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DETERMINISTIC AND STOCHASTIC VECTOR DIFFERENTIAL EQUATIONS APPLIED IN TECHNICAL SYSTEMS THEORY

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Section D

1. This paper will give a choice from different *systems* of technics, physics, fight, astronautic etc. treated mainly by vector differential equations (vDE) in our papers, books, bulletins [1-13], having various results. These can illustrate the mathematical and technical variety and complexity of such problems, of course, without the claim to totality. A firm use of matrix analysis & algebra will accompany the following research.

1.1. As well known [4,14], the models of (deterministic) danymic systems are often described in the state (S) space by its SvDE and by its output algebraic one OvAE (which remains here in background):

$$\dot{\underline{z}} = \underline{f}[\underline{z},\underline{x}(t),t], \quad \underline{y} = \underline{g}[\underline{z},\underline{x}(t),t], \qquad (1,11a,b)$$

$$\langle t_0 \le t,\ \underline{z}_0(t_0) = \underline{z}_0;\ \ \underline{z}_0(t) = ?\ \ \underline{y}_0(t) = ?\ \rangle$$

where SvDE is supposed as satisfying the existency & unicity conditions. This generally non-linear (nl.) SvDE can have special forms occasionally, namely [4-14]

$\dot{\underline{z}} = \underline{f}[\underline{z},\underline{x}(t)]$ time-invariant (t.inv.), $\dot{\underline{z}} = \underline{f}(z)$ autonomous, $\dot{\underline{z}} = \underline{\underline{A}}(t)\underline{z} + \underline{\underline{B}}x(t)$ linear (l.), $\dot{\underline{z}} = \underline{\underline{A}}\underline{z} + \underline{\underline{B}}\underline{x}(t)$ l.t.inv., $\dot{\underline{z}} = [-\underline{e}_n\underline{p}^*(t) + \underline{\underline{K}}]\ z + \underline{e}_n x(t)\ \langle z_i = z^{(i-1)},\ \underline{\underline{K}} = [\delta_{i,j-1}], z^{(n)} + \sum_{l=o}^{n-1} p_l(t) z^{(l)} = x(t)\rangle$

phase-vDE of a l.DE. etc. and similarly the OvAE.

1.2. A system can have better and worse models [4a] so such SvDEs too, according to more, or less abstractions from the reality; but the truth of model to reality and the mathematical handling of SvDE is often compatible by compromise only.

1.3. As a hepl for the further treatment, let be mentioned our dynamic transform algorithm (DTA) for a matrix $\underline{\underline{A}} \hat{=} \underline{\underline{A}}_0$ in p(< r) steps

(in a spring: [2,5])

$$\underline{\underline{A}}_p \underset{(D)}{=} \underline{\underline{A}}_0 - \sum_{q=0}^{p-1} \gamma_q (\underline{a}_{l_q}^{(q)} - \underline{e}_{k_q})(\underline{a}_{(q)}^{k_q} + \underline{e}^{l_q}) =$$

$$= \underline{\underline{A}}_0 - (\underline{\underline{A}}_L - \underline{\underline{E}}_K)\underline{\underline{\Gamma}}_{KL}(\underline{\underline{A}}^K + \underline{\underline{E}}^L) = \begin{bmatrix} \underline{\underline{\Gamma}}_{KL} & \underline{\underline{\Gamma}}_{KL}\underline{\underline{A}}_{KJ} \\ -\underline{\underline{A}}_{IL}\underline{\underline{\Gamma}}_{KL} & \underline{\underline{\Omega}}_{IJ} \end{bmatrix}.$$

($\forall k_q \in K$ different, $\forall l_q \in L$ too;

$$\forall a^{(q)}_{k_q l_q} = 1/\gamma_q \neq 0,\ |\underline{\underline{\Gamma}}_{KL}| = \prod_{q=0}^{q-1} \gamma_q \neq 0,\ \underline{\underline{\Gamma}}_{KL} \mathrel{\hat{=}} \underline{\underline{A}}^{-1}_{KL} \ ) \qquad (1,31)$$

- The *rank* $\rho(\underline{\underline{A}})$ is r, if the p = rth step let vanish the *free* block: $\underline{\underline{\Omega}}_{IJ} = \underline{\underline{O}}_{ij}$....(1.32) - For a *regular* $\underline{\underline{A}}$ having n = m = r, the p = nth step furnishes (with $\forall a_{k_q k_q} \neq 0$ at different $k_q \in K$) its *inverse* matrix: $\underline{\underline{A}}_n = \underline{\underline{A}}^{-1} = \underline{\underline{\Gamma}}$....(1.33) - This DTA is suitable in algebras (A) (D) to solve arbitrary l.vAE [5], l.programming [2] and is generalized (gDTA) to solve nl.vAE [5] too. ...(1,34a-c) - If $\underline{\underline{A}}' = \underline{\underline{A}}_0 + \underline{bd}^* = \underline{\underline{A}} + \underline{\underline{A}}\beta.\delta^*\underline{\underline{A}} = \underline{\underline{A}}(\underline{\underline{E}} + \beta\underline{d}^*)$, then $\underline{\underline{A}}'^{-1} \mathrel{\hat{=}} \underline{\underline{\Gamma}}'$ can be found [15] in the form and with the scalar factor

$$\underline{\underline{\Gamma}}' = \underline{\underline{\Gamma}} - x\underline{\beta\delta}^*(\underline{\underline{E}} - x\underline{\beta d}^*)\underline{\underline{\Gamma}},\ x = (1 + \underline{d}^*\underline{\beta})^{-1}\ (\underline{d}^*\underline{\beta} \neq -1), \qquad (1.35)$$

as easy to control ($\underline{\underline{A}}'\underline{\underline{\Gamma}}'$ =...= $\underline{\underline{E}}$). - We have created a set of matrix algorithmic methods (MAM) [5,6,8,13] for various purpose; e.g. STA, SMA, OMA, TAD, IAℵ, ITA, OTA, FA, SoTA etc.

