ \frac{\partial u}{\partial \nu} = \lambda \ \text{on the free boundary} \end{aligned} \tag{1}$$

where c, λ are constants. If we take gravity into account, then λ is replaced by $\sqrt{a + gy}$ $(a > 0,\ g > 0)$ where the gravitational force is in the upward direction.

In addition to (1) we must impose boundary conditions

$$u = u_0 \quad \text{or} \quad \frac{\partial u}{\partial \nu} = u_1 \tag{2}$$

on the fixed boundary

as well as a condition at infinity. For example

(i) for a symmetric jet flow from a nozzle ℓ we have:

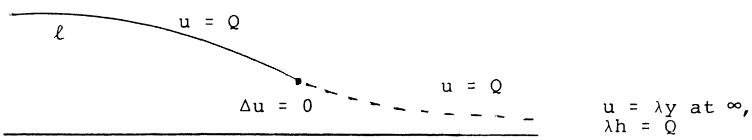

Figure 1

where h is the asymprotic height of the free boundary as $x \to \infty$;

(ii) for a symmetric cavitational flow with nose ℓ we have:

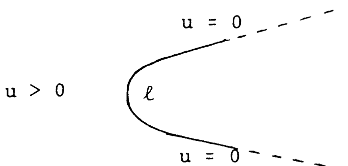

Figure 2

$u = y(1 + o(1))$. $\nabla u = \vec{e}(1 + o(1))$ where $\vec{e} = (0,1)$ and $o(1) \to 0$ if $x^2 + y^2 \to \infty$.

Problems such as (i), (ii) have been solved by several methods over the last 100 years. The general procedure has been to use conformal mappings or the hodograph transformation in order to reduce problems such as (i), (ii) to nonlinear integral equations (of a rather complicated type) and then apply the Leary-Schauder fixed point theorem; for details see [14][23][24] and the references in [12],[22]. Another approach based on a variational principle was developed in [19],[20].

In the last few years Alt, Caffarelli and Friedman have developed a new variational approach to establish existence of solutions for free boundary problems of general ideal fluids [2-4,8,9]. This work has also been extended to two fluids (flowing side-by-side)[2-4,8,9]. We shall explain the essence of the method in the simplest case (i)(Figure 1, above).

Consider the functional

$$J(v) = \int_{\Omega_\mu} |\nabla v - \lambda \vec{e} \chi_{\{v<Q\}} \chi_{E_\mu}|^2 dxdy, \quad v \in K_\mu$$

where

$$\ell : y = g(x), \ -\infty < x < 0, \quad g \text{ monotone } (b = g(0),\ A = (0,b)),$$

$$E_\mu = \{(x,y);\ -\mu < x < \infty,\ 0 < y < b\},\ \Omega_* = \{(x,y);\ 0 < y < g(x),\ -\infty < x \le 0\},$$

$$\mathbf{R}^2_+ = \{(x,y);\ x > 0,\ y > 0\},$$

$$\Omega = \Omega_* \cup \mathbf{R}^2_+,\ \Omega_\mu = \Omega \cap \{x > -\mu\},$$

$$K_\mu = \{v \in H^1(\mathbf{R}^2),\ v = Q \text{ if } y \ge g(x),\ x < 0 \text{ or } y \ge b,\ x > 0;\ v(x,0) = 0 \text{ if } -\infty < x < \infty,\ v(-\mu,y) = h_\mu(y), \text{ and } 0 \le v \le Q \text{ a.e.}\};$$

here $h_\mu(y)$ is a given function monotone in y, $h_\mu(0) = 0$, $h_\mu(g(-\mu)) = Q$.

Consider the problem: Find $v = v_{\lambda,\mu}$ in K_μ such that

$$\min_{v \in K_\mu} J(v) = J(u_{\lambda,\mu}).$$

It is easily seen that this problem has a solution. The solution is harmonic in $\Omega_* \setminus E_\mu$ and is a local minimizer in E_μ of

$$\tilde{J}(v) \equiv \int (|\nabla v|^2 + \lambda^2 \chi_{\{v<Q\}}) dxdy.$$

Alt and Caffarelli [1] studied the local minimizer $\tilde{v}$ of $\tilde{J}$ and proved that $\tilde{v}$ is Lipschitz continuous and that the free boundary $\partial\{\tilde{v} < Q\} \cap E_\mu$ is locally analytic.

LEMMA 1. The minimizer is unique.

