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SOME PROBLEMS CONCERNING THE EQUIVALENCES OF TWO SYSTEMS OF DIFFERENTIAL EQUATIONS

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Consider two systems

(1) $\dot{x}(t) = f(t,x_t)$

(2) $\dot{y}(t) \in f(t,y_t) + g(t,y_t)$

where $t \in J = \langle 0,\infty)$, $x : R \to R^n$, $y : R \to R^n$, $x_t = x(t+s)$, $y_t = y(t+s)$, $s \in (-\infty,0\rangle$. Denote by $C = C(-\infty,0;R^n)$ the space of all functions $\varphi : (-\infty,0\rangle \to R^n$ which are bounded and continuous with sup norm $\|.\|$. Then $f : JxC \to R^n$, $g : JxC \to$ {the set of all nonempty subsets of R^n}. Further properties of f and g will be given later. However, we will still assume that f and g are such that the existence of the solutions of (1) and (2) is guaranteed on J. $|.|$ is the vector norm in R^n. If $A \subset R^n$, then $|A| \doteq \sup\{|a| : a \in A\}$.

Our aim is to estabilish the conditions which give the possibility of pairing of the solutions x(t) of (1) and y(t) of (2) in such a way that we will be able to say something about the asymptotic behaviour of the difference $y(t) - x(t) = z(t)$. Assume that x(t) is given. Then, proceeding formally, substituting y(t) by $z(t) + x(t)$ in (2), we get

(3) $\dot{z}(t) \in -f(t,x_t) + f(t,z_t + x_t) + g(t,z_t + x_t)$

We have to prove the existence of such solution z(t) to the functional differential inclusion (3) that $\lim z(t) = 0$ as $t \to \infty$ (the case of asymptotic equivalence) or that $z(t) \in L_p(J)$, $p \geq 1$ (the case of p-integral equivalence). There are many methods how to do it, e.g. use the viability theory, method of fixed point, method of Liapunov function.

First we will use the viability theory.

Theorem 1. a) Let be $f : JxC \to R^n$ continuous and let it satisfy the Lipschitz condition

(4) $|f(t,\varphi_1) - f(t,\varphi_2)| \leq L(t)\|\varphi_1 - \varphi_2\|, \quad L(t) \in L_1(J)$

for each (t,φ_1), $(t,\varphi_2) \in JxC$.

b) Let g be an upper semicontinuous map from JxC to the nonempty compact convex subsets of R^n and let

$|g(t,\varphi)| \leq G_0(t,\|\varphi\|)$ a.e. on J where $G_0(t,u) : JxJ \to J$ is monotone nondecreasing in u for each fixed $t \in J$ and is integrable on J for each fixed $u \in J$.

c) Let $x : R \to R^n$ be a bounded solution of (1).

d) Let there exist a solution $u : J \to J$ to the differential equation

(5) $\dot{u}(t) = -L(t)u - G_0(t,u + \|x_t\|) \doteq -G(t,u),\ u(0) > 0$

e) Let be

$K(t) \doteq \{x \in R^n: |x| \le u(t)\},\quad t \in J$

f) Let be

$\forall t \in J,\ \mathcal{K}(t) \doteq \{\varphi \in C : \varphi(0) \in K(t)\}$

g) Let be the image of the graph ($\mathcal{K}$) by the map

$F(t,\varphi) \doteq -f(t,x_t) + f(t,\varphi + x_t) + g(t,\varphi + x_t)$

relatively compact.

h) Let for

$\forall t \in J, \forall \varphi$ such that $\varphi(0) \in K(t), \forall x \in K(t)$

$F(t,\varphi) \cap DK(t,\varphi(0))(1) \neq \phi$

where $DK(t,\varphi(0))$ is the contingent derivative of K at $(t,\varphi(0))$. Then for each $\varphi \in \mathcal{K}(0)$ there exists a solution z(t) to the functional inclusion such that

(6) for almost all $t \in J$, $\dot{z}(t) \in F(t,z_t)$

$(z)_0 = \varphi$

which is viable in the sense that

(7) $\forall t \in J,\ z(t) \in K(t) \qquad (|z(t)| \le u(t))$

Remark 1. Evidently, if $\lim u(t) = 0$ as $t \to \infty$, then also $\lim z(t) = 0$ as $t \to \infty$ and if $u(t) \in L_p(J)$, $p \ge 1$, then also the restriction $z(t)|_J \in L_p(J)$ holds.