2. Let us make some remarks on *the non-linear SDEs* and their solutions!

2.1. Such ones can be solved *exactly* in exceptional cases only. - A such problem in [4e,14] is the pursuiting motion of an averting rocket R_1 in trace of an attacking on R_2 (in the vertical plane); namely - at radial velocity v, distance $r = \overline{R_1R_2}$ and incl. angle φ of R_1, at horizontal velocity c of R_2 and at ratio $m \mathrel{\hat{=}} v/c > 1$ - the SDEs are as follow:

$$(\underline{0} \geq)\dot{\underline{z}} \mathrel{\hat{=}} \begin{bmatrix}\dot{r}\\ \dot{\varphi}\end{bmatrix} = \begin{bmatrix} c\,\frac{\cos\varphi}{r} & -\frac{v}{\varphi} \\ -c\,\frac{\sin\varphi}{r} & 0\end{bmatrix}\begin{bmatrix} r\\ \varphi\end{bmatrix} \mathrel{\hat{=}} \underline{\underline{A}}(\underline{z})\underline{z} = \begin{bmatrix} c\cos\varphi - v \\ -c\sin\varphi + 0\end{bmatrix} \mathrel{\hat{=}} f(\underline{z}),\ \underline{z}(t_0) = \begin{bmatrix} r_0 \\ \pi/2\end{bmatrix} \quad (2.11)$$

and the exact solution:

$$\frac{r}{r_0} = e^{-\int_{\pi/2}^{\varphi} \frac{c\cos\varphi' - v}{c\sin\varphi'} d\varphi'} = \frac{\mathrm{tg}^m \frac{\varphi}{2}}{\sin\varphi} \quad (m > 1). \qquad (2.12)$$

2.2. A nl.SvDE is often solved approximately by (local) *linearization* around its equilibry (EL) points. - It is proposed in our [4e] for the growth vDE of two rival rasses with z_i populations

$$\dot{\underline{z}} \mathrel{\hat{=}} \begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \begin{bmatrix} K_1z_1 - M_1z_1z_2 \\ -M_2z_1z_2 + K_2z_2\end{bmatrix} = \begin{bmatrix} K_1z_1(1 - z_2/n_1) \\ K_2z_2(1 - z_1/n_2)\end{bmatrix} \mathrel{\hat{=}} \underline{f}(\underline{z}) \ (M_i > 0, K_i > 0; \ (1 <)K_2/K_1 = k, n_i = K_i/M_i) \qquad (2.21)$$

with EL-situations (K = I,II)

$$\dot{\underline{z}}_K = \underline{f}(\underline{z}_K) = \underline{0} \ \text{at} \ \underline{z}_I = \underline{0} \ \text{and} \ \underline{z}_{II} = \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}. \qquad (2.22)$$

The T_1 Taylor-polynom gives the approximate SvDE at $\underline{z}_K$

$$\dot{\underline{z}}_K \approx \underline{\dot{0}} + \underline{\underline{F}}(\underline{z}_K)d\underline{z}_K \doteq \begin{bmatrix}(n_1-z_2)M_1 - M_1z_1 \\ -M_2z_2 & (n_2-z_1)M_2\end{bmatrix}\begin{bmatrix}dz_1 \\ dz_2\end{bmatrix}_K \; ; \; \underline{\underline{F}}_I = \begin{bmatrix}K_1 & 0 \\ 0 & K_2\end{bmatrix},$$

$$\underline{\underline{F}}_{II} \hat{=} \begin{bmatrix}0 & -M_1n_2 \\ -M_2n_1 & 0\end{bmatrix}; \; (K = I, II; \; \underline{\underline{F}}(\underline{z}_K) = \underline{\underline{F}}_K),$$

then the eigen-values (1.at 3,2) and approximate solutions around $\underline{z}_K$

$|\lambda\underline{\underline{E}} - \underline{\underline{F}}_I| = (\lambda - K_1)(\lambda - K_2) = 0, \; \lambda_i = K_i > 0$ | $|\lambda\underline{\underline{E}} - \underline{\underline{F}}_{II}| \hat{=} \lambda^2 - K_1K_2 = 0, \; \lambda_i = \pm K \lessgtr 0$

(i=1,2): $\underline{z}_I$ is labil node point; | $(K=\sqrt{K_1K_2})$: $\underline{z}_{II}$ is saddle point;

$dz_2 = ce^{K_2 t} = ce^{(K_1 t)} = c(dz_1)^k \quad (k > 1)$ | $dz_2 = \pm\frac{m}{\sqrt{K}}dz_1$ and $d\zeta_2 d\zeta_1 = e^{Kt}e^{-Kt}c = c$.

These asymptotes through $\underline{z}_{II}$-and the smoothing hyperboles too - show the limit $z_2 \to 0$, or $z_1 \to 0$ at $t \to \infty$, so a rasse will be died. - However, <u>the fight of two uniformly armed forces and areas</u> - with Lancaster's and Diener's components [4e,14] (and at $-K_1, -K_2$) - is a math. analogous problèm... ... (2,24)

2.3. The analytical difficulty of a nl. SvDE (1,11a) can sometimes invert to an algebraical facility by the *difference method*, as approximation, or if the problem itself has a structure of difference. - This is the case at <u>the bending and moving equations of a *chain* bridge</u> treated in our [4e,1]. The chain connected with links from rigid bars let be characterized at the end links by horizontal strains h/H = = h + Δh, at link $x_i = \sum_{j=1}^{i} \Delta x_i = ia$ by hangs down $y_i \in \underline{y}/y_i + v_i \in \underline{y} + \underline{v}$, loads $p = P/a \in \underline{p}$ (own)/$\tilde{q}_i = \tilde{Q}_i/a \in \tilde{\underline{q}}$ (useful); its *balance vEs* (using the continuant matrix $\underline{\underline{C}} = [\underline{c}^i]_{n-1}^1 = [0,\dots,0,1,2,-1,\dots,0]$, the vector $\underline{e} = [1]^1$ and the fact $-\frac{1}{2}\underline{\underline{C}}\underline{y} = \left[\frac{\Delta^2 y_i}{\Delta \underline{x}_i^2}\right]$ and remarking in ⟨...⟩ the (at 1/n = a → 0) correspondant DEs appear as follow [1]:

$$-\frac{1}{a^2}\underline{\underline{C}}\underline{y} = \frac{1}{h}\underline{p} \;\langle -y'' = \frac{p}{h}\rangle, \; \frac{1}{a^2}\underline{\underline{C}}(\underline{y} + \underline{v}) = \frac{1}{H}(\underline{p} + \underline{q}) \;\langle -(y''+v'') = \frac{1}{H}[p + \tilde{q}(x)]\rangle.$$

