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FREE BOUNDARY PROBLEMS FOR STOKES' FLOWS AND FINITE ELEMENT METHODS

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Abstract:

In two dimensions a Stokes' flow is considered symmetric to the abscissa $\eta = 0$ and periodic with respect to ξ. On the free boundary $|\eta| = S(\xi)$ the conditions are: (i) the free boundary is a streamline, (ii) the tangential force vanishes, (iii) the normal force is proportional to the mean curvature of the boundary. By straightening the boundary, i. e. by introducing the variables $x = \xi$, $y = \eta/S(\xi)$, the problem is reduced to one in a fixed domain. The underlying differential equations are now highly nonlinear: They consist in an elliptic system coupled with an ordinary differential equation for S. The analytic properties of the solution as well as the convergence of the proposed finite element approximation are discussed.

1. In accordance to the restrictions formulated in the abstract the problem under consideration is: We ask for the free boundary $\eta = S(\xi)$, 1-periodic in ξ, such that there exists a solution pair $\underline{U} = (U_1, U_2)$ and P with the properties:

(i_1) In the domain $\Omega = \{ (\xi,\eta) \mid |\eta| < S(\xi) \}$ the system of differential equations

(1. 1) $$\sigma_{ik|k} = F_i$$

hold true with

(1. 2) $$\sigma_{ik} = U_{i|k} + U_{k|i} - P\delta_{ik}.$$

(i_2) In the domain Ω the incompressibility condition

(1. 3) $$\nabla \cdot \underline{U} = U_{1|\xi} + U_{2|\eta} = 0$$

holds true.

(ii_1) The free boundary $\eta = \pm S(\xi)$ is streamline, i. e.

(1. 4) $$U_2 - S' U_1 = 0 \quad \text{for } \eta = \pm S(\xi) .$$

(ii_2) On the free boundary the shear-force vanishes, i. e.

(1. 5) $$\sigma_{ik} t_i n_k = 0$$

with $\underline{t} = (t_1,t_2)$ and $\underline{n} = (n_1,n_2)$ being the tangential resp. normal unit vectors.

(ii_3) The normal-force is proportional to the mean curvature, i. e.

(1. 6) $$\sigma_{ik} n_i n_k = \kappa H .$$

We will consider fluid motions only "not too far" from $\underline{U}^0 = (1,0)$. Together with $P^0 = 0$ and $S^0 = 1$ the triple $\{\underline{U}^0, P^0, S^0\}$ is a solution to the problem stated above with $\underline{F}^0 = \underline{0}$. - The main idea of our analysis is the "straigthening" of the free boundary, quite often used. This consists in introducing new variables

(1. 7) $$x = \xi \quad , \quad y = \eta / S(\xi) .$$

Since we are looking for solutions $\{\underline{U},P,S\}$ near to $\{\underline{U}^0,P^0,S^0\}$ we replace $\underline{U}$, P and S - depending on ξ,η - by $(1+u_1,u_2)$, p and 1+s depending on x,y . This leads to a nonlinear problem in the new variables but now in the fixed domain

(1. 8) $$Q_{\pm} = \{ (x,y) \mid |y| < 1 \} .$$

Because of our setting all functions are assumed to be 1-periodic in x. For functions $\underline{F}$ resp. in the new variables $\underline{f}$ symmetric with respect to $y = 0$, i. e. $f_1(x,-y) = f_1(x,y)$ and $f_2(x,-y) = -f_2(x,y)$, the solution also will be symmetric to $y = 0$. Hence we can restrict ourselves to the unit square

(1. 9) $$Q = \{ (x,y) \mid 0 < x, y < 1 \} .$$

The condition of symmetry implies the boundary conditions

(1.10) $$\begin{aligned} u_2(x,0) &= 0 \quad , \\ u_{1|y}(x,0) &= 0 \quad . \end{aligned}$$

By linearizing, i. e. by spltting into linar and nonlinear terms, we get from (1.1) the system

(1.11) $$\begin{aligned} \partial_x(2u_{1|x} - p) + \partial_y(u_{1|y} + u_{2|x}) &= \partial_x \Sigma_{11} + \partial_y \Sigma_{12} + f_1 \quad , \\ \partial_x(u_{1|y} + u_{2|x}) + \partial_y(2u_{2|y} - p) &= \partial_x \Sigma_{21} + \partial_y \Sigma_{22} + f_2 \quad . \end{aligned}$$

Here $\Sigma_{ik} = \Sigma_{ik}(\underline{u},p,s)$ are at least quadratic in their arguments, for example it is

$$(1.12) \qquad \Sigma_{12} = -2ys'u_{1|x} + 2(1+s)^{-1}(1+y^2s'^2)u_{2|y} - (1+s)^{-1}ys'u_{2|y} + ys'p .$$

