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UNIQUENESS WITHOUT CONTINUOUS DEPENDENCE

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1. Introduction. In the classical theory of ordinary differential equations if solutions of a system

$$x' = h(t,x) \tag{1}$$

are uniquely determined by initial conditions, then the solutions are continuous in the initial conditions. But the situation is much different for differential equations in infinite dimensional spaces. Sufficient conditions for this to hold have been discussed in [8-16] and [19-20]. Recently, Schäffer [18] constructed a fairly abstract example of a differential equation in the Banach space ℓ^∞ of bounded sequences with the supremum norm in which solutions are unique but are not continuous in initial conditions.

We present a simple example of the same behavior and point out that the real difficulty is that there are many topologies for the initial condition space.

2. Continuity in initial conditions. Consider the system

$$x' = h(t,x) + \int_{-\infty}^{t} q(t,s,x(s))ds \tag{2}$$

in which $h\colon (-\infty,\infty) \times R^n \to R^n$, $q\colon (-\infty,\infty) \times (-\infty,\infty) \times R^n \to R^n$, with h and q continuous pointwise. To fix the function space we suppose all solutions start at $t_0 = 0$. Then, to specify a solution of (2) we require a continuous initial function $\phi\colon (-\infty,0] \to R^n$ such that

$$\Phi(t) \overset{\text{def}}{=} \int_{-\infty}^{0} q(t,s,\phi(s))ds$$

is continuous for $t \geq 0$. We may then use the Schauder fixed point theorem to show that the system

$$x' = h(t,x) + \int_{0}^{t} q(t,s,x(s))ds + \Phi(t), \quad x(0) = \phi(0) \tag{3}$$

has a solution $x(t,0,\phi)$ satisfying (3) on an interval $[0,\beta)$, for some $\beta > 0$, with $x(t,0,\phi) = \phi(t)$ on $(-\infty,0]$.

System (2) is well defined using pointwise continuity in R^n and there is an initial function set X consisting of continuous functions ϕ for which Φ is continuous for $t \geq 0$ (X may be empty). Without putting any topology at all on X the problems of existence, uniqueness, and continuation of solutions are well-defined. But to complete a classical fundamental theory for (2) we want to say that for each $\phi \in X$ if there is a unique solution $x(t,0,\phi)$ on $[0,\beta]$ and if $\{\psi_n\}$ is a sequence in X converging to ϕ then solutions $x(t,0,\psi_n)$ converge to $x(t,0,\phi)$ on $[0,\beta]$. While we are quite willing to accept any type of convergence of $x(t,0,\psi_n)$ to $x(t,0,\phi)$ on $[0,\beta]$, the meaning of ψ_n converging to ϕ must be specified.

In a given problem we frequently have a wide degree of freedom in our choice of topology for the initial condition space. Recent problems call for unbounded initial functions, plentiful compact subsets of these initial functions, and continuity of the translation map. These requirements lead us to a locally convex topological vector space (Y,ρ) with $\phi \in Y$ if $\phi: (-\infty,0] \to R^n$ is continuous and for $\phi,\psi \in Y$ then

$$\rho(\phi,\psi) = \sum_{k=1}^{\infty} 2^{-k}[\rho_k(\phi,\psi)/(1+\rho_k(\phi,\psi))] \tag{4}$$

where $\rho_k(\phi,\psi) = \max_{-k\leq s\leq 0} |\phi(s) - \psi(s)|$ and $|\cdot|$ is any norm on R^n. For motivations see [1 - 5] and [7]. Problems are also effectively treated using a Banach space with weighted norm as the same references show.

EXAMPLE 1. Consider the linear scalar equation

$$x' = x + \int_{-\infty}^{t} [x(s)/(t-s+1)^3]ds \tag{5}$$

which has the zero solution and it is unique. In fact, if $\phi: (-\infty,0] \to R^n$ is any continuous function for which $\Phi(t) = \int_{-\infty}^{0} [\phi(s)/(t-s+1)^3]ds$ is continuous for $t \geq 0$, then there is one and only one solution $x(t,0,\phi)$ defined on $[0,\infty)$. Note that the set X is not empty. We now show that solutions $x(t,0,\phi)$ are not continuous in (Y,ρ).

PROOF. Define a sequence $\{\phi_n\} \subset X$ by

$$\phi_n(s) = \begin{cases} 0 & \text{if} \quad -n \leq s \leq 0 \\ -n(s+n) & \text{if} \quad s \leq -n. \end{cases}$$

Notice that

$$\rho(\phi_n,0) = \sum_{k=1}^{\infty} 2^{-k}\rho_k(\phi_n,0)/[1+\rho_k(\phi_n,0)]$$

$$\leq \sum_{k=n}^{\infty} 2^{-k} \to 0 \quad \text{as} \quad n \to \infty$$

and so $\{\phi_n\}$ converges to the zero function in (Y,ρ). Now, for $0 \leq t \leq 1$

and $n \geq 2$ we have

$$\phi_n(t) = \int_{-\infty}^{0} [\phi_n(s)/(t - s + 1)^3]ds$$

$$\geq -n \int_{-\infty}^{-n} [(s + n)/(-s + 2)^3]ds \geq 1/16.$$

Hence, we are considering the equation

$$x' = x + \int_0^t [x(s)/(t - s + 1)^3]ds + \phi_n(t)$$

$$\geq x + (1/16)$$

so that continuity of $x(t,0,\phi)$ in ϕ fails.