The <u>beam</u> of rigidity EI carries at x_i a (useful) load $q_i - \tilde{q}_i = (Q_i - \tilde{Q}_i)/a \in \underline{q} - \tilde{\underline{q}}$, $m_i \in \underline{m}$ bending strain and has the *balance* vE (DE)

$$\frac{1}{a^2}\underline{\underline{C}}\,\underline{m} = \underline{q} - \tilde{\underline{q}} \quad \langle -m'' = q(x) - \tilde{q}(x)\rangle. \tag{2,32}$$

The rigid <u>pendant bars</u> at x_i transfer the emotions v_i of the loaded chain to the beam and establish with m_i the *relation* [1]

$$\frac{1}{EI}\underline{\underline{K}}\underline{m} \hat{=} \frac{1}{EI}(\underline{\underline{E}} - \frac{1}{6}\underline{\underline{C}})\underline{m} = \frac{1}{a^2}\underline{\underline{C}}\underline{v} \;\langle \frac{1}{a^2 EI}(m + \frac{a^2}{6}m'') = -v''\rangle. \tag{2,33}$$

With union of (2,31a-2,33), the *basic vE of bending* for the chain will be formed as follows:

$$\frac{1}{a^2}\underline{\underline{L}}(\underline{\underline{C}})\ \underline{m} \hat{=} \frac{1}{a^2}\left[\underline{\underline{C}} + \frac{Ha^2}{EI}(\underline{\underline{E}} - \frac{1}{6}\underline{\underline{C}})\right]\underline{m} = (\underline{q}-\tilde{\underline{q}}) + \overbrace{(\tilde{\underline{q}} - \frac{\Delta h}{h}\underline{p})}^{\underline{\underline{C}}\underline{v}/a^2} = \underline{q} - \frac{\Delta h}{h}\underline{p} \hat{=} \underline{r} \qquad (2,34)$$

$$\langle L(-d^2)m \hat{=} -m'' + \frac{H}{EI}(m + \frac{a^2}{6}m'') \equiv (\frac{Ha^2}{EI} - 1)m'' + \frac{H}{EI}m = q(x) - \frac{\Delta h}{h}p \hat{=} r(x)\ \rangle .$$

Its formal solution $\underline{m} = a^2\underline{\underline{L}}^{-1}(\underline{\underline{C}})\underline{r}$ can be surely realized, because the eigen-values -vectors are well known [16]:

$$\lambda_i = 2\cos\frac{i\pi}{n},\quad \underline{u}_i = \sqrt{\frac{2}{n}}[\sin\frac{ik\pi}{n}]_{n-1}^{k=1}\ (\underline{\underline{C}}\underline{u} = \lambda_i\underline{u}, \underline{u}_i^*\underline{u}_j = \delta_{ij}), \qquad (2,35a,b)$$

$$(\forall i \in \{1,2,\dots,n-1\})$$

then $L^{-1}(\lambda) \hat{=} [\lambda + \frac{Ha^2}{EI}(1 - \frac{\lambda}{6})]^{-1}$ is rational function, finally the form $\underline{r} = \sum_i(\underline{u}_i^*\underline{r})\underline{u}_i \hat{=} \sum_i \rho_i\underline{u}_i$ is ready, so the s.c. <u>canonical formed bending vector</u> $\underline{m}$ can be writte as follows [4e]:

$$\underline{m} = \sum_{i=1}^{n-1} L^{-1}(\lambda_i)\underline{u}_i(\underline{u}_i^*\underline{r}) \equiv \sum_{i=1}^{n-1}\frac{a^2\rho_i\underline{u}_i}{\lambda_i + \frac{Ha^2}{EI}(1 - \frac{\lambda_i}{6})} . \qquad (2,36)$$

Having it, the solving <u>emotion vector</u> appears so:

$$\underline{v} = \frac{a^2}{EI}\underline{\underline{C}}^{-1}\underline{\underline{K}}\underline{m} = \frac{a^2}{EI}\underline{\underline{C}}^{-1}(\underline{\underline{E}} - \frac{1}{6}\underline{\underline{C}})\underline{m} . \qquad (2,37)$$

Remarkable, that the our upper procedure (1. in $\underline{m}$) is *more simple and suitable* as other ones (nl. in $\underline{v}$) [15].

2.4. Omitting the various numerical methods of Euler, Euler-Adams, Milne, predictor-corrector etc. [17], let be mentioned only the Runge -Kutta method (RKM) to solve the nl. SvDE (1,11a). Namely - for an interval $[t,t+\tau]$ of length $\tau \hat{=} dt \sim 0$ and with signs $\underline{\xi} \hat{=} d\underline{x} \sim 0$, $\zeta_1 \hat{=} \hat{=} d\underline{z} \equiv \dot{\underline{z}}dt = \underline{f}(\underline{z},\underline{x},t).\tau$ - <u>a procedure step</u> advances as an algorithm of 4 substeps e.g. by our *recurrent formula* [4e].

