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ORDINARY LINEAR DIFFERENTIAL EQUATIONS – A SURVEY OF THE GLOBAL THEORY

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I. History

Investigations of linear differential equations from the point of their transformations, canonical forms and invariants started in the last century. In 1834 E.E. Kummer [6] studied transformations of the second order equations in the form involving a change of the independent variable and multiplication of the dependent variable. Till the end of the last century several mathematicians dealt also with higher order equations. Let us mention at least E. Laguerre, A.R. Forsyth, F.Brioschi, G.H.Halphen from many others. Perhaps the most known result from this period is the so called Laguerre-Forsyth canonical form of linear differential equations characterized by the vanishing of the coefficients of the (n - 1)st and (n - 2)nd derivatives.

However as late as in 1892 P. Stäckel (and one year later independently S.Lie) proved that the form of transformation considered by Kummer (as well as all his successors) is the most general pointwise transformation that converts solutions of any linear homogeneous differential equation of the order greater than one into solutions of an equation of the same kind. In fact, only this result justified backwards the whole previous investigations.

Already in 1910 G.D. Birkhoff [1] pointed out that the investigations, considered in the real domain, were of local character. He presented an example of the third order linear differential equation that cannot be transformed into any equation of the Laguerre-Forsyth canonical form on its whole interval of definition.

The local nature of methods and results is not suitable for dealing with problems of global character, as boundedness, periodicity, asymptotic or oscillatory behavior and other properties of solutions that necessarily involve investigations on the whole intervals of definition.

Only to demonstrate that even in the middle of this century there were just isolated results of a global character and no systematic theory, let me mention G. Sansone's example of the third order linear differential equation with all oscillatory solutions. This result occured as late as in 1948 in spite of the fact that the question about the existence or nonexistence of such an equation is as old as the problem of factorization of linear differential operators.

It is now some 35 years ago that O. Borůvka started the systematic study of global properties of the second order linear differential equations. He deeply developed this theory and summarized his original methods and results in his monograph [3] that appeared in 1967 in Berlin and in an extension version in 1971 in London.

For linear differential equations of the second and higher orders there have occurred results of a global character in papers of several mathematicians. Let me mention at least N.V.Azbelev, J.H.Barrett, E. Barvínek, L.M.Berkovič, T.A.Burton, Z.B.Caljuk, T.A.Chanturija, W.A. Coppel, W.N.Everitt, M.Greguš, H.Guggenheimer, G.B.Gustafson, M.Hanan, Z.Hustý, I.P.Kiguradze, V.A. Kondratjev, M.K. Kwong, M.Laitoch, A.C. Lazer, A.Ju.Levin, W.T.Patula, M.Ráb, G.Sansone, S.Staněk, J.Suchomel, C.A.Swanson, V.Šeda, M.Švec, M.Zlámal from several others. However, there was still no unified and systematic theory of global properties of linear differential equations of an arbitrary order enabling us to fortell what can and what cannot happen in global behavior of solutions.

In the last 15 years we discovered enough general approach and methods, we introduced new useful notions and derived results giving answers to substantial questions and solving basic problems in the area of global properties of linear differential equations of an arbitrary order. O.Borůvka's methods and results for the second order equations were at the beginning of our approach to equations of arbitrary orders and they still play an important role in the whole theory. We cannot see the possibility how to handle the general situation without having had his results at our disposal.

Algebraic, topological, analytical and geometrical tools together with methods of the theory of dynamical systems and functional equations make it possible to deal with problems concerning global properties of solutions by contrast to the previous local investigations or isolated results. Theory of categories, Brandt and Ehresmann groupoids, Cartan's moving-frame-of-reference method among other differential geometry methods, and functional equations are some of the means used in our approach.

The theory in question includes also effective methods for solving several special problems, e.g. concerning the global equivalence of two given equations, or from the area of questions on distribution of zeros of solutions, disconjugacy, oscillatory behavior, etc.

