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LINEAR PERTURBATIONS OF GENERAL DISCONJUGATE EQUATIONS

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Suppose that $p_1,\dots,p_{n-1},\ q \in C[a,\infty]$, $p_i > 0$, and

(1) $$\int^{\infty} p_i\,dt = \infty, \quad 1 \le i \le n-1,$$

and define the quasi-derivatives

(2) $$L_0 x = x; \quad L_r x = \frac{1}{p_r}(L_{r-1}x)', \quad 1 \le r \le n$$

(with $p_n = 1$). We will give conditions which imply that the equation

(3) $$L_n u + q(t)u = 0$$

has solutions which behave as $t \to \infty$ like solutions of the equation $L_n x = 0$.

Let $I_0 = 1$ and

$$I_j(t,s;\,q_j,\dots,q_i) = \int_s^t q_j(w)I_{j-1}(w,s;q_{j-1},\dots,q_i)\,dw, \quad j \ge 1.$$

Then a principal system [2] for $L_n x = 0$ is given by

$$x_i(t) = I_{i-1}(t,a;p_1,\dots,p_{i-1}), \quad 1 \le i \le n\ ;$$

in fact,

(4) $$L_r x_i(t) = \begin{cases} I_{i-r-1}(t,a;p_{r+1},\dots,p_{i-1}), & 0 \le r \le i-1\ , \\ 0, & i \le r \le n-1\ . \end{cases}$$

We also define

$$y_i(t) = I_{n-i}(t,a;p_{n-1},\dots,p_i), \quad 1 \le i \le n,$$

and

(5) $$d_{ir}(t) = \begin{cases} L_r x_i(t), & 0 \le r \le i-1, \\ 1/I_{r-i+1}(t,a;p_r,\dots,p_i), & i \le r \le n\ . \end{cases}$$

We give sufficient conditions for (3) to have a solution u_i such that

(6) $$L_r u_i = L_r x_i + o(d_{ir}) \quad (t \to \infty), \quad 0 \le r \le n-1,$$

for some given i in $\{1,\dots,n\}$. This formulation of the question is due to Fink and Kusano, and the best previous result on this question is the following special case of a theorem obtained by them in [1].

THEOREM 1. *If*

(7) $\int^{\infty} x_i y_i |q| ds < \infty,$

then (3) *has a solution* u_i *which satisfies* (6).

Our results require less stringent integrability conditions. We need the following lemma from [4].

LEMMA 1. *Suppose that* $Q \in C[t_0,\infty)$ *for some* $t_0 \ge a$, *that* $\int^{\infty} y_i Q dt$ *converges (perhaps conditionally), and that*

$$\sup_{\tau\ge t} \left| \int_{\tau}^{\infty} y_i Q ds \right| \le \psi(t), \quad t \ge t_0,$$

where ψ *is nonincreasing and continuous on* $[t_0\infty)$. *Define*

$$K(t;Q) = \int_t^{\infty} I_{n-i}(t,s;p_i,\ldots,p_{n-1})Q(s)ds,$$

and, for $t \ge t_0$, *let*

$$J(t;Q) = K(t;Q) \quad \text{if} \quad i = 1;$$

or

$$J(t;Q) = \int_{t_0}^{t} p_1(s)K(s;Q)ds = I_1(t,t_0;p_1K(\ ;Q))ds$$

if $i = 2$; *or*

$$J(t;Q) = I_{i-1}(t,t_0;p_1,\ldots,p_{i-1}K(\ ;Q))$$

if $3 \le i \le n$.

Then

(8) $L_n J(t;Q) = -Q(t), \ t \ge t_0,$

and

$$|L_r J(\ ;Q)| \le \begin{cases} \Psi(t_0)d_{ir}(t), & 0 \le r \le i-2, \\ 2\Psi(t)d_{ir}(t), & i-1 \le r \le n-1, \end{cases} \quad t \ge t_0;$$

moreover, if $\lim_{t\to\infty} \Psi(t) = 0$, *then also*

$$L_r(J(t;Q)) = o(d_{ir}(t)), \quad 0 \le r \le i-2.$$

The following assumption applies throughout.