$$\underline{\zeta}_s \hat{=} \underline{\underline{E}}\underline{s} \equiv \tau.\sum_{i=1}^{4}\underline{f}(\hat{\underline{y}}_{i-1}^*)\ s_i \equiv \tau.\sum_{i=1}^{4}\underline{f}(\hat{\underline{y}}_0^* + 1_i\hat{\underline{n}}_{i-1}^*)\ s_i \qquad (2,41)$$

$$\langle \underline{s} \hat{=} \frac{1}{6}[1,2,2,1]^*,\ \underline{1} \hat{=} [0,1/2,1/2,1]^*,\ \hat{\underline{y}}_0^* \hat{=} [\underline{z},\underline{x},t],\ \hat{\underline{n}}_{i-1}^* \hat{=} [\underline{\zeta}_{i-1},\underline{\xi},\tau]\rangle$$

having - verificably [17] - the excellent *accuracy* /(with. suppl. 0-s)

$$|\zeta_\tau - \underline{\zeta}_s| \hat{=} |\underline{f}(\hat{\underline{y}}_0^* + \hat{n}_1^*) - \underline{f}(\hat{\underline{y}}_0^* + \hat{\underline{n}}_s^*)|.|\tau| = c.|\tau|^5 . \qquad (2,42)$$

2,5. Look at a problematics, where the former RKM is often used. - This is <u>the motor vehicle as complex vibrating system</u> treated in our

[9]. Its (deterministic) model can be characterized very generally by the Lagrangean vDE of motion (one of second kind)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\underline{q}}} - \frac{\partial L}{\partial \underline{q}} + \frac{\partial R}{\partial \dot{\underline{q}}} = \underline{p}(t) \text{ at } \delta\int_{t_0}^{t} L(\underline{q},\dot{\underline{q}},t')dt' = 0 \text{ for } \int_{t_0}^{t} L(\underline{q},\dot{\underline{q}},t')dt' = \text{Extr!}, \qquad (2,51)$$

where (2,51a) is just the Eulerian vDE (as necessary condition) of the variation and extremum problem (2,51b,c), namely with T/U kinetic/ potential energy, L = T - U L-function, R dissipation, $\underline{q} = {}_1[q_1]_f^*$ vector of generalized coordinates, $\underline{p}(t) = {}_1[p_1(t)]_f^*$ external forces, f degree of freedom (e.g. at a car model can be f = 7). - The detailed form of (2.51) will be - after total derivation - as follows [10]:

$$\frac{\partial^2 L}{\partial\dot{\underline{q}}\partial\dot{\underline{q}}^*}\ddot{\underline{q}} + \frac{\partial^2 L}{\partial\dot{\underline{q}}\partial\underline{q}^*}\dot{\underline{q}} + \frac{1}{\underline{q}^*\underline{q}}\left(\frac{\partial^2 L}{\partial\dot{\underline{q}}\partial t} - \frac{\partial L}{\partial \underline{q}} + \frac{\partial R}{\partial\dot{\underline{q}}}\right)\underline{q}^*\underline{q} = \underline{p}(t), \qquad (2,52a)$$

which can be translated - by the signs for coefficient matrices $\underline{\underline{A}}_k(\underline{q},\dot{\underline{q}},t) = \underline{\underline{A}}_k + \Delta\underline{\underline{A}}_k(\underline{q},\dot{\underline{q}},t)$ at k = 0,1,2 - into the quasilinear (ql.) form

$$\underline{\underline{A}}_2\ddot{\underline{q}} + \underline{\underline{A}}_1\dot{\underline{q}} + \underline{\underline{A}}_0\underline{q} - \underline{\underline{L}}\underline{x}(t) = \sum_{k=0}^{2}\Delta\underline{\underline{A}}_k(\underline{q},\dot{\underline{q}},t)\underline{q}^{(k)} \mathrel{\hat{=}} \Delta\dot{\underline{z}}(\underline{q},\dot{\underline{q}},\ddot{\underline{q}},t), \qquad (2,52b)$$

or - with $\underline{\underline{A}}_2 \mathrel{\hat{=}} \underline{\underline{M}}$ inertia, $\underline{\underline{A}}_1 \mathrel{\hat{=}} \underline{\underline{D}}$ damping and $\underline{\underline{A}}_0 \mathrel{\hat{=}} \underline{\underline{S}}$ stiffnes matrices of *linear model* (got: $\Delta\underline{a} = \underline{0}$), then supposing $\underline{\underline{M}}^{-1}$ and $\underline{x}(t)$ as reduced $\underline{p}(t)$ to 4 wheels - into the *hyper-vector form* [10]

$$\dot{\underline{z}} \mathrel{\hat{=}} \begin{bmatrix}\dot{\underline{q}}\\ \ddot{\underline{q}}\end{bmatrix} = \begin{bmatrix}0 & \underline{\underline{E}}\\ -\underline{\underline{M}}^{-1}\underline{\underline{S}} & -\underline{\underline{M}}^{-1}\underline{\underline{D}}\end{bmatrix} \cdot \begin{bmatrix}\underline{q}\\ \dot{\underline{q}}\end{bmatrix} + \begin{bmatrix}0\\ +\underline{\underline{M}}^{-1}\end{bmatrix}(\underline{\underline{L}}\underline{x} + \Delta\dot{\underline{z}}) = [\underline{\underline{A}}\underline{z} + \underline{\underline{B}}\underline{x}(t)] + \underline{\underline{C}}\Delta\dot{\underline{z}}(\underline{z},\dot{\underline{z}},t) \equiv \qquad (2,52c)$$

$$\equiv \underline{l}[\underline{z},\underline{x}(t)] + \underline{n}(\underline{z},\underline{z},t) \mathrel{\hat{=}} \underline{f}[\underline{z},\underline{z},x(t),t].$$

This is solved every now as *nl*.SvDE e.g. by the upper RKM [17,4e], then as *ql*. one: firstly $\dot{\underline{z}} = \underline{l}[\underline{z},\underline{x}(t)]$ by 4,1 to $\underline{z}_0(t)$, secondly $\dot{\underline{z}} = \underline{n}[\underline{z}_0(t),\dot{\underline{z}}_0(t),t]$ by integration to $\Delta\underline{z}_0(t)$ etc.