Remark 2. It follows from the properties of f and g that F is an upper semicontinuous map from JxC to the nonempty compact convex subsets of R^n and

(8) $|F(t,\varphi)| \le L(t)\|\varphi\| + G_0(t,\|\varphi\| + \|x_t\|) \doteq G(t,\|\varphi\|)$

Evidently, $G : JxJ \to J$ is nondecreasing in u for each fixed $t \in J$ and integrable in t for each fixed $u \in J$.

Remark 3. In our case the basic space is R^n. The set valued map K defined by e) is upper semicontinuous and therefore its graph is closed.

The proof of the Theorem 1. follows immediatly from the time dependent Viability Theorem. (See e.g. [1].)

Remark 4. The most important condition is the condition h) which is necessary in our case because R^n has a finite dimension. This follows from the viability theory. If $t > 0$ and $|\varphi(0)| < u(t)$ then $(t,\varphi(0)) \in \text{int}\,(\text{graph}\,(K))$ and therefore the contingent cone $T_{\text{graph}(K)}(t,\varphi(0)) = R^{n+1}$. Evidently, in this case the condition h) is satisfied.

As to what concerns the existence of the solution u(t) from the condition d) we have the following lemma.

Lemma 1. Let be satisfied a) from the Theorem 1. Let i) $G_0(t,c) \in L_1(J)$ for each $c \geq 0$; ii) $\liminf_{c \to \infty}(c^{-1} \int_0^\infty G_0(s,c)ds) = 0$. Then there exists a solution $u : J \to J$ of the equation (5) such that $\lim u(t) = 0$ as $t \to \infty$. If, moreover, iii) $tG_0(t,c) \in L_1(J)$ for each $c \geq 0$, then this solution $u(t) \in L_p(J)$, $p \geq 1$.

The proof of this Lemma 1 can be made via Schauder fixed point theorem.

Theorem 2. Let be satisfied a) and b) from the Theorem 1. c') Let $y : R \to R^n$ be a bounded solution to the functional differential inclusion (2).

d') Let $u : J \to J$ be a solution of the equation

(9) $\dot{u}(t) = -L(t)u - G_0(t,\|y_t\|) \doteq -G_1(t,u)$, $u(0) > 0$

e') Let be

$K_1 \doteq \{x \in R^n: |x| \leq u(t)\}$, $t \in J$

f') Let be

$\forall t \in J$, $\mathcal{K}_1 \doteq \{\varphi \in C : \varphi(0) \in K_1(t)\}$

g') Let be the image of the graph $(\mathcal{K}_1)$ by the map

$F_1(t,\varphi) \doteq f(t,\varphi + y_t) - f(t,y_t) - g(t,y_t)$

relatively compact.

h') Let for

$\forall t \in J, \forall\varphi$ such that $\varphi(0) \in K_1(t), \forall x \in K_1(t)$

$F_1(t,\varphi) \cap DK_1(t,\varphi(0))(1) \neq \phi$

where $DK_1(t,\varphi(0))$ is the contingent derivative of K_1 at $(t,\varphi(0))$. Then for each $\varphi \in \mathcal{K}_1(0)$ there exists a solution z(t) to the functional differential inclusion such that

(10) for almost all $t \in J$, $\dot{z}(t) \in F_1(t,z_t)$

$(z)_0 = \varphi$

which is viable in the sense that

$t \in J$, $z(t) \in K_1(t)$ $\qquad (|z(t)| \leq u(t))$

The similar remarks as Remark 1 - 4 hold also in this case. The proof of the Theorem 2 follows also immediatly from the time dependent Viability Theorem.

