Neil S. Trudinger

Classical boundary value problems for Monge-Ampère type equations

In: Jaromír Vosmanský and Miloš Zlámal (eds.): Equadiff 6, Proceedings of the International Conference on Differential Equations and Their Applications held in Brno, Czechoslovakia, Aug. 26 - 30, 1985. J. E. Purkyně University, Department of Mathematics, Brno, 1986. pp. 251--258.

Persistent URL: http://dml.cz/dmlcz/700147

CLASSICAL BOUNDARY VALUE PROBLEMS FOR MONGE-AMPÈRE TYPE EQUATIONS

N. S. TRUDINGER
Centre for Mathematical Analysis, Australian National University
Canberra, A.C.T. Australia

This report is concerned with recent work on the solvability of classical boundary value problems for elliptic Monge-Ampère type equations with particular attention to that of the author, P-L. Lions and J.I.E. Urbas [20] on Neumann type problems. The Dirichlet problem for these equations,

$$\det D^2u = f(x,u,Du) \quad \text{in} \quad \Omega \ , \tag{1}$$

$$u = \phi \quad \text{on} \quad \partial\Omega \ , \tag{2}$$

in convex domains Ω in Euclidean n space $\mathbb{R}^n$, has received considerable attention in recent years. For the standard Monge-Ampère equation,

$$\det D^2u = f(x) \quad \text{in} \quad \Omega \ , \tag{3}$$

Pogorelev [21,22] and Cheng and Yau [7] proved the existence of a unique convex solution $u \in C^2(\Omega) \cap C^{0,1}(\bar\Omega)$, provided Ω is a uniformly convex $C^{1,1}$ domain in $\mathbb{R}^n$ and the functions $\phi, f \in C^{1,1}(\bar\Omega)$ with f positive in Ω. Their methods depended on establishing interior smoothness of the generalized solutions of Aleksandrov [1]. These results were extended to equations of the more general form by P-L. Lions [17,18] using a direct PDE approach. Lions' approach led to the following classical existence theorem of Trudinger and Urbas [26], which we formulate explicitly for comparison with later results. Here we assume that the function f in equation (1) belongs to the space $C^{1,1}(\bar\Omega\times\mathbb{R}\times\mathbb{R}^n)$, is positive and non-decreasing in z, for all $(x,z,p) \in \Omega\times\mathbb{R}\times\mathbb{R}^n$ and satisfies the following growth limitations:

$$f(x,N,p) \le g(x)/h(p) \tag{4}$$

for all $(x,p) \in \Omega\times\mathbb{R}^n$, where N is some constant and $g \in L^1(\Omega)$, $h \in L^1_{loc}(\mathbb{R}^n)$ are positive functions such that

$$\int_\Omega g < \int_{\mathbb{R}^n} h \quad ; \tag{5}$$

$$f(x,N',p) \leq K[\mathrm{dist}(x,\partial\Omega)]^\alpha (1+|p|^2)^{\delta/2} \tag{6}$$

for all $x \in N$, $p \in \mathbb{R}^n$ where $N' = \max_{\partial\Omega} \phi$, K, α and δ are non-negative constants such that $\delta \leq n+1+\alpha$ and N is some neighbourhood of $\partial\Omega$. Then we have

Theorem 1 [26] *Let* Ω *be a uniformly convex* $C^{1,1}$ *domain in* $\mathbb{R}^n$, $\phi \in C^{1,1}(\bar\Omega)$ *and suppose that* f *satisfies the above hypotheses. Then there exists a unique convex solution* $u \in C^2(\Omega) \cap C^{0,1}(\bar\Omega)$ *of the Dirichlet problem* (1), (2).

Conditions (4) and (6) were introduced by Bakelman [2] in his treatment of generalized solutions and they are both sharp [2],[26]. For the special case of the equation of prescribed Gauss curvature,

$$\det D^2u = K(x)(1+|Du|^2)^{(n+2)/2} , \tag{7}$$

conditions (5) and (6) become respectively,

$$\int_\Omega K < \omega_n , \tag{8}$$

$$K = 0 \quad \text{on} \quad \partial\Omega. \tag{9}$$

Moreover condition (8) is necessary for a $C^{0,1}(\bar\Omega)$ solution of equation (7) to exist [9],[26] while if condition (9) is violated there exist arbitrarily smooth boundary values ϕ for which the classical Dirichlet problem (7), (2) is not solvable, [26].