2,6. The stability of a nl. system is often contolled by Ljupanov's *direct method* [18].- We use here it for an astronave (N) with linear help-rocket (R) treated in [4e]. N is considered as a *spin*, whose known autonomous nl.SvDE - at R's l.vAE (of $\underline{\underline{C}}_\backslash > \underline{\underline{0}}$)- follows here: $\dot{\underline{z}} = \underline{z} \times \underline{\underline{I}}_\backslash^{-1}\underline{z} + \underline{\underline{I}}_\backslash^{-1}\underline{x}(t) \equiv \underline{z} \times \underline{\underline{I}}_\backslash^{-1}\underline{z} - \underline{\underline{I}}_\backslash^{-1}\underline{\underline{C}}_\backslash\underline{z} \equiv (\underline{z} \times \underline{\underline{I}}_\backslash^{-1} - \underline{\underline{I}}_\backslash^{-1}\underline{\underline{C}}_\backslash)\underline{z} \mathrel{\hat{=}} \underline{\underline{F}}(\underline{z})\underline{z}$ with the angular velocity's vector $\underline{z} \equiv \underline{\omega} \in E_3$ (through the centre of mass) and with the main inertia moments $\langle I_1, I_2, I_3\rangle = \underline{\underline{I}}_\backslash$. Choosing $\underline{\underline{P}}_\backslash = \underline{\underline{I}}_\backslash^2$, the L.-function $V(\underline{z})$ and its derivative $W(\underline{z}) \mathrel{\hat{=}} \dot{V}(\underline{z}) = \underline{grad}^*V(\underline{z})$. $\dot{\underline{z}}$ will be: $V(\underline{z}) = \underline{z}^*\underline{\underline{P}}_\backslash\underline{z} > 0$ (for $\forall\underline{z} \neq 0$, def. pos.), $W(\underline{z}) = \dot{\underline{z}}^*\underline{\underline{P}}_\backslash\underline{z} + \underline{z}^*\underline{\underline{P}}_\backslash\dot{\underline{z}} = \underline{z}^*[\underline{\underline{F}}^*(z)\underline{\underline{P}}_\backslash + \underline{\underline{P}}_\backslash\underline{\underline{F}}(\underline{z})]\underline{z} \mathrel{\hat{=}} \underline{z}^*\underline{\underline{N}}_\backslash(\underline{z})\underline{z} = -2\underline{z}^*\underline{\underline{C}}_\backslash\underline{\underline{I}}_\backslash\underline{z} < 0$

($\forall \underline{z} \neq 0$), which last quadratic form is def. nagative. Consequently, the equilibry point $\underline{z} = \underline{0}$ has globally asymptotic stability.

3. The point 2 had shown, there is a natural gravitation into the direction of linearity at the SvDE, for its relative simplicity (e.g. for the superponability ets.).

3,1. The *homogeneous* form of *general* 1. SvDE (1,12c) $\dot{\underline{z}} = \underline{\underline{A}}(t)\,\underline{z}$ can be solved simply in possession of a *basic matrix* (bM) $\underline{\underline{Z}}(t) = {}_1[\underline{z}_j(t)]_n$ $\langle |\underline{\underline{Z}}(t)| \mathrel{\hat{=}} Z(t) \neq 0$ for $\forall t \in T = [t_0, \infty)\rangle$, when its general and a particular solution appears as

$$\underline{z}(t) = \underline{\underline{Z}}(t)\underline{c} \text{ and } \underline{z}_0(t) = \underline{\underline{Z}}(t)\underline{c}_0 = \underline{\underline{Z}}(t)\underline{\underline{Z}}^{-1}(t_0)\underline{z}_0 \mathrel{\hat{=}} \mathrel{\hat{=}} \tilde{\underline{\underline{Z}}}(t,t_0)\underline{z}_0 \langle \underline{z}(t_0) = \underline{z}_0 \rangle \qquad (3,11)$$

with the (by $\tilde{\underline{\underline{Z}}}(t_0,t_0) = \underline{\underline{E}}$) *normed* (n.) bM [6].- Having a *phase* SvDE (1,12e) $\dot{\underline{z}} + \underline{\underline{P}}(t)\underline{z} = \underline{0}$, or $L_n[z] = z^{(n)} + \underline{p}^*(t)\underline{z} = 0$ with $\underline{z} = [z_i]_n^1 = [z^{(i-1)}]_n^1$, the bM is $\underline{\underline{Z}}(t)\underline{r}_n(t_0)$ (Green vector) $\langle$at $\underline{r}_n(t_0) \subset \underline{\underline{R}}(t_0) \mathrel{\hat{=}} \mathrel{\hat{=}} \underline{\underline{Z}}^{-1}(t)\rangle$ [6].- Our algorithm SoTA [13] advances e.g. from a $L_4[z] = 0$ - by transforms $z = z_1 \int u dt$, $u = u_2 \int v dt$, $v = v_3 \int w dt$ - into $L_1[w] = 0$, giving the factorization $Z_4(w) = c \,.\, z_1^4(t)u_2^3(t)v_3^2(t)w_4(t)$ etc. (3.13)

3,2. At a *time-invariant* hom. form of 1.vDE (1,12d) $\dot{\underline{z}} - \underline{\underline{A}}\underline{z} = \underline{0}$, so at the ql. motor vehicle problem of 2,5 (at $\underline{x} = \underline{0}$, $\Delta\dot{\underline{z}} = \underline{0}$) too [9], exponential solutions $\underline{z} = e^{\lambda t}\underline{u}$ are supposed, which guides to the eigen-value problem [4c]

$$(\lambda\underline{\underline{E}} - \underline{\underline{A}})\underline{u} = \underline{0} \text{ at } D_n(\lambda) \mathrel{\hat{=}} |\lambda\underline{\underline{E}} - \underline{\underline{A}}| = \prod_{\sigma=1}^{s} (\lambda-\lambda_\sigma)^{\alpha_\sigma} = 0 \text{ and } M_\nu(\lambda) \mathrel{\hat{=}} \mathrel{\hat{=}} \prod_{\sigma=1}^{s} (\lambda-\lambda_\sigma)^{\beta_\sigma} = 0, \ldots \quad (3,21) \text{ where the case } \forall(\alpha_\sigma \geq)\beta_\sigma = 1$$