II. Global Transformations

For $n \geq 2$, let $P_n(y,x;I)$ denote a linear homogeneous ordinary linear differential equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_o(x) = 0 ,$$

where $p_i \in C^o(I)$, $i = 0,1,\ldots,n-1$, are real continuous functions defined on an open interval I of reals. Similarly, $Q_n(z,t;J)$ denotes

$$z^{(n)} + q_{n-1}(t)z^{(n-1)} + \ldots + q_o(t) = 0 , \quad q_i \in C^o(J) ,$$

$i = 0,1,\ldots,n-1$, $J \subset \mathbf{R}$ being an open interval.

We say that $P_n(y,x;I)$ is *globally transformable* into $Q_n(z,t;J)$ if there exist

a function $f \in C^n(J)$, $f(t) \neq 0$ on J, and

a C^n-diffeomorphism h of J into I,

such that

$z(t) = f(t) . y(h(t))$, $t \in J$

is a solution of $Q_n(z,t;J)$ whenever y is a solution of $P_n(y,x;I)$.

This definition complies with the most general form of a pointwise transformation derived by Stäckel. The bijectivity of h guarantees the transformation of solutions on their whole intervals of definition, i.e. the globality of the transformation. Let me remark also, that recently M.Čadek derived Stäckel's result without any differentiability assumption, [4].

It appears to be convenient to write the global transformation in the following form. Let $\mathbf{y} = (y_1,\ldots,y_n)^T$ be the vector column function whose coordinates y_i are linearly independent solutions of the equation $P_n(y,x;I)$ for $i = 1,\ldots,n$. Let us call the $\mathbf{y}$ a fundamental solution of $P_n(y,x;I)$. Similarly, let $\mathbf{z}$ denote a fundamental solution of the equation $Q_n(z,t;J)$. Then there exists a nonsingular n by n constant matrix C such that

(α) $\quad \mathbf{z}(t) = C.f(t).\mathbf{y}(h(t))$, $t \in J$.

The global transformation expressed explicitely by this formula will be denoted by $\alpha = \langle Cf,h \rangle_{\mathbf{y}}$, and we shall write

$$P_n(y,x;I)\alpha = Q_n(z,t;J) ,$$

or shortly

$P\alpha = Q$.

The relation of global transformability is an *equivalence relation*. Hence the set $\mathbf{A}$ of all linear homogeneous differential equations of all orders greater than and equal to two, is decomposed into the classes of globally equivalent equations.

Let $\mathbf{B}$ be one of the classes of the equivalence. For each three equations P, Q and T of the class $\mathbf{B}$ there exist global tranformations α and β such that

$P\alpha = Q$ and $Q\beta = T$.

If we define a composition $\alpha\beta$ of the tranformations α and β by

$(P\alpha)\beta = P(\alpha\beta) = T$,

we introduce a certain algebraic structure into each class $\mathbf{B}$ of globally equivalent equations. This algebraic structure considered on the whole set $\mathbf{A}$ is a special *category*, called the *Ehresmann groupoid*. Linear differential equations are objects and global transformations are morphisms of the category. The same algebraic structure restricted to any class $\mathbf{B}$ of globally equivalent equations is a special Ehresmann groupoid, called the *Brandt groupoid*.

The basic (and in fact, the only) structural notion of a Brandt groupoid is the so called *stationary group* of any of its objects. In our case of differential equations, the stationary group G(P) of an equation P is formed by all global transformations that transform the equation P into itself, i.e.

$G(P) = \{\alpha;\ P\alpha = P\}$.

It can be shown that the stationary groups of any two equations P and Q from the same equivalent class $\mathbf{B}$ are conjugate:

if $P\alpha = Q$ then $G(P) = \alpha G(Q)\alpha^{-1}$.

Having a special (*canonical*) object (*equation*) $S_{\mathbf{B}}$ in the class $\mathbf{B}$ of equivalent equations, all global transformations transforming P into Q are described by the formula

$\gamma^{-1}G(S_{\mathbf{B}})\delta$, where $P = S_{\mathbf{B}}\gamma$ and $Q = S_{\mathbf{B}}\delta$.