The above developments shed no light on the global regularity of solutions beyond being uniformly Lipschitz in Ω. This was an open problem, in more than two dimensions, for many years and was finally settled, for *uniformly* positive f, through the contributions of Ivochkina [10], who proved global bounds for second derivatives for arbitrary $\phi \in C^{3,1}(\bar\Omega)$, $\partial\Omega \in C^{3,1}$, Krylov [14],[15] and Caffarelli, Nirenberg and Spruck [5] who independently discovered the hitherto elusive global Hölder estimates for second derivatives. As a particular consequence of this work, we can infer the following existence theorem for globally smooth solutions of the classical Dirichlet problem.

Theorem 2 *Let* Ω *be a uniformly convex* $C^{3,1}$ *domain in* $\mathbb{R}^n, \phi \in C^{3,1}(\bar\Omega)$ *and suppose that* $f \in C^{1,1}(\bar\Omega\times\mathbb{R}\times\mathbb{R}^n)$ *is positive and non-decreasing with respect to* z, *for all* $(x,z,p) \in \bar\Omega\times\mathbb{R}\times\mathbb{R}^n$ *and satisfies conditions* (4) *and* (6) *with* $\alpha=0$. *Then there* *exists a unique convex solution* $u \in C^{3,\gamma}(\bar{\Omega})$ *for all* $\gamma < 1$ *of the Dirichlet problem* (1),(2).

More general results are in fact formulated in [5], [12] but the condition $\delta \leq n+1$ cannot be improved [26]. The situation with regard to *oblique* boundary value problems of the form

$$\beta \cdot Du = \phi(x,u) \quad \text{on } \partial\Omega , \tag{10}$$

where $\beta \cdot \nu > 0$ on $\partial\Omega$ and ν denotes the unit inner normal to $\partial\Omega$, turned out to be more satisfactory in that condition (6) is not required for the estimation of first derivatives. For the case $\beta = \nu$, that is for the usual Neumann case,

$$\nu \cdot Du = \phi(x,u) \quad \text{on } \partial\Omega , \tag{11}$$

we proved in collaboration with Lions and Urbas in [20], the following existence theorem,

<u>Theorem 3</u> *Let* Ω *be a uniformly convex* $C^{3,1}$ *domain in* $\mathbb{R}^n$ *and* $\phi \in C^{2,1}(\bar{\Omega}\times\mathbb{R})$ *satisfy*

$$\phi_z(x,z) \geq \gamma_0 \tag{12}$$

for all $x,z, \in \partial\Omega\times\mathbb{R}$ *and some positive constant* γ_0. *Then if* $f \in C^{1,1}(\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n)$ *is positive and non-decreasing with respect to* z *for all* $(x,z,p) \in \bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n$ *and satisfies condition* (5), *there exists a unique convex solution* $u \in C^{3,\gamma}(\bar{\Omega})$ *for all* $\gamma < 1$ *of the boundary value problem* (1),(11).

Further regularity of the solutions in Theorems 2 and 3 follows by virtue of the Schauder theory of linear equations [9], when $\partial\Omega, \phi$ and f are appropriately smooth. In particular when $\partial\Omega \in C^\infty$, $\phi \in C^\infty(\partial\Omega\times\mathbb{R})$ and $f \in C^\infty(\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n)$ we deduce $u \in C^\infty(\bar{\Omega})$. The proofs of Theorems 2 and 3 both depend, through the method of continuity as described for example in [9], on the establishment of global $C^{2,\alpha}(\bar{\Omega})$ estimates for solutions of related problems. However the techniques employed by us to obtain these estimates in the Neumann boundary value case differ considerably from those used for the Dirichlet problem, particularly with respect to the estimation of first and second derivatives. For the estimation of sup norms we make use of the following maximum principle which does include that of Bakelman [2,3] for the Dirichlet problem as a special case.