furnishes to an eigen value λ_σ 1. independent eigen-vectors $\underline{u}_{\sigma\alpha}$ ($\forall\alpha\leq\alpha_\sigma$) and solutions

$$\underline{z}(t) = \sum_{\sigma=1}^{s} \underline{\underline{U}}_\sigma e^{\underline{\underline{\Lambda}}_\sigma t} \underline{c}_\sigma = \underline{\underline{U}}\, e^{\underline{\underline{\Lambda}}\, t}\, \underline{c} = e^{\underline{\underline{A}}t}\underline{c} \text{ and } \underline{z}_0(t) = \underline{\underline{U}}e^{\underline{\underline{\Lambda}}t}\underline{c}_0 = = \underline{\underline{U}}\, e^{\underline{\underline{\Lambda}}(t-t_0)}\underline{\underline{U}}^{-1}\underline{z}_0 = e^{\underline{\underline{A}}(t-t_0)}\,\underline{z}_0 \qquad (3,22)$$

with exponential nbM $\tilde{\underline{\underline{Z}}}(t,t_0) = \tilde{\underline{\underline{Z}}}(t-t_0)$. (L. at [40] for $\exists\beta_\sigma > 1$.)
For the stability, all $\lambda_\sigma = \mu_\sigma + i\nu_\sigma$ must have $\mu_\sigma \mathrel{\hat{=}} \mathrm{Re}\lambda_\sigma < 0$. (3,23)

3,3. A problem of type 3,2 can be also very complicated one. This is illustrated by the rotating system of a rotor (R) and n ~ n axles (A_i), as a *turbine's* model reached by matrix method in our [7]. It was our lecture's theme at the Equadiff-6; so let be enough here to refer it only!

3,4. In the general case of 3,1 $\dot{\underline{z}} = \underline{\underline{A}}(t)\ \underline{z}$, there *is'nt* generally an exponential bM $\underline{\underline{Z}}(t) = e^{\int \underline{\underline{A}}(\tau)d\tau} = e^{\hat{\underline{\underline{A}}}(t)}$ (because $\dot{\underline{z}} \hat{=} e^{\hat{\underline{\underline{A}}}}\underline{\underline{A}}\underline{c} \neq \underline{\underline{A}}e^{\hat{\underline{\underline{A}}}}\underline{c} \hat{=}$ $\hat{=} \underline{\underline{A}}\underline{z}$, gen.) To find *a bM* for the SvDE or for its matrix variant $\dot{\underline{z}}_j(\tau) = \underline{\underline{A}}(\tau)\underline{z}_j(\tau)$ (for $\forall j \in \{1,2,\ldots,n\}$, so $\dot{\underline{\underline{Z}}}(\tau) = \underline{\underline{A}}(\tau)\underline{\underline{Z}}(\tau)$ (3,41)
and $\dot{\tilde{\underline{\underline{Z}}}}(\tau,t_0) = \underline{\underline{A}}(\tau)\underline{\underline{Z}}(\tau,t_0)$, the *integral equation* of Volterra-type [4c]

$$\tilde{\underline{\underline{Z}}}(t,t_0) = \underline{\underline{E}} + \int_{t_0}^{t} \underline{\underline{A}}(\tau)\tilde{\underline{\underline{Z}}}(\tau,t_0)d\tau \qquad (3,42\text{-}43)$$

will be solved by the Picard-*iteration* $(\forall t,t_0 \in T)$

$$\tilde{\underline{\underline{Z}}}_0(t,t_0) = \underline{\underline{E}},\ \tilde{\underline{\underline{Z}}}_{k+1}(t,t_0) = \underline{\underline{E}} + \int_{t_0}^{t}\underline{\underline{A}}(\tau)\underline{\underline{Z}}_k(\tau,t)d\tau \quad (k=0,1,2,\ldots,n,\ldots)(3,44)$$

<u>obtaing</u> so the Neuman-series $\tilde{\underline{\underline{Z}}}(t,t_0) = \underline{\underline{E}} + \sum_{K=I}^{N} \hat{\underline{\underline{A}}}^K(t,t_0) \approx \tilde{\underline{\underline{Z}}}(t,t_0)$.

- A regular transform $\underline{z} = \underline{\underline{U}}(t)\underline{v}$ $\langle U(t) \neq 0$ for $\forall t \in T\rangle$ and a l. MDE $\dot{\underline{\underline{U}}}(t) = \underline{\underline{Q}}(t)\underline{\underline{U}}(t)$ sometimes guide to a diagonalized form [13] $\dot{\underline{v}} =$ $= \underline{\underline{U}}^{-1}(\underline{\underline{A}} - \underline{\underline{Q}})\underline{\underline{U}}\underline{v} \hat{=} \underline{\underline{\Lambda}}_{\setminus}(t)$, so to an exp. nbM $\tilde{\underline{\underline{V}}}(t,t_0) =$ $= e^{\int_{t_0}^{t}\underline{\underline{\Lambda}}_{\setminus}(\tau)d\tau}$,... (3,45) if one can solve the eigen-value problem $[\underline{\underline{A}}(t) - \underline{\underline{Q}}(t) - \lambda_j(t)\underline{\underline{E}}]\underline{u}_j(t) = \underline{0}$, $\forall j \in N$.