We could observe that in each area of mathematics where a structure of an Ehresmann groupoid occurs as it is also in our case, the following basic problems have to be solved in order to describe the structure of sets of objects and transformations in this area, and in this manner, to form a foundation of the corresponding theory:

1. Find sufficient and/or necessary conditions (if even effective, the better) under which two given objects, two given equations are

equivalent, i.e. *criterion of global equivalence.*

2. Characterize all possible *stationary groups* according to the classes of equivalence.

3. Find, construct *canonical objects*, equations in *each* class of equivalent equations.

In what follows we shall answer the mentioned questions for linear differential equations of arbitrary orders.

First, let us introduce also a *geometrical representation* of our global transformations very useful in the sequel when different geometrical approaches are applied.

Again, let an equation P be represented by its (arbitrary, but fixed) fundamental solution y, considered now as a curve in n-dimensional vector space V_n, the independent variable x ranging through the interval I and being the parameter of the curve. Due to the form (α) of a global transformation,

the change $x = h(t)$ is only a reparametrization,

the factor $f(t)$ selects only another curve but on the same cone K formed by straight lines going through the origin $0 \in V_n$ and all points of the original curve y,

the matrix C performs a centroaffine mapping.

We may conclude that each fundamental solution, or curve z of any equation Q globally equivalent to the equation P is a section of a cone in n-dimensional vector space obtained as a centroaffine image of a fixed cone determined by a fixed curve y.

Now, let us come to answer the above mentioned basic questions.

III. Global Equivalence

A sufficient and necessary condition for global equivalence of the *second order* linear differential equations was found by O. Borůvka [3] in the sixties. First, some definitions:

The maximal number of zeros of nontrivial solutions of an equation of the second order P_2 gives the *type* of the equation: either *finite*, an integer m, or *infinite*. Moreover, the equation P_2 being of finite type m is called of *general kind*, if it admits two linearly independent solutions with m - 1 zeros, everything considered on the whole interval of definition. Otherwise, P_2 being of finite type m is called of *special kind*. If the equation P_2 is of infinite type then its kind is either *one-side oscillatory* or *both-side oscillatory*.

Now Borůvka's criterion reads as follows:

Two second order linear differential equations are globally equivalent if and only if they are of the same type and at the same *time of the same kind.*

Our criterion of global equivalence of equations of higher orders needs the following notion. Let

(p) $u'' + p(x)u = 0$

be an equation of the second order whose coefficient p belongs to the class $C^{n-2}(I)$, and let u_1 and u_2 denote two of its independent solutions. Define n functions

$$y_1 := u_1^{n-1},\ y_2 := u_1^{n-2}.u_2, \dots, y_n := u_2^{n-1} .$$

These functions are of the class $C^n(I)$ and they are linearly independent. Hence they can be considered as solutions of the uniquely determined n-th order linear differential equation, called the *iterative equation* iterated from the equation (p). We denote the iterative equation by $p^{[n]}(y,x;I)$, or simply by $p^{[n]}$. The differential expression of the iterative equation normalized by the unit leading coefficient will be denoted as $|p^{[n]}|$. It can be shown (e.g. [5]) that

$$|p^{[n]}| = y^{(n)} + \binom{n+1}{3}p(x)y^{(n-2)} + 2\binom{n+1}{4}p'(x)y^{(n-3)} + \dots .$$

In order to find whether two given linear differential equations of the n-th order, $P_n(y,x;I)$ and $Q_n(z,t;J)$ with sufficiently smooth coefficients are globally equivalent, we rewrite them in the form

$$P_n(y,x;I) = |p^{[n]}| + r_{n-3}(x)y^{(n-3)} + r_{n-4}(x)y^{(n-4)} + \dots = 0$$

and

$$Q_n(z,t;J) = |q^{[n]}| + s_{n-3}(t)z^{(n-3)} + s_{n-4}(t)z^{(n-4)} + \dots = 0 ,$$

where the first three coefficients of P_n and Q_n coincide with the coefficients of the iterative expressions $|p^{[n]}|$ and $|q^{[n]}|$, respectively. *If the equation P_n is globally transformable into the equation Q_n by means of a global transformation with the change $x = h(t)$, then*