In the new variables condition (1. 3) becomes

$$(1.13) \qquad \begin{aligned} u_{1|x} + u_{2|y} &=: \mathbf{D} \\ &= (1+s)^{-1}(ys'u_{1|y} + su_{2|y}) \end{aligned}$$

The boundary condition (1. 4) may be used as defining relation for $s = s(x)$:

$$(1.14) \qquad \begin{aligned} s' &= (1+u_1)^{-1}u_2 \\ &=: u_2 + \mathbf{P} . \end{aligned}$$

(1. 5) leads to a boundary condition of the type

$$(1.15) \qquad u_{1|y} + u_{2|x} = \mathbf{T}_1 .$$

The mean curvature H of the free surface depends on the second derivative S'' resp. s''. This quantity may be computed from (1.14). In this way (1. 6) leads to the second boundary condition of the type

$$(1.16) \qquad 2u_{2|y} - p + \kappa u_{2|x} = \mathbf{T}_2 .$$

The $\mathbf{T}_i = \mathbf{T}_i(\underline{u},p,s)$ are at least quadratic in their arguments. Similar to the Σ_{ik} they depend only on the functions themselves and their first derivatives. Since s is assumed to be 1-periodic we have $\int s' = 0$. Here $\int w$ resp. later $\int\int w$ are abbreviations defined by

$$(1.17) \qquad \int w = \int_0^1 w(x,1)dx , \qquad \int\int w = \int\int_Q w(x,y)dxdy$$

In view of the boundary condition (1.10) we get from (1.13) $\int\int \mathbf{D} = -\int u_2$. Therefore the quantity

$$(1.18) \qquad \begin{aligned} \gamma &= \int\int \mathbf{D} - \int \mathbf{P} \\ &=: \gamma(\underline{u},p,s) \end{aligned}$$

will be zero. Hence we may replace in (1.13) the right hand side $\mathbf{D}$ by

$$(1.19) \qquad \tilde{\mathbf{D}} = \mathbf{D} - \gamma .$$

In the new variables we have the

Problem:

Given the vector $\underline{f}$ defined in Q (1. 9) and 1-periodic in x. Find $\underline{u}$, p, s 1-periodic in x, fulfilling the differential equations (1.11), (1.13) in Q, and the boundary condition (1.10) on y=0 as well as (1.14), (1.15), and (1.16) on y=1.

<u>2.</u> The idea of proving the existence of a solution of the problem as well as deriving a ***finite element method*** in order to approximate this solution is as follows: We consider the quadruple $\mathbf{m} = \{u_1, u_2, p, s\}$ as an element of a linear space $\mathfrak{M}$ equipped with an appropriate norm. The geometric boundary condition (1.10_1) has to be imposed on u_2. Obviously u_1 as well as s are defined up to a constant only. Therefore we nomalize u_1, s according to $\int\int u_1 = 0$, $\int s = 0$. The correspondent restriction of the space $\mathfrak{M}$ will be denoted by ${}^{\bullet}\mathfrak{M}$. Similarily we consider the octuple $\mathbf{n} = \{\Sigma_{11}, \Sigma_{12}, \Sigma_{21}, \Sigma_{22}, \tilde{D}, P, T_1, T_2\}$ as an element of a linear space $\mathfrak{N}$, also equipped with a norm. By (1.12), (1.13) etc. the mapping $\mathbf{A} : \mathfrak{M} \mapsto \mathfrak{N}$ is defined. The mapping $\mathbf{B} : \mathfrak{N} \mapsto {}^{\bullet}\mathfrak{M}$ which associates the solution of the boundary value problem to the right hand sides is constructed by the natural weak formulation of the problem: If $\mathbf{m} \in {}^{\bullet}\mathfrak{M}$ is the solution then with any $\boldsymbol{\mu} = \{\underline{v}, q, r\} \in {}^{\bullet}\mathfrak{M}$ the variational equations hold:

$$
(2.2)\qquad
\begin{aligned}
a(\mathbf{m},\boldsymbol{\mu}) + b(\mathbf{m},\boldsymbol{\mu}) &= L_1(\mathbf{n},\boldsymbol{\mu}) + F(\underline{f},\boldsymbol{\mu}) \quad ,\\
b(\boldsymbol{\mu},\mathbf{m}) &= L_2(\mathbf{n},\boldsymbol{\mu}) \quad ,\\
c(\mathbf{m},\boldsymbol{\mu}) - \int u_2 r' &= \int P r' \quad .
\end{aligned}
$$

Here L_i, F, a, b, c are bilinear functionals; especially a(.,.), b(.,.), and c(.,.) are defined by

$$
(2.3)\qquad
\begin{aligned}
a(\mathbf{m},\boldsymbol{\mu}) &= \int\int\left[2u_{1|x}v_{1|x} + (u_{1|y}+u_{2|x})(v_{1|y}+v_{2|x}) + 2u_{2|y}v_{2|y}\right] - \kappa\int u_{2|x}v_2 \quad ,\\
b(\mathbf{m},\boldsymbol{\mu}) &= \int\int\; q\left\{u_{1|x} + u_{2|y}\right\}\\
c(\mathbf{m},\boldsymbol{\mu}) &= \int\; s' r'
\end{aligned}
$$

The standard inf-sup condition is valid for the form b(.,.), because of Korn's second inequality a(.,.) may be extended to a bounded and coercive bilinear form in the Sobolev space $H_1(Q)\times H_1(Q)$. In connection with the normalisation of u_1 and s uniqueness of the mapping $\mathbf{B}$ is guaranteed.