ASSUMPTION A. Let $\int^{\infty} y_i x_i q ds$ converge (perhaps conditionally), and suppose that

(9) $E(t) = \int_t^{\infty} y_i x_i q ds = O(\varphi(t))$

with φ nonincreasing on $[a,\infty)$, and

(10) $$\lim_{t\to\infty} \varphi(t) = 0 .$$

If $t_0 > a$, let $B(t_0)$ be the set of functions h such that $L_0 h,\dots,L_{n-1} h \in C[t_0,\infty)$ and

$$L_r h = \begin{cases} O(d_{ir}), & 0 \le r \le i-2, \\ & \qquad\qquad t \ge t_0 , \\ O(\varphi d_{ir}), & i-1 \le r \le n-1 , \end{cases}$$

with norm $\| \ \|$ defined by

(11) $$\|h\| = \sup_{t\ge t_0} \max\left\{ \frac{|L_r h(t)|}{\varphi(t_0) d_{ir}(t)} \ (0\le r\le i-2),\ \frac{|L_r h(t)|}{2\varphi(t) d_{ir}(t)} \ (i-1\le r\le n-1) \right\}$$

Then Lemma 1 with $Q = qv$ and $\Psi = K\varphi$ implies the following lemma.

LEMMA 2. *If* $v \in C[t_0,\infty)$ *and*

$$\left| \int_t^\infty y_i q v\, ds \right| \le K\varphi(t), \quad t \ge t_0 ,$$

then

$$J(\ ;qv) \in B(t_0)$$

and

$$\|J(\ ;qv)\| \le K .$$

Now define the transformation T by

(12) $$(Th)(t) = J(t;qx_i) + J(t;qh) .$$

Lemma 2 and Assumption A imply that $J(\ ;qx_i) \in B(t_0)$ for all $t_0 > a$. We need only impose further conditions which will imply that $\int^\infty y_i qh ds$ converges (perhaps conditionally) if $h \in B(t_0)$, and that

$$\left| \int_t^\infty y_i qh ds \right| \le \|h\| \sigma(t;t_0)\varphi(t), \quad t \ge t_0 ,$$

where σ does not depend on h, and

(13) $$\sup_{t\ge t_0} \sigma(t;t_0) = \theta < 1$$

if t_0 is sufficiently large. Lemma 2 will then imply that T is a contraction mapping of $B(t_0)$ into itself, and therefore that there is an h_i in $B(t_0)$ such that $Th_i = h_i$. It will then follow from (8) and (12) that $u_i = x_i + h_i$ is a solution of (3). Moreover, Lemma 3 with $Q = qu_i$ will imply that

(14) $$L_r u_i - L_r x_i = \begin{cases} o(d_{ir}), & 0 \le r \le i-2 \\ O(\varphi d_{ir}), & i-1 \le r \le n-1 . \end{cases}$$

The next lemma can be obtained from (9) and integration by parts.

See [3] for the proof of the special case where $p_1 = \dots = p_n = 1$.

LEMMA 3. *Let*

(15) $H_0 = y_i q;\ H_j(t) = \int_t^\infty p_{j-1} H_{j-1}\,ds,\quad 1 \le j \le i \quad (p_0 = 1).$

Then (9) implies that

(16) $H_j = O(\varphi / L_{j-1} x_i),\quad 1 \le j \le i,$

and that the integrals

(17) $\int^\infty p_j (L_j x_i) H_j\,ds,\quad 0 \le j \le i-1,$

all converge. Moreover, if the convergence is absolute for some $j = k$ with $0 \le k \le i-2$, then it is absolute for $k \le j \le i-1$.

THEOREM 2. *If*

(18) $\overline{\lim_{t\to\infty}}\, (\varphi(t))^{-1} \int_t^\infty p_{i-1} |H_{i-1}| \varphi\,ds = A < \frac{1}{2},$

then (3) *has a solution* u_i *which satisfies* (14).