A. *the second order equation $u'' + p(x)u = 0$ on I is globally transformable into $v'' + q(t)v = 0$ on J with the same change $x = h(t)$ of the independent variable,*

B. *the following relations are satisfied*

$r_{n-3}(h(t))h'^3(t) = s_{n-3}(t)$ *on J*

$r_{n-4}(h(t))h'^4(t) = s_{n-4}(t)$ *on J where* $s_{n-3}(t) = 0$,

$r_{n-5}(h(t))h'^5(t) = s_{n-5}(t)$ *on J where* $s_{n-3}(t) = s_{n-4}(t) = 0$,

etc.

Due to condition B the criterion is *in general effective*, that means, that it is expressible in terms of quadratures of coefficients of given equations. Let us recall that for the second order equations the criterion is not effective in this sense, since it requires the number of zeros of solutions.

IV. Stationary Groups

Stationary groups for the second order equations, called groups of *dispersions*, were studied and completely described by O. Borůvka [3] in the sixties. Some results on stationary groups of linear differential equations of an arbitrary order were obtained in 1977 mainly by using the theory of functional equations [11].

In 1979 J. Posluszny and L.A. Rubel [15] characterized (up to conjugacy) those transformations, called *motions*, of a linear differential equation into itself that consist in a change of the independent variable only.

Finally, in 1984 on the basis of our criterion of global equivalence a *complete characterization* of all possible stationary groups was derived [14]. Here is the list of the groups up to conjugacy of linear differential equations of all orders considered with respect to global transformations in the most general form, i.e., involving changes both the independent and the dependent variables:

1. The functions $h : \mathbf{R} \to \mathbf{R}$, $h(x) = \text{Arctan}\,\frac{a \tan x + b}{c \tan x + d}$, $|ad - bc| = 1$

2. $h : \mathbf{R}_+ \to \mathbf{R}_+$, $h(x) = \text{Arctan}\,\frac{a \tan x}{c \tan x + 1/a}$, $a \neq 0$

3m. For each positive integer m, $h : (0, m\pi) \to (0, m\pi)$,

$$h(x) = \text{Arctan}\,\frac{a \tan x}{c \tan x \pm 1/a}, \quad a \neq 0$$

4m. For each positive integer m, $h : (0, m\pi - \pi/2) \to (0, m\pi - \pi/2)$,
$h(x) = \text{Arctan}(k \tan x)$ and $h(x) = \text{Arctan}(k \cot x)$, $k > 0$

5. The functions $h : \mathbf{R} \to \mathbf{R}$, $h(x) = x + c$ and $h(x) = -x + c$, $c \in \mathbf{R}$

6. The increasing functions from 5

7. The functions $h : \mathbf{R} \to \mathbf{R}$, $h(x) = x + k$ and $h(x) = -x + k$, $k \in \mathbf{Z}$

8. The increasing functions from 7

9. $id_{\mathbf{R}}$ and $-id_{\mathbf{R}}$

10. Only $id_{\mathbf{R}}$.

These groups range from the maximal one, a three-parameter group in case 1, through an infinite cyclic group in case 8, to the trivial group in case 10 consisting from the identity only. Let me point out that the maximal group has already occured as the fundamental group in

Borůvka's investigations of the second order equations.

For each case of the stationary groups we can characterize the corresponding equations and each of the cases listed here actually occurs. E.g., the case 1 takes place exactly when the equation is an iterative equation of an arbitrary order iterated from a both-side oscillatory second order equation.

Let us note that if we consider global transformations with only *increasing* changes of the independent variable then, up to conjugacy, there are 5 *possible cases* of stationary groups *with respect to the number of parameters* as announced in 1982 [12].