<u>3.</u> Since the mapping $\mathbf{A}$ is nonlinear we will work with Hölder-spaces: We equip the spaces ${}^{\bullet}\mathfrak{M}$ and $\mathfrak{N}$ in the following way with norms, in these topologies they are Banach-spaces:
For $\boldsymbol{\mu} = \{\underline{v}, q, r\} \in {}^{\bullet}\mathfrak{M}$ we define

$$
(3.1)\qquad
\begin{aligned}
|||\boldsymbol{\mu}||| &:= |||\boldsymbol{\mu}|||_{{}^{\bullet}\mathfrak{M}}\\
&= \sum \|v_i\|_{C_{1\cdot\lambda}(Q)} + \|q\|_{C_{0\cdot\lambda}(Q)} + \|r\|_{C_{2\cdot\lambda}(I)}
\end{aligned}
$$

Here $\|.\|_{C_{k\cdot\lambda}(.)}$ denote the usual Hölder-norms with $\lambda \in (0,1]$, I is the unit interval.

For $\mathbf{v} = \{\Sigma_{11}, \Sigma_{12}, \Sigma_{21}, \Sigma_{22}, \tilde{D}, P, T_1, T_2\} \in \mathfrak{n}$ we define

$$\begin{aligned} \|\mathbf{v}\| &:= \|\mathbf{v}\|_{\mathfrak{n}} \\ &= \sum \|\Sigma_{ik}\|_{C_{0\cdot\lambda}(Q)} + \|\tilde{D}\|_{C_{0\cdot\lambda}(I)} + \|P\|_{C_{1\cdot\lambda}(Q)} + \sum \|T_i\|_{C_{0\cdot\lambda}(I)} \end{aligned} \tag{3.2}$$

Now we consider elements μ in the ball $B_\delta({}^*\mathfrak{m}) := \{\mu \mid \mu \in {}^*\mathfrak{m} \wedge \|\mu\| \leq \delta\}$ with $\delta \leq \delta_0 < 1$ and δ_0 fixed. Obviously the two estimates are valid:

$$\begin{aligned} \|A\mu\|_{\mathfrak{n}} &\leq c\delta\|\mu\|_{\mathfrak{m}} \\ \|A\mu^1 - A\mu^2\|_{\mathfrak{n}} &\leq c\delta\|\mu^1 - \mu^2\|_{\mathfrak{m}} \end{aligned} \tag{3.3}$$

Here "c" denotes a numerical constant depending only on δ_0 which may differ at different places.

It can be shown: The mapping **B** is bounded, i. e. for $\mathbf{m} = \mathbf{Bn}$ the estimate

$$\|\mathbf{m}\| \leq c\|\mathbf{n}\| + \sum \|f_i\|_{C_{0\cdot\lambda}(Q)} \tag{3.4}$$

is valid. Thus the Banach Fixed Point Theorem leads to: For $\|f_i\|$ sufficiently small and δ chosen appropriately the mapping

$$T := B * A \tag{3.5}$$

possesses an unique fixed point in the ball $B_\delta({}^*\mathfrak{m})$. It turns out that the quantity γ (1.18) vanishes. This implies that the fixed point corresponds to the solution of the original problem.

4. Now let ${}^*\mathfrak{m}_h$ be an appropriate finite element approximation space. By restricting in (2.2) the elements $\mu = \mu_h \in {}^*\mathfrak{m}_h$ and looking for the solution $\mathbf{m}_h \in {}^*\mathfrak{m}_h$ the mapping $\mathbf{B}_h$ and consequently also T_h (see (3.5)) is defined.

It can be shown: Under certain conditions concerning the approximation spaces, especially the Brezzi condition is needed, the mapping $\mathbf{B}_h$ is bounded, i. e. an inequality of the type (3.4) holds true. This finally leads to almost best error estimates: Let $\mathbf{m} \in {}^*\mathfrak{m}$ and $\mathbf{m}_h \in {}^*\mathfrak{m}_h$ be the solution of the analytic problem resp. the finite element solution then

$$\|\mathbf{m} - \mathbf{m}_h\| \leq C \inf\{\|\mathbf{m} - \mu_h\| \mid \mu_h \in {}^*\mathfrak{m}_h\}\ . \tag{4,1}$$

The proofs and the complete bibliography will appear elsewhere. Here we refer only to

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