4. Let pass over to *linear homogeneous systems*.

4,1. In the general case of (1,12c), the solution (3,11a) of hom. l.SvDE $\underline{z}(t) = \underline{\underline{Z}}(t)\underline{c}$ will be applied - by variation of constant $\underline{c}$ into $\underline{c}(t) = ?$ - to the *inhom.* one [4] (at $\underline{\underline{Z}}(t) = {}_1[\underline{z}_j(t)]_n$, $Z(t) \neq 0$).

$$(\dot{\underline{\underline{Z}}} - \underline{\underline{A}}\underline{\underline{Z}})\underline{c} + \underline{\underline{Z}}\dot{\underline{c}} = \underline{\underline{B}}\underline{x},\ \dot{\underline{c}} = \underline{\underline{Z}}^{-1}\underline{\underline{B}}\underline{x},\ \text{so}\ \underline{z}_n(t) \hat{=} \underline{\underline{Z}}(t)\underline{c}(t) = \int_{t_0}^{t}\underline{\underline{Z}}(t,\tau)\underline{\underline{B}}(\tau)\underline{x}(\tau)d\tau$$

appears as (at t_0 with $\underline{0}$ conditioned) *particular solutions*. - In the *phase* case (1,12e) and (3,12), the (4,11c) formula is simplified [4] to the form (at $\underline{\underline{R}}(\tau) \hat{=} \underline{\underline{Z}}^{-1}(\tau)$ and $\tilde{\underline{z}}_n(\tau,\tau) = \underline{e}$)

$$\underline{z}_n(t) = \int_{t_0}^{t}\underline{\underline{Z}}(t)\underline{\underline{R}}_{\setminus}(\tau)\underline{e}_n x(\tau)d\tau = \int_{t_0}^{t}\underline{\underline{Z}}(t)\underline{r}_n(\tau)x(\tau)d\tau \hat{=} \hat{=} \int_{t_0}^{t}\tilde{\underline{z}}_n(t,\tau)x(\tau)d\tau . \qquad (4,12)$$

4,2. In the time-invariant case of (1,12d) and 3,2, so at the <u>motor vehicle problem</u> of 2,5 and 3,2, the nb.M $\tilde{\underline{\underline{Z}}}(t,\tau) = e^{\underline{\underline{A}}(t-\tau)}$ let write the ordinary and eigen forms (with $\underline{s}_n(t) = \underline{\underline{U}}^{-1}\underline{z}_n(t)$, $\underline{\underline{B}}_s = \underline{\underline{U}}^{-1}\underline{\underline{B}}$):

$$\underline{z}_n(t) = \underline{\underline{U}}\underline{s}_n(t) = \int_{t_0}^{t} e^{\underline{\underline{A}}(t-\tau)}\underline{\underline{B}}\underline{x}(\tau)d\tau,\ \text{or}\ \underline{s}_n(t) = \int_{-\infty}^{t} e^{\underline{\underline{\Lambda}}(t-\tau)}\underline{\underline{B}}_s\underline{x}(\tau)\ \langle \text{at}\ \underline{s}(-\infty) = 0\rangle . \qquad (4,21)$$

4,3. Let us treat - following [4e,18] - the dynamical optimalization of a linear control system on the basis of *quadratical* criterium (QC). - Here must minimalize a Ljapunov-function of QC $V(\underline{z}^0)$ beside the 1.SvDE^0 $\dot{\underline{z}}^0 = \underline{\underline{A}}\underline{z}^0 + \underline{\underline{B}}\underline{x}^0(t)$ at an optimal feed-back $\underline{x}^0(t) = -\underline{\underline{K}}\underline{z}^0(t)$

(with $\underline{\underline{K}} = ?$): $V(\underline{z}^0) \hat{=} \int_t^{\infty}(\underline{z}^{0*}\underline{\underline{P}}\underline{z}^0 + \underline{x}^0\underline{\underline{Q}}\underline{x}^0)d\tau = \int_t^{\infty}\underline{z}^{*0}(\underline{\underline{P}}+\underline{\underline{K}}^*\underline{\underline{Q}}\underline{\underline{K}})\underline{z}^0 d\tau = \text{Min!}$ (4,31)

$(\underline{\underline{P}}=\underline{\underline{P}}^*,\ \underline{z}^*\underline{\underline{P}}\underline{z}^0 \geq 0$ (+s.def.); $\underline{\underline{Q}}=\underline{\underline{Q}}^*,\ \underline{x}^{*0}\underline{\underline{Q}}\underline{x}^0 > 0$ at $\underline{x}^0 \neq \underline{0}$ (+def.), so $\underline{\underline{Q}}^{-1}$).

Supposing $V(\underline{z}^0) = \underline{z}^{*0}\underline{\underline{R}}z^0 > 0$ at $\underline{\underline{R}}=\underline{\underline{R}}^*=?$ and $\underline{z}^0 \neq \underline{0}$(+def.), its derivative has double form: $W(\underline{z}^0) = -\underline{z}^{*0}(\underline{\underline{P}}+\underline{\underline{K}}^*\underline{\underline{Q}}\underline{\underline{K}})\underline{z}^0 = \dot{\underline{z}}^{*0}\underline{\underline{R}}\underline{z}^0 + \underline{z}^{*0}\underline{\underline{R}}\dot{\underline{z}}^0 =$ (4,32)
$= \underline{z}^{*0}(\underline{\underline{A}}-\underline{\underline{B}}\underline{\underline{K}})\underline{\underline{R}}+\underline{\underline{R}}(\underline{\underline{A}}-\underline{\underline{B}}\underline{\underline{K}})^*\underline{z}^0$ (< 0 for $\forall\underline{z}^0$: asympt. stab. supposed), similarly the coefficient matrix too:

$$\underline{\underline{W}}(\underline{\underline{K}}) \hat{=} (\underline{\underline{A}}-\underline{\underline{B}}\underline{\underline{K}})^*\underline{\underline{R}} + \underline{\underline{R}}(\underline{\underline{A}}-\underline{\underline{B}}\underline{\underline{K}}) = -(\underline{\underline{P}}+\underline{\underline{K}}^*\underline{\underline{Q}}\underline{\underline{K}}), \tag{4,33}$$

where from $\partial\underline{\underline{W}}/\partial\underline{\underline{K}}^* = -\underline{\underline{B}}^*\underline{\underline{R}} = -\underline{\underline{Q}}\underline{\underline{K}}$ follows $\underline{\underline{K}} = \underline{\underline{Q}}^{-1}\underline{\underline{B}}^*\underline{\underline{R}}$ $(\underline{\underline{R}} = ?)$, as the optimal feed-back matrix. With this $\underline{\underline{K}}(\underline{\underline{R}})$, one obtains