V. Canonical Forms

The next important notion is the notion of canonical forms of linear differential equations. Such forms were studied from the early beginning of investigation of the equations in the middle of the last century.

We have mentioned that already in 1910 G.D. Birkhoff pointed out that the so called *Laguerre-Forsyth canonical form is not global*. It can be shown [13] that also the other canonical form that has occurred in the literature, the so called *Halphen canonical form is not global either*.

For constructions of *global canonical forms* we may proceed in two ways, either we use a certain geometrical approach, or we may apply the criterion of global equivalence.

First let us explain shortly our geometrical approach. We have seen that fundamental solutions $\mathbf{z}$, considered as curves in an n-dimensional vector space, corresponding to all equations globally equivalent to one equation with a fundamental solution y, a curve y, are obtained as sections of a cone determined by the curve y. To find a canonical, that means, a special equation in the class of equivalent equations, we need a special section of the cone. By applying Cartan's moving-frame-of-reference method we come unfortunately again to the Halphen forms that are not global. However, if we consider the euclidean n-dimensional space and take the central projection of our curves and then their length parametrization, we obtain special sections of the cone, special curves. Fortunately, this can be done without any restrictions on the whole intervals of definition. Then by using differential geometrical methods the explicit forms of the special, canonical equations corresponding to the special curves are obtained.

These global canonical forms are

$n = 2$: $\quad y'' + y = 0$ on (different) $I \subset \mathbf{R}$,

n = 3: $y''' - (p'(x)/p(x))y'' + (1 + p^2(x))y' - (p'(x)/p(x))y = 0$ on $I \subset \mathbf{R}$,

(one) arbitrary function $p \in C^1(I)$, $p(x) \neq 0$ on I,

etc.

For n = 2 the canonical equations coincide with the canonical forms studied by O.Borůvka.

There is also another procedure producing global canonical forms. This procedure is analytical and the construction is based on our criterion of global equivalence. Among many different global canonical forms obtained by this approach [13] the following equations

$$y^{(n)} + 0.y^{(n-1)} + 1.y^{(n-2)} + p_{n-3}(x)y^{(n-3)} + \ldots + p_0(x)y = 0, I \subset \mathbf{R},$$

are global canonical forms for equations with sufficiently smooth coefficients. They are characterized by their first three coefficients

1, 0, 1 .

Comparing with the local Laguerre-Forsyth canonical forms having the corresponding sequence

1, 0, 0 ,

we may conclude that if Laguerre and Forsyth had taken 1 as the coefficient of the (n-2)nd derivative instead of their zero they would have got global forms instead of their local.

VI. Invariants

Invariants of linear differential equations with respect to transformations have been derived from the middle of the last century either directly, or mainly on the basis of the Halphen canonical forms. These invariants are local.

A *global invariant* of the *second order* linear differential equations is in fact their *type*:finite (a positive integer) or infinite, and their *kind*, as introduced and derived by O.Borůvka in the sixties.

Due to the criterion of global equivalence we have now also *global invariants* for equations of an *arbitrary order*. Indeed, the *type and kind of the equation* (p): $u'' + p(x)u = 0$ *on* I *is a global invariant of the n-th order equations* P_n *rewritten in the form*

$$P_n(y,x;I) = |p^{[n]}(y,x;I)| + r_{n-3}(x)y^{(n-3)} + \ldots = 0 .$$

Another interesting invariants have occurred recently. It is a bit misleading fact that each second order equation with only continuous coefficients can be globally transformed into an equation with even analytic coefficients, e.g., into $y'' + 1.y = 0$ on some $I \subset \mathbf{R}$. For higher order equations the degree of the smoothness of their coefficients is in some respect an invariant property. From many results of this kind let me introduce at least the following simplest one:

If the coefficients of the equation $P_n(y,x;I)$ satisfy

$$p_{n-1} \in C^{n-2}(I),\ p_{n-2} \in C^{n-3}(I),\ldots,\ p_j \in C^{j-1}(I) \text{ for some } j \leq n-1\,,$$

then the *coefficients of any globally equivalent equation to the* $P_n(y,x;I)$ *have the same order of differentiability.*

VII. Equations with Solutions of Prescribed Properties

The main idea how to construct linear differential equations with solutions of some prescribed properties is based on the following "*coordinate approach*".