Schäffer suggests that the absence of continuity in his example may be the result of his space, ℓ^∞, being neither separable nor reflexive. But our sequence $\{\phi_n\}$ is contained in a compact subset of (Y,ρ) so the subset is separable and it may be embedded in a Banach space. One can show that (Y,ρ) is not reflexive. However, since (Y,ρ) is Frechet it is barreled (cf. [17; p. 60]).

PROPOSITION 1. Let $\{\phi_n\}$ be the sequence of Example 1 in (Y,ρ). Then $\{\phi_n\}$ is contained in a compact subset of (Y,ρ).

PROOF. Define a continuous function $g\colon (-\infty,0] \to [0,\infty)$ by $g(s) = \sup_n \phi_n(s)$. Then g is a continuous piecewise linear function. Moreover, if $s \geq -n$, then $g(s)$ is Lipschitz with constant n. Let $\alpha\colon (-\infty,0] \to [0,\infty)$ be the piecewise continuous linear function defined by $\alpha(-n) = n$. Then the set

$$S = \{\phi \in Y \mid |\phi(s)| \leq g(s) \quad \text{on} \quad (-\infty, 0],$$

$$|\phi(u) - \phi(v)| \leq \alpha(|u| + |v| + 1)|u - v|\}$$

is compact in (Y,ρ) (cf. [7; p. 2]) and contains $\{\phi_n\}$. This completes the proof.

To see that S can be embedded in a compact subset of a Banach space, for the function g defined in the proof of Proposition 1, define $\tilde{g}(s) = [g(s) + 1]^2$. Then define the Banach space $(Z,|\cdot|_{\tilde{g}})$ by $\phi \in Z$ if $\phi \in Y$ and if

$$|\phi|_{\tilde{g}} = \sup_{-\infty < s \leq 0} |\phi(s)|/\tilde{g}(s)$$

exists. This is a Banach space and S is compact in it.

PROPOSITION 2. The set S in the proof of Proposition 1 is contained in a reflexive subspace of Y.

PROOF. Let $Q = L(S)$ be the linear hull of S (i.e., Q is the space formed by taking linear combinations from S.). Now Q is a subspace of the locally convex metric space Y and so Q is a locally convex metric space. Moreover, as S is closed, Q is closed and complete. Hence, Q is a Frechet space and is barreled. Q is not compact since Q is unbounded in the sense of Treves [21; pp. 136-7]. However, closed and (Treves) bounded subsets of Q satisfy boundedness and Lipschitz conditions similar to those of S, and so must be compact. Therefore, Q is a reflexive space (cf. Treves [21; p. 373]). This completes the proof.

Hence, continuity in initial conditions is not guaranteed by the separability and reflexivity of the space.

3. Fading memory. In general, (2) makes sense only when there is a fading memory. Consider the scalar equation

$$x' = A(t)x + \int_{-\infty}^{t} C(t-s)x(s)ds.$$

At the very least we wish to consider all bounded continuous $\phi \in Y$. Since we want $\Phi(t)$ to be continuous for $t \geq 0$ we need to ask that $\int_0^\infty |C(u)|du < \infty$. Then by [6] there is a continuous increasing function $r\colon [0,\infty) \to [1,\infty)$ such that $r(t) \to \infty$ as $t \to \infty$ and $\int_0^\infty |C(u)|r(u)du < \infty$. We take $g(s) = r(-s)$ so that if $|\phi(s)| \leq \gamma g(s)$ for some $\gamma > 0$, then

$$\int_{-\infty}^{0} |C(t - s)\phi(s)|ds \leq \gamma \int_t^\infty |C(u)|r(u)du$$

and this tends to 0 as $t \to \infty$. In summary, if we admit bounded ϕ then we can admit unbounded ϕ and the memory of ϕ fades in $\Phi(t)$ as $t \to \infty$. A similar result holds for nonlinear systems as may be seen in [5].

The function g is central to the study of delay equations and its role may be seen in [1 - 5] and [8]. For (2) to have meaning we expect to be able to require Φ to be continuous in t for bounded continuous ϕ. But in many problems one quickly learns that unbounded ϕ are needed; however, we show in [5] that if bounded ϕ make Φ continuous then so do certain classes of unbounded ϕ. And this gives rise to the function g which is the weight for a Banach space $(X, |\cdot|_g)$. In order to have a unified theory of existence, continuity, boundedness, stability, and periodicity we work entirely in this Banach space. The importance of that unity is illustrated in [4].