<u>Theorem 4</u> [20] *Let* Ω *be a* C^1 *bounded domain in* $\mathbb{R}^n$ *and* $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *a convex solution of the boundary problem* (1),(10) *in* Ω *where* f *satisfies condition* (5),

$\beta\cdot\nu \geqq 0$ *on* $\partial\Omega$ *and* ϕ *satisfies* (12). *Then we have the estimate*

$$\min\left\{N, -\sup_{\partial\Omega}\phi^+(x,0)/\gamma_0 - (\beta_1/\gamma_0+d)R_0\right\} \leqq u \leqq \sup_{\partial\Omega}\phi^-(x,0)/\gamma_0 \tag{13}$$

where $d = \operatorname{diam}\Omega, \beta_1 = \sup_{\partial\Omega}|\beta|$, *and* R_0 *is given by*

$$\int_\Omega g = \int_{|p|<R_0} h$$

The gradient estimation in the oblique boundary condition case is a consequence of convexity as *any* convex $C^1(\bar\Omega)$ function satisfies an estimate

$$\sup_\Omega |Du| \leqq C \tag{14}$$

where C depends on $\beta_0, \beta_1, |u|_{0;\Omega}$, $\sup_{\partial\Omega}|\beta\cdot Du|$ and Ω, provided $\beta\cdot\nu \geqq \beta_0$ where β_0 is a positive constant and $\Omega \in C^{1,1}$[20]. In contrast, a gradient estimate for solutions of the Dirichlet problem (1),(2) holds provided $\partial\Omega \in C^{1,1}$, $\phi \in C^{1,1}(\bar\Omega)$ and condition (6) is fulfilled [26].

In both Dirichlet and Neumann problems the global estimation in Ω of *second derivatives* is reduced to considerations at the boundary $\partial\Omega$, by means of an approach which goes back to Pogorelev [21], although its implementation in the Neumann case [20] is substantially more involved than in the Dirichlet case [9], [5]. The boundary considerations are different as the Dirichlet problem is handled through barrier constructions [10], [5], whereas in [20] we employ different techniques including a device which necessitates our restriction of the vector β to the normal vector. The consequent estimates may be formulated as follows.

Theorem 5 *Let* Ω *be a* $C^{3,1}$ *uniformly convex domain in* $\mathbb{R}^n$ *and* $u \in C^4(\Omega) \cap C^3(\bar\Omega)$ *a convex solution of the boundary value problem* (1),(11) *where* $f \in C^{1,1}(\bar\Omega\times\mathbb{R}\times\mathbb{R}^n)$ *is positive and* $\phi \in C^{2,1}(\partial\Omega\times\mathbb{R})$ *satisfies* (12). *Then we have*

$$\sup_\Omega |D^2u| \leqq C \tag{15}$$

where C *depends on* n,Ω,f,ϕ *and* $|u|_{1;\Omega}$ *A similar estimate holds for solutions of the Dirichlet problem* (1),(2) *provided* $\phi \in C^{3,1}(\bar\Omega)$.

We remark that the restriction (12) can be weakened to $\phi_z \geq 0$ and the case when $f^{1/n}$ is convex with respect to p is simpler. We do not know whether one need only assume $\phi \in C^{2,1}(\bar\Omega)$ in the Dirichlet case. Once the second derivatives of solutions of the boundary value problems (1), (2), (10) are bounded, we obtain a control on the uniform ellipticity of equation (1) , and further estimation hence follows from the theory of fully nonlinear uniformly elliptic equations. In particular interior $C^{2,\alpha}(\Omega)$ estimates were derived by Calabi [4] for Monge-Ampère equations and by Evans [8] and Krylov [13] for general uniformly elliptic equations. Global $C^{2,\alpha}(\bar{\Omega})$ estimates for the Dirichlet problem then arose from combination with key boundary estimates discovered by Krylov [14] and Caffarelli, Nirenberg and Spruck [5]. Global $C^{2,\alpha}(\bar{\Omega})$ estimates for oblique boundary value problems were proved by Lions and Trudinger [19], with more general results being given by Lieberman and Trudinger [16] and Trudinger [24]. The global estimates of Krylov [14] and Trudinger [24] are also applicable to classical solutions of uniformly elliptic Hamilton-Jacobi-Bellman equations. We may in fact formulate these estimates for general second order equations of the form

$$F[u] = F(x,u,Du,D^2u) = 0 \quad \text{in} \quad \Omega \quad , \tag{16}$$

subject to general boundary conditions

$$G[u] = G(x,u,Du) = 0 \quad \text{on} \quad \partial\Omega \quad , \tag{17}$$

where either G is *oblique* so that

$$G_p \cdot \nu > 0 \tag{18}$$

for all $(x,z,p) \in \partial\Omega\times\mathbb{R}\times\mathbb{R}^n$, or G is *Dirichlet* so that