Proof. Integration by parts yields

(19) $\int_t^T y_i q h\,ds = -\sum_{j=1}^{i-1} H_j (L_{j-1} h) \Big|_t^T + \int_t^T p_{i-1} H_{i-1} (L_{i-1} h)\,ds$

if $h \in B(t_0)$ and $2 \le i \le n$; if $i = 1$, then the sum on the right is vacuous and (19) is trivial. (Recall (2) and (15).) Now (5),(9), (11),(18), and Lemma 3 imply that we can let $T \to \infty$ in (19) and infer (13) with

(20) $$\sigma(t;t_0) = \varphi(t_0)(\varphi(t))^{-1} \sum_{j=1}^{i-1} |H_j(t)| L_{j-1} x_i(t) + 2(\varphi(t))^{-1} \int_t^\infty p_{i-1} |H_{i-1}| \varphi\,ds .$$

From (16), the sum on the right side of (20) is bounded on $[a,\infty)$; hence, (10) and (18) imply (13) for t_0 sufficiently large. This completes the proof.

With $i = 1$, (18) reduces to

$$\overline{\lim_{t\to\infty}}\, (\varphi(t))^{-1} \int_t^\infty y_1 |q| \varphi\,ds < \frac{1}{2},$$

which is weaker than (7), since $x_1 = 1$. The next two corollaries show that (18) is also weaker than (7) if $2 \le i \le n$.

COROLLARY 1. *If* $2 \le i \le n$ *and*

(21) $\int^\infty p_k (L_k x_i)(L_{k-1} x_i)^{-1} \varphi\,dt < \infty$

for some k *in* $\{1,\dots,i-1\}$, *then* (3) *has a solution* u_i *which* *satisfies* (14).

Proof. From (16),

$$p_k(L_k x_i)|H_k| \le Mp_k(L_k x_i)(L_{k-1}x_i)^{-1}\varphi \tag{22}$$

for some constant M, so (21) implies that (17) with $j = k$ converges absolutely. From the closing sentence of Lemma 3, this means that

$$\int^{\infty} p_{i-1}|H_{i-1}|ds < \infty ,$$

which obviously implies (18) with $A = 0$.

COROLLARY 2. *If* $2 \le i \le n$ *and*

$$\int_t^{\infty} p_{i-1}(s)\left(\int_a^s p_{i-1}(w)dw\right)^{-1}\varphi^2(s)ds = o(\varphi(t)), \tag{23}$$

then (3) *has a solution* u_i *which satisfies* (13).

Proof. From (22) with $k = i - 1$ and (4), (23) implies (18) with $A = 0$.

THEOREM 3. *If* $1 \le i \le n - 1$ *and*

$$\overline{\lim_{t\to\infty}}\, (\varphi(t))^{-1}\int_t^{\infty}\varphi(s)p_i(s)\left(\int_a^s p_i(w)dw\right)^{-1}|H_i(s)|ds = B < \frac{1}{2}, \tag{24}$$

then (3) *has a solution which satisfies* (14).

Proof. Lemma 3 and our present assumption enable us to continue the integration by parts in (19) by one more step, to obtain

$$\int_t^{\infty} y_i qh ds = \sum_{j=1}^{i} H_j(t)L_{j-1}h(t) + \int_t^{\infty} p_i H_i(L_i h)ds.$$

Because of (5) (with $r = i$) and (11), this yields

$$\sigma(t;t_0) = \varphi(t_0)(\varphi(t))^{-1}\sum_{j=1}^{i-1}|H_j(t)|L_{j-1}x_i(t) + 2H_i(t) +$$
$$+ 2(\varphi(t))^{-1}\int_t^{\infty}\varphi(s)p_i(s)\left(\int_a^s p_i(w)dw\right)^{-1}|H_i(s)|ds.$$

Now (10) and (16) imply (20) for t_0 sufficiently large. This completes the proof.

COROLLARY 3. *If* $1 \le i \le n - 1$ *and*

$$\int_t^{\infty} p_i(s)\left(\int_a^{\infty} p_i(w)dw\right)^{-1}\varphi^2(s)ds = o(\varphi(t)), \tag{25}$$

then (3) *has a solution* u_i *which satisfies* (14).

Proof. From (16) with $j = i$, it follows that (25) implies (24) with $B = 0$.

References.

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