In a private communication Kaminogo informs us that he improved our Example 1 by using bounded initial functions and has obtained continuous dependence results for bounded initial functions.

As a step toward completion of a unified theory we now present a result on continuous dependence on initial conditions using unbounded initial functions. In preparation for that result we now suppose that for the equation (2) there is a continuous function $g\colon (-\infty,0] \to [1,\infty)$ which is decreasing, $g(0) = 1$, and $g(r) \to \infty$ as $r \to -\infty$. Form the Banach space $(X,|\cdot|_g)$ with $\phi \in X$ if $\phi\colon (-\infty,0] \to R^n$ is continuous,

$$|\phi|_g \overset{\text{def}}{=} \sup_{-\infty<t\leq 0} |\phi(t)|/g(t)$$

exists, and there is a nonempty subset $U \subset X$ for which the following definition holds.

DEF. A set $U \subset X$ is an existence set for (2) if $\phi \in U$ implies $\phi(t)$ is continuous for $t \geq 0$.

DEF. Let U be an existence set for (2). Then (2) has a fading memory with respect to U if for each $\phi \in U$, each $J > 0$, and each $\varepsilon > 0$ there is a $\delta > 0$, a $D > 0$, and an $M > 0$ such that if $\psi \in U$, $|\phi-\psi|_g < \delta$, and $0 \leq t \leq J$ then

(a) $\int_{-\infty}^{-D} |q(t,s,\psi(s)) - q(t,s,\phi(s))|ds < \varepsilon$ and

(b) $|\int_{-\infty}^{0} q(t,s,\psi(s))ds| \leq M$.

THEOREM. Let U be an existence set for (2) and let (2) have a fading memory with respect to U. Suppose there is a $\phi \in U$ such that $x(t,0,\phi)$ is unique on some interval $[0,t_1]$. Then $x(t,0,\phi)$ is continuous in ϕ in the following sense: If $\{\psi_n\} \subset U$ and $|\phi - \psi_n|_g \to 0$ as $n \to \infty$, then $|Q\phi - Q\psi_n|_g \to 0$ as $n \to \infty$ where $(Q\phi)(t) = x(t+t_1,0,\phi)$ for $-\infty < t \leq 0$ and $x(t,0,\psi_n)$ is any solution of (2) with initial function ψ_n.

PROOF. Let $x(t,0,\phi)$ be defined on $[0,t_1]$ and suppose it is not continuous in ϕ. Then for some $\varepsilon > 0$ and for each $\delta_k > 0$ there exists $\psi_k \in U$ and $t_k \in [0,t_1]$ with $|x(t_k,0,\phi) - x(t_k,0,\psi_k)| \geq \varepsilon$. We may assume $t_k \to S \in [0,t_1]$ by picking a subsequence if necessary. Moreover, we may assume the t_k chosen so that $\{x(t,0,\psi_k)\}$ is bounded on $[0,S]$. Thus, $\{x'(t,0,\psi_k)\}$ is bounded on $[0,S]$ and so $\{x(t,0,\psi_k)\}$ is an equicontinuous sequence with a convergent subsequence, say $\{x(t,0,\psi_k)\}$ again, with limit $\eta(t)$. We may write

$$x_k(t) = x(t,0,\psi_k) = \psi_k(0) + \int_0^t h(s,x_k(s))ds$$
$$+ \int_0^t \int_0^u q(u,s,x_k(s))ds\,du + \int_0^t \int_{-\infty}^0 q(u,s,\phi(s))ds\,du$$
$$+ \int_0^t \int_{-\infty}^{-D} [q(u,s,\psi_k(s)) - q(u,s,\phi(s))]ds\,du$$
$$+ \int_0^t \int_{-D}^0 [q(u,s,\psi_k(s)) - q(u,s,\phi(s))]ds\,du$$

for any $D > 0$. Let $\varepsilon_1 > 0$ be given and let $0 \le t \le S$. Then there is a $D > 0$ such that

$$\left|\int_0^t \int_{-\infty}^{-D} [q(u,s,\psi_k(s)) - q(u,s,\phi(s))]ds\,du\right| < \varepsilon_1.$$

For this $D > 0$, then $\{\psi_k(s)\}$ converges uniformly to $\phi(s)$ on $[-D,0]$. Hence, we may take the limit as $k \to \infty$ and find that $x_k(t) \to \eta(t)$ and

$$\eta(t) = \phi(0) + \int_0^t h(s,\eta(s))ds$$
$$+ \int_0^t \int_0^u q(u,s,\eta(s))ds\,du + \int_0^t \int_{-\infty}^0 q(u,s,\phi(s))ds\,du.$$

Thus, η and $x(t,0,\phi)$ satisfy the same equation. Since that equation has a unique solution, $\eta(t) = x(t,0,\phi)$. This contradicts $|x(t_k,0,\phi) - x(t_k,0,\psi_k)| \ge \varepsilon$ and completes the proof.

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