Indeed, suppose u_1, u_2 are two minimizers and introduce $u_1^\varepsilon(x,y) = u_1(x - \varepsilon,y)$ and

$$v_1 = u_1^\varepsilon \wedge u_2, \quad v_2 = u_1^\varepsilon \vee u_2 .$$

Denote by J^ε the functional $J = J_{\lambda,\mu}$ corresponding to the translation $x \to x + \varepsilon$ of Ω_μ, K_μ. One verifies that

$$J^\varepsilon(u_1^\varepsilon) + J(u_2) = J^\varepsilon(v_1) + J(v_2) ,$$

which implies that $J(u_2) = J(v_2)$, i.e., $u_1^\varepsilon \vee u_2$ is a minimizer. Consequently $u_1^\varepsilon \vee u_2$ is smooth, which implies that either $u_1^\varepsilon \geq u_2$ or $u_1^\varepsilon \leq u_2$ everywhere. Since $u_1^\varepsilon < u_2$ near the boundary, we deduce that $u_1^\varepsilon < u_2$ throughout the domain. Taking $\varepsilon \to 0$ we get $u_1 \leq u_2$, and similarly $u_2 \leq u_1$.

Taking $u_1 = u_2$ in the above argument we get:

$$\frac{\partial}{\partial x} u_{\lambda,\mu} \geq 0 .$$

Thus the analytic free boundary $\Gamma = \Gamma_{\lambda,\mu}$ has the form

$$\Gamma_{\lambda,\mu} : x = f_{\lambda,\mu}(y) .$$

Next we have:

LEMMA 2. $f_{\lambda,\mu}(y)$ is continuous and finite if and only if $h < y < b$, where $h = Q/\lambda$.

LEMMA 3. $\lambda \to f_{\lambda,\mu}(b)$ is continuous ($f_{\lambda,\mu}(b) = f_{\lambda,\mu}(b + 0)$).

LEMMA 4. If λ is sufficiently small then $f_{\lambda,\mu}(b) < 0$; if $\lambda < Q/b$ and $|\lambda - Q/b|$ is small enough, then $f_{\lambda,\mu}(b) > 0$.

From Lemmas 3, 4 we deduce that there is a value $\lambda = \lambda(\mu)$ such that $f_{\lambda,\mu}(b) = 0$, i.e., there is a "continuous fit" at A. Fro this value of λ, $(u_{\lambda,\mu}, \Gamma_{\lambda,\mu})$ "almost" solves the jet problem. In order to complete the construction of a solution we let $\mu \to \infty$, $\lambda(\mu) \to \lambda$ and denote the limiting $u_{\lambda,\mu}$, $\Gamma_{\lambda,\mu}$ by u, Γ.

LEMMA 5. Continuous fit implies smooth fit.

More precisely the curve $\ell \cup \Gamma$ is not only continuous at the point of detachment A, but it is also C^1 at A, and ∇u is uniformly C^1 in $\{u < Q\}$-neighborhood of A.

THEOREM. There exists a unique classical solution of the symmetric jet problem (i).

Existence was already outlined above; uniqueness is proved by a comparison argument [21].

The above procedure has been extended to three-dimensional axially symmetric jets [2], 2-dimensional asymmetric flows [3], to flows in a gravity field [4], to rotational flows [16] and to compressible fluids [8][9]; some cavity problems are treated in [13][18].

Two-fluid problems are treated in [5-7]. Here u^+ and u^- are harmonic and

$$\frac{\partial u^+}{\partial \nu} - \frac{\partial u^-}{\partial \nu} = \lambda \quad \text{on the free boundary.} \tag{3}$$

In a two-fluid flow in a porous media of a rectangular dam, the free boundary condition can be reduced to

$$\frac{\partial u^+}{\partial \nu} - \frac{\partial u^-}{\partial \nu} = \cos(x,\nu) \tag{4}$$

which is similar to (3); in (3) λ is not a priori given, whereas in (4) a degeneracy occurs at points where $\cos(x,\nu) = 0$. Problem (4) is studied in [10] where existence of a solution is proved having a C^1 free boundary.

Lemma 5 has been extended in [11] to quasilinear elliptic operators and to more general boundary conditions $\partial u/\partial \nu = f$. The assertion is that either $\Gamma \cup \ell$ is C^2 at A or it is only in $C^{3/2}$ and the curvature of Γ goes to $\pm \infty$ as $x \downarrow 0$.

Other physical flow problems lead to free boundary conditions as above. We mention the problem of freezing in a channel because of heat sink at the origin [25]. Thus

$$\Delta u = -M\delta \quad \text{in} \quad \{u > 0\}$$

where δ = Dirac measure, $-u$ is the temperature, and

$$u > 0 \quad \text{in the ice,}$$

$$\frac{\partial u}{\partial \nu} = \lambda \text{ on the free boundary;}$$

λ and M are given positive constants. Assuming that the channel Ω is symmetric with respect to teh y-axis it was recently proved by Friedman and Stojanovic [17] that the problem has a unique solution with free boundary concave to the ice. This implies that if $\partial\Omega$ consists of p curves ℓ_i convex to Ω then the free boundary consists of at most p arcs ("fingers") concave to Ω, each connecting an adjacent pair ℓ_i, ℓ_{i+1}.

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