Lemma 2. Let be satisfied a) from the Theorem 1 and i) from Lemma 1. Then

$$u(t) = \exp(-\int_0^t L(s)ds) \int_t^\infty \exp(\int_0^s L(v)dv) G_0(s, \|y_s\|)ds$$

is a solution of (9) such that lim u(t) = 0 as $t \to \infty$. If, moreover, iii) from Lemma 1 holds, then also $u(t) \in L_p(J)$, $p \geq 1$ holds true.

The proof can be made immediately.

From Theorem 1, Theorem 2, Lemma 1 and Lemma 2 we get

Theorem 3. Let be satisfied all conditions of Theorem 1 and 2. Then the conditions i) and ii) from Lemma 1 guarantee the asymptotic equivalence between the set of all bounded solutions of (1) and the set of all bounded solutions of (2). If, moreover, the condition iii) from Lemma 1 is satisfied, then there exists also the p- integral equivalence, $p \geq 1$, between the above mentioned sets of solutions.

Now, we will consider the same problem of asymptotic and integral equivalences for the systems (1) and (2) by use of fixed point method. Henceforth we will assume that the following assumptions are satisfied:

(F) $|f(t,\varphi_1) - f(f,\varphi_2)| \leq L(t)w(\|\varphi_1 - \varphi_2\|)$

where $L(t) \in L_1(J)$, $w : J \to J$ is a continuous function, $\int_0^\infty L(t)dt = S$, $\sup_{\langle 0,u\rangle} w(r) \leq S^{-1}\alpha u$, $\alpha < 1$;

(H_1) $g(t,\varphi)$ is nonempty compact convex subset of R^n for each $(t,\varphi) \in \in J \times C$;

(H_2) for every fixed $t \in J$ $g(t,\varphi)$ is upper semicontinuous in φ;

(H_3) for each measurable function $z : R \to R^n$ such that $z|_{(-\infty,0\rangle} \in C$ there exists a measurable selector $v : J \to R^n$ such that

$v(t) \in g(t,z_t)$ a.e. on J

We set $M(z(t)) \doteq$ {all measurable selectors belonging to z(t)}.

(H_4) there exists a function $G_0 : J \times J \to J$ such that α) $G_0(t,u)$ is monotone nondecreasing in u for each fixed $t \in J$ and $G_0(t,u) \in L_1(J)$ for any fixed $u \in J$; β) $|g(t,\varphi)| \leq G_0(t,\|\varphi\|)$ a.e. on J; γ) $\liminf_{u \to \infty} (u^{-1} \int_0^\infty G_0(t,u)dt) = 0$ uniformly for $t \in J$.

Lemma 3. Let $z : R \to R^n$ be a measurable and bounded function. Then for each $v(t) \in M(z(t))$ we have $v(t) \in L_1(J)$.

Proof. It follows from (H_4).

Lemma 4. Let be satisfied (F), (H_1) - (H_4). Let be $B = \{z : R \to R^n$: continuous and bounded} and $B_u = \{z \in B : \|z\| \leq u\}$. Let $x : R \to R^n$ be a bounded solution of (1) and let be $\varphi \in C$ given. Then the operator T defined on B by the relation : for $z \in B$ it is

$$(Tz)(t) = \{-\int_t^{\infty}[f(s,z_s+x_s) - f(s,x_s)]ds - \int_t^{\infty}v(s)ds : v(t) \in$$

$$\in M(z(t)+x(t))\} \text{ for } t \in J$$

$$(Tz)_0 = \{\varphi(t) - \varphi(0) - \int_0^{\infty}[f(s,z_s+x_s) - f(s,x_s)]ds -$$

$$- \int_0^{\infty}v(s)ds\} , \quad \text{for} \quad t \le 0$$

maps $B \to 2^B$, is compact and upper semicontinuous in B and there exists such $u \in J$ that T maps B_u into $cf(B_u)$. ($cf(B_u)$ is the set of all closed and convex subsets of B_u.)