$$G(x,z,p) = z-\phi(x) \tag{19}$$

for some function $\phi \in C^{2,1}(\partial\Omega)$. Here $F \in C^{1,1}(\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n\times U)$, $G \in C^{1,1}(\partial\Omega\times\mathbb{R}\times\mathbb{R}^n)$ where U is some open convex subset of the linear space $\$^n$ of $n\times n$ real symmetric matrices, and F is: (i) *elliptic* so that the matrix,

$$F_r = [F_{r_{ij}}] > 0 \ , \tag{20}$$

for all $(x,z,p) \in \bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n$, $r = [r_{ij}] \in U$; and (ii) *concave* with respect to r for all $(x,z,p) \in \bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n$, $r \in U$. Then we have

Theorem 6 *Let Ω be a bounded $C^{3,1}$ domain in $\mathbb{R}^n$ and $u \in C^3(\bar{\Omega})\cap C^4(\Omega)$ a solution of the boundary value problem* (16),(17) *such that $D^2u(\Omega) \subset U$. Then we have*

$$[D^2u]_{\alpha;\Omega} \leq C \tag{21}$$

where $\alpha < 1$ *and* C *are positive constants depending only on* $n, \Omega, |u|_{2;\Omega}$ *and the first and second derivatives of* F *and* G *(excluding* F_{rr}*), (and* $|\phi|_3$ *in the Dirichlet case).*

We remark here that the solution u in Theorem 6 need only lie in the space $C^{1,1}(\bar{\Omega})$ and the smoothness of $\partial\Omega$, G, F, ϕ can be reduced, [25]. For application to Monge-Ampère type equations the convex set $\mathcal{U}$ becomes the set of positive symmetric matrices.

Finally we note that the sharpness of condition (8) is strikingly demonstrated by the following result of Urbas [28] concerning extremal domains for the equation of prescribed Gauss curvature.

Theorem 7 *Let* Ω *be a uniformly convex domain in* $\mathbb{R}^n$ *and* $K \in C^{1,1}(\bar{\Omega})$ *be positive in* Ω *and satisfy*

$$\int_\Omega K = \omega_n \tag{22}$$

Then there exists a convex solution $u \in C^2(\Omega)$ *of equation* (7) *in* Ω . *Furthermore the function* u *is vertical at* $\partial\Omega$ *and is unique up to additive constants. If* K *is positive in* $\bar{\Omega}$ *, then the solution* u *is bounded; if* K *vanishes on* $\partial\Omega$ *then the solution* u *approaches infinity at* $\partial\Omega$.

REFERENCES

[1] A.D. Aleksandrov, Dirichlet's problem for the equation $\mathrm{Det}\|z_{ij}\| = \varphi(z_i, \ldots, z_n z, x_1, \ldots, x_n)$, *Vestnik Leningrad Univ.* 13 (1958), 5-24, [Russian].

[2] I. Ya. Bakel'man, The Dirichlet problem for equations of Monge-Ampère type and their n-dimensional analogues, *Dokl. Akad. Nauk, SSSR* 126 (1959), 923-926, [Russian].

[3] I. Ya. Bakel'man, The Dirichlet problem for the elliptic n-dimensional Monge-Ampere equations and related problems in the theory of quasilinear equations, *Proceedings of Seminar on Monge-Ampère Equations and Related Topics,* Firenze 1980), Instituto Nazionale di Alta Matematica, Roma, (1982), 1-78.

[4] E. Calabi, Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens, *Michigan Math. J.* 5 (1958), 105-126.

[5] L. Caffarelli, L. Nirenberg, J. Spruck, The Dirichlet problem for nonlinear second order elliptic equations, I. Monge-Ampère equation, *Comm. Appl. Math.* 37 (1984), 369-402.

[6] L. Caffarelli, J.J. Kohn, L. Nirenberg, J. Spruck, The Dirichlet problem for nonlinear second order elliptic equations II. Complex Monge-Ampère and uniformly elliptic equations, *Comm. Pure Appl. Math.* 38 (1985), 209-252.

[7] S.-Y. Cheng, S.-T. Yau. On the regularity of the solution of the n-dimensional Minkowski problem, *Comm. Pure Appl. Math.* 29 (1976), 495-516.