$$(\underline{\underline{W}}[\underline{\underline{K}}(\underline{\underline{R}})] - \underline{\underline{W}}[\underline{\underline{K}}(\underline{\underline{R}})] \equiv)\ \underline{\underline{Q}}(\underline{\underline{R}}) \hat{=} (\underline{\underline{R}}\underline{\underline{A}}+\underline{\underline{A}}^*\underline{\underline{R}}) - \underline{\underline{R}}\underline{\underline{B}}\underline{\underline{Q}}^{-1}\underline{\underline{B}}^*\underline{\underline{R}} + \underline{\underline{P}} = \underline{\underline{0}}\quad (=-\dot{\underline{\underline{R}}}),$$

as a degenerated Riccatian MDE ($\underline{\underline{R}}$ const.) being nl. (quadr.) MAE and with its solution $\underline{\underline{R}}$ (e.g. by our gDTA of (3,14c) [5]) the optimal control in final form: $\underline{x}^0(t) = -\underline{\underline{Q}}^{-1}\underline{\underline{B}}^*\underline{\underline{R}}\ \underline{z}(t) = -\underline{\underline{K}}\underline{z}^0(t)$.

5. Finally, let us turn shortly to the *stochastic systems!*

5,1. To avoide the complications of stochastic analysis, there is advantageous to transform linearly an arbitrary $\underline{\xi}(t)$ into its random basic product (Rbp) [8](with ordinary coordinate factor $\underline{\underline{X}}(t)$)

$$\underset{(m)}{\underline{\xi}(t)} = \underline{m}_\xi(t) + \sum_{l=1}^{\infty} x_l(t)\xi_l = \underline{m}_\xi(t) + \underset{(m\times\infty)(\infty)}{\underline{\underline{X}}(t)\ \underline{\xi}}\quad \langle \mathring{\underline{\xi}}(t) \hat{=} \underline{\xi}(t)-\underline{m}_\xi(t)\rangle \tag{5,11}$$

where $\underline{m}_\xi(t) \hat{=} M[\underline{\xi}(t)]$; $\underline{m}_\xi \hat{=} M(\underline{\xi})=0$, $\underline{\underline{C}}_{\xi\xi} \hat{=} M(\underline{\xi}\underline{\xi}^*)=M\langle\xi_j^2\rangle \hat{=} \underline{\underline{V}}_\xi \langle V_\xi = \prod_j \sigma_j^2 > 0\rangle$,

so $\underline{\xi}(t)$ consist of incovariant components (white noises) $\underline{x}_1(t)\xi_1$ and has the covariance functions: $\underline{\underline{C}}_{\xi\xi'}(t,t') = \underline{\underline{X}}(t)\underline{\underline{V}}_\xi\overline{\underline{\underline{X}}}^*(t')$, $\underline{\underline{X}}(t)=\underline{\underline{C}}_\xi(t)\underline{\underline{V}}_\xi$. - One uses it in an finite (approximate) form (but with former $\underline{x}_1(t)$).

$$\underset{(m)}{\mathring{\underline{\xi}}(t)} = \sum_{l=1}^{\mu}\underline{x}_l(t)\xi_l + \underline{\rho}_\mu(t) = \underset{(m\times\mu)(\mu)}{\underline{\underline{X}}(t)\ \ \underline{\xi}} + \underline{\rho}_\mu(t) \approx \underline{\underline{X}}(t)\underline{\xi}\ \langle \text{at } \sigma^2_{\rho_\mu} = \underline{\underline{V}}_\xi(t) - \underline{\underline{X}}(t)\underline{\underline{V}}_\xi\overline{\underline{\underline{X}}}^*(t)$$

and the suitable random vector $\underline{\xi}$ can be realized e.g. by our algorithms ITA (probable) or OTA (statistical) [8], e.g. giving for $T_{\cdot\mu} = {}_1\{t_\lambda(<t_{\lambda+\mu}\}$ the exact (sample) values $\mathring{\underline{\xi}}^*(T_{\cdot\mu}) = \underline{\xi}^*\underline{\underline{X}}_{\nabla}(T_{\cdot\mu}),\ldots$ (5,15) but for $t \in T - T_{\cdot\mu}$ the approximates ones only.

5,2. This method can be used also at our motor vehicle problem of

4,2, namely with the Rbp-form (5,11) and with (4,22), its stoch. one:

$$\underline{\zeta}_n(t) = \int_{-\infty}^{t} e^{\underline{\underline{A}}(t-\tau)}\underline{\underline{B}}[\underline{m}_\xi(\tau) + \underline{\underline{X}}(\tau)\underline{\xi}]d\tau = m_\zeta(t) + \underline{\underline{Z}}(t)\underline{\xi}. \qquad (5,21a,b)$$

From our [10], its covariance is for a general and stac. & ergodic case:

$$\underline{\underline{C}}_{\xi\xi'}(t,t') = \underline{\underline{Z}}(t)\underline{\underline{V}}_\xi \overline{\underline{\underline{Z}}}^*(t'),\ \underline{\underline{C}}_{\xi\xi'}(\tau) = \frac{1}{2T_\infty}\int_{-T_\infty}^{T_\infty} \underline{\zeta}(t)\underline{\zeta}^*(t+\tau)dt, \qquad (5,22)$$

then the spectral density matrix and its inverse, by Fourier-∫ &-inverse

$$\underline{\underline{S}}_\zeta(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega\tau}\underline{\underline{C}}_{\xi\xi'}(\tau)d\tau,\ \underline{\underline{C}}_{\xi\xi'}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\tau}\underline{\underline{S}}_\zeta(\omega)d\omega, \qquad (5,23)$$

whose approximate form (at $\tau = 0$) $\underline{\underline{C}}_{\xi\xi}(0) \approx \frac{1}{2\pi}\int_{\omega_1}^{\omega_2}\underline{\underline{S}}_\zeta(\omega)d\omega$, can be applied as criterium of optimality, e.g. an element of it will be min.

5,3. At the end, let us mention the Markov-chains treated by matrix analysis in our bulletin [11] with problems of mass service, demography, random walk etc., then our investigations [13] on parametrical and noisy Gaussian process and white noise, which promis an advance at the optimalization of noisy control systems.

R e f e r e n c e s (Look at the literary dates of these works too!)

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