Proof. Let be $z(t) \in B$. Then $\|z\| = \beta < \infty$ and by (F) we have

$$\int_0^{\infty}|f(s,z_s+x_s) - f(s,x_s)|ds \le \int_0^{\infty} L(s)w(\|z_s\|)ds \le$$

$$\le \max_{0\le\tau\le\beta} w(\tau) \int_0^{\infty} L(s)ds < \infty .$$

By Lemma 3 $v(t) \in M(z(t)+x(t))$ is from $L_1(J)$. Thus the operator T is well defined. Evidently, for $z(t) \in B$ $(Tz)(t)$ is a subset of B.

Let be $\|x(t)\| = \rho$. Consider the set B_u. Let be $z(t) \in B_u$ and let be $\xi(t) \in (Tz)(t)$. Then there exists such $v(t) \in M(z(t)+x(t))$ that

$$\xi(t) = -\int_t^{\infty}[f(s,z_s+x_s) - f(s,x_s)]ds - \int_t^{\infty}v(s)ds, \quad t \in J$$

$$\xi(t) = \varphi(t) - \varphi(0) - \int_0^{\infty}[f(s,z_s+x_s) - f(s,x_s)]ds -$$

$$- \int_0^{\infty}v(s)ds, \quad t \in \langle -\infty, 0\rangle$$

and

$$(*) \qquad |\xi(t)| \le \max_{0\le r\le u} w(r) \int_0^{\infty} L(s)ds + \int_0^{\infty} G_0(s,\rho+u)ds = K < \infty, \quad t \in J$$

Thus the functions $\xi(t) \in (Tz)(t)$ are uniformly bounded by the constant K and because for each $z(t) \in B_u$ we get the same constant K, we may conclude that TB_u is the set of continuous and uniformly bounded functions.

Let be $0 \le t_1 < t_2$. Then for $\xi(t) \in (Tz)(t)$, $z(t) \in B_u$ we have

$$|\xi(t_2) - \xi(t_1)| \le \int_{t_1}^{t_2}|f(s,z_s+x_s) - f(s,x_s)|ds + \int_{t_1}^{t_2}|v(s)|ds \le$$

$$\leq \max_{0\leq r\leq u} w(r) \int_{t_1}^{t_2} L(s)ds + \int_{t_1}^{t_2} G_0(s,\rho + u)ds$$

From this we conclude that all functions from TB_u are equicontinuous on J. Moreover, to each $\varepsilon > 0$ there exists $t_0(\varepsilon) > 0$ such that for $t_0(\varepsilon) \leq t_1 < t_2$ we have

$$|\xi(t_2) - \xi(t_1)| \leq \max_{0\leq r\leq u} w(r) \int_{t_0}^{\infty} L(s)ds + \int_{t_0}^{\infty} G_0(s,\rho + u)ds < \varepsilon$$

Then from this, from the uniform boundedness and from the equicontinuity of all functions of TB_u it follows that TB_u is compact in the topology of uniform convergence.

Evidently, to each bounded set $A \subset B$ there exists such $u \in J$ that $A \subset B_u$ and $TA \subset TB_u$. From this it follows that T is compact in B.