Having global canonical forms (the globality is essential), each linear differential equation P of an arbitrary order can be "coordinated" by a couple $\{S,\alpha\}$ consisting of its global canonical form S and of the global transformation α converting S into P, i.e., $P = S\alpha$.

If we succeed to reformulate a given property of solutions of P equivalently into properties of S and α, we may construct all required equations. Also problems concerning relations among certain properties are then converted into (sometimes simple, or even already solved) problems from the theory of functions.

By using this approach there were constructed linear differential equations that have important applications in differential and integral geometries. E.g., it was possible *to generalize Blaschke's and Santalõ's isoperimetric theorems*, [8].

Connections between boundedness of solutions and their L^2-properties were easily explained by the above methcd [7].

Relations between *distributions of zeros* and *asymptotic behavior* of the solutions were also deeply studied by means of the coordinate approach.

There is also another way, a geometrical one, how to see what happens with zeros of solutions and how to construct equations with prescribed distribution of zeros of their solutions.

VIII. Zeros of Solutions

This geometrical approach is based on the representation of a fundamental solution y of an equation $P_n(y,x;I)$ as a curve in n-dimensional vector or even euclidean space V_n mentioned in the previous sections.

Let the curve v be the central projection of the curve y onto the unit sphere S_{n-1} in the space V_n without a change of parameter x. Each solution y of $P_n(y,x;I)$ can be written as a scalar product $c \cdot y$ where c is a nonzero constant vector in V_n. Let H(y) denote the hyperplane

$$H(y) := \{d \in V_n;\ c \cdot d = 0\}$$

going through the origin and corresponding to the vector c. Evidently

$$0 = y(x_0) = c \cdot y(x_0) = c \cdot v(x_0)|y(x_0)| \Leftrightarrow c \cdot v(x_0) = 0$$

since $|y(x_0)| \neq 0$. Thus we have shown that

to each solution y *of the equation* P_n *there corresponds a hyperplane* H(y) *in* V_n *going through the origin such that*

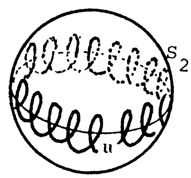
zeros of the solution y *occur as parameters of intersections of the particular hyperplane* H(y) *with the curve* v, *and vice versa.*

Multiplicities of zeros occur as orders of contacts, [9].

Let us recall that all this happens on the unit sphere, a compact space, where strong topological tools are at our disposal.

Several open problems were solved and many complicated constuctions were easily explained by using this approach, [10]. As a simple demonstration of the method let us present Sansone's result by constructing a *third order linear differential equation with all oscillatory solutions.*

For this purpose it is sufficient to have an enough smooth (of the class C^3) curve u on the unit aphere S_2 in 3-dimensional space without points of inflexion (that means, that Wronskian of u is nonvanishing) such that each plane going through the origin intersects u for infinitely many values of parameter. The picture of a closed "prolonged cycloid" infinitely many times surrounding the equator as its parameter ranges from $-\infty$ to $+\infty$ may serve as an example of a curve with the required property.



IX. Applications

To the end of my survey let me mention some fruitful applications of the presented theory.

The above methods were succesfully applied to *systems of linear differential equations*. E.g., construction of certain second order systems with only periodic solutions, [10], plays an important role in geometry of manifolds whose all geodesics are closed [2].

By using the above approach there were solved some problems concerning *linear and nonlinear differential equations and systems with one or several delays*. There are useful applications in generalized differential equations and *linear differential expressions with quasi-derivatives* as well. Last but not least, there are many fruitful connections with the *theory of functional equations*.

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Details will appear in

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