[8] L.C. Evans, Classical solutions of fully nonlinear, convex, second order elliptic equations, *Comm. Pure Appl. Math.* 35 (1982), 333-363.

[9] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, Second Edition, 1983.

[10] N.M. Ivochkina, Construction of a priori bounds for convex solutions of the Monge-Ampère equation by integral methods, *Ukrain. Mat. Ž.* 30 (1978), 45-53, [Russian].

[11] N.M. Ivochkina, An apriori estimate of $\|u\|_{C^2(\bar{\Omega})}$ for convex solutions of the Dirichlet problem for the Monge-Ampère equations, *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 96 (1980), 69-79, [Russian]. English translation in *J. Soviet Math.* 21 (1983), 689-697.

[12] N.M. Ivochkina, Classical solvability of the Dirichlet problem for the Monge-Ampère equation, *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 131 (1983), 72-79.

[13] N.V. Krylov, Boundedly inhomogeneous elliptic and parabolic equations, *Izv, Akad. Nauk. SSSR* 46 (1982), 487-523, [Russian]. English translation in *Math. USSR Izv.* 20 (1983), 459-492.

[14] N.V. Krylov, Boundedly inhomogeneous elliptic and parabolic equations in a domain, *Izv. Akad. Nauk. SSSR* 47 (1983), 75-108, [Russian].

[15] N.V. Krylov, On degenerative nonlinear elliptic equations, *Mat. Sb. (N.S.)* 120 (1983), 311-330, [Russian].

[16] G.M. Lieberman, N.S. Trudinger, Nonlinear oblique boundary value problems for nonlinear elliptic equations. Aust. Nat. Univ. Centre for Math. Anal. Research Report R24 (1984).

[17] P.-L. Lions, Sur les équations de Monge-Ampère I, *Manuscripta Math.* 41 (1983), 1-44.

[18] P.-L. Lions, Sur les équations de Monge-Ampère II, *Arch. Rational Mech. Anal.* (to appear).

[19] P.-L. Lions, N.S. Trudinger, Linear oblique derivative problems for the uniformly elliptic Hamilton-Jacobi-Bellman equation, *Math. Zeit.* (to appear).

[20] P.-L. Lions, N.S. Trudinger, J.I.E. Urbas, The Neumann problem for equations of Monge-Ampère type. Aust. Nat. Univ. Centre for Math. Anal. Research Report R16 (1985).

[21] A.V. Pogorelov, The Dirichlet problem for the n-dimensional analogue of the Monge-Ampère equation, *Dokl. Akad. Nauk. SSSR* 201 (1971), 790-793, [Russian]. English translation in *Soviet Math. Dokl.* 12 (1971), 1727-1731.

[22] A.V. Pogorelov, *The Minkowski multidimensional problem,* J. Wiley, New York, 1978.

[23] F. Schultz, A remark on fully nonlinear, concave elliptic equations, *Proc. Centre for Math. Anal.* Aust. Nat. Univ. 8, (1984), 202-207.

[24] N.S. Trudinger, Boundary value problems for fully nonlinear elliptic equations. *Proc. Centre for Math. Anal.* Aust. Nat. Univ. 8 (1984), 65-83.

[25] N.S. Trudinger, Regularity of solutions of fully nonlinear elliptic equations. *Boll. Un. Mat. Ital.,* 3-A (1984), 421-430. II Aust. Nat. Univ. Centre for Math. Anal. Research Report R38 (1984).

[26] N.S. Trudinger, J.I.E. Urbas, The Dirichlet problem for the equation of prescribed Gauss curvature, *Bull. Austral. Math. Soc.* 28 (1983), 217-231.

[27] N.S. Trudinger, J.I.E. Urbas, On second derivative estimates for equations of Monge-Ampère type, *Bull. Austral. Math. Soc.* 30 (1984), 321-334.

[28] J.I.E. Urbas, Some recent results on the equation of prescribed Gauss curvature, *Proc. Centre for Math. Anal.* Aust. Nat. Univ. 8 (1984), 215-220.

[29] J.I.E. Urbas, The equation of prescribed Gauss curvature without boundary conditions, *J. Differential Geometry* (to appear).

[30] J.I.E. Urbas, The generalized Dirichlet problem for equation of Monge-Ampère type. *Ann. L'Inst. Henri Poincaré,* (to appear).