Let be $z_n(t)$, $z(t) \in B$ and let $\{z_n(t)\}$ converge to $z(t)$ in B, i.e. uniformly on R. Therefore, the set $\{z_n(t), z(t), n = 1,2,..\}$ is bounded in B. Thus there exists $u \geq 0$ such that $z_n(t) \in B_u$, $z(t) \in B_u$ and TB_u is a compact set. Let $h_n(t) \in (Tz)(t)$, $n = 1,2,...$ Evidently $h_n(t) \in TB_u$, $n = 1,2...$ The set TB_u being compact there exists a subsequence $\{h_{n_i}(t)\}$ of $\{h_n(t)\}$, which converges uniformly to a function $h(t) \in TB_u$. Then to each $h_n(t)$ there exists $v_n(t) \in M(z_n(t) + x(t))$, $n = 1,2,...$ such that

$$h_n(t) = -\int_t^{\infty} [f(s,(z_n)_s + x_s) - f(s,x_s)]ds - \int_t^{\infty} v_n(s)ds,$$

$$t \in J, \quad n = 1,2,...$$

$$h_n(t) = \varphi(t) - \varphi(0) - \int_0^{\infty} [f(s,(z_n)_s + x_s) - f(s,x_s)]ds -$$

$$- \int_0^{\infty} v_n(s)ds, \quad t \in (-\infty,0)$$

By Lemma 3 we have

$$\|v_n(t)\|_1 \leq \int_0^{\infty} G_0(s,u + \rho)ds < \infty$$

It means that the sequence $\{v_n(t)\}$ is bounded in $L_1(J)$. Furthermore, if $\{E_k\}$, $E_k \subset J$, is a nonincreasing sequence of sets such that $\bigcap_{k=1}^{\infty} E_k = \phi$, then

$$\lim_{k\to\infty} |\int_{E_k} v_n(s)ds| \leq \lim_{k\to\infty} \int_{E_k} |v_n(s)|ds \leq \lim_{k\to\infty} \int_{E_k} G_0(s,u + \rho)ds = 0$$

Then (see [2], Th. IV. 8.9) it is possible to choose from $\{v_n(t)\}$ a subsequence $\{v_{n_k}(t)\}$ which weakly converges to some $v(t) \in L_1(J)$.

Now, because $\{z_{n_k}(t)\}$ converges to $z(t)$ in B and $v_{n_k}(t) \in g(t,z_{n_k}(t))$, $k = 1,2,\ldots$, using (H_2), to given $\varepsilon > 0$ and $t \in J$ there exists $N = N(t,\varepsilon)$ such that for any $n_k \geq N$ we have

$$g(t,z_{n_k}(t)) \subset 0_\varepsilon(g(t,z(t)))$$

where $0_\varepsilon(g(t,z(t)))$ is ε-neighbourhood of the set $g(t,z(t))$. It means that for all $n_k \geq N$ $v_{n_k}(t) \in 0_\varepsilon(g(t,z(t)))$.

Consider the sequence $\{v_{n_k}(t)\}$, $n_k \geq N$. Then (see [2], Corollary V.3.14) it is possible to construct such convex combinations from v_{n_k}, $n_k \geq N$, denote them $g_m(t)$, $m = 1,2,\ldots$ that the sequence $\{g_m(t)\}$ converges to $v(t)$ in $L_1(J)$. Then by Riesz theorem there exists a subsequence $\{g_{m_i}(t)\}$ of $\{g_m(t)\}$ which converges to $v(t)$ a.e. on J. From the convexity of $0_\varepsilon(g(t,z(t)))$ and from the fact that $v_{n_k}(t) \in$ $\in 0_\varepsilon(g(t,z(t)))$ it follows that $g_{m_i}(t) \in 0_\varepsilon(g(t,z(t)))$, $i = 1,2,\ldots$ and, therefore, $v(t) \in \overline{0}_\varepsilon(g(t,z(t)))$. For $\varepsilon \to 0$ we get that $v(t) \in g(t,z(t))$.

Recall that t was a fixed point and that $g(t,z(t))$ was a compact convex subset of R^n.

Thus

$$h(t) = - \int_t^\infty [f(s,z_s + x_s) - f(s,x_s)]ds - \int_t^\infty v(s)ds$$

is well defined and $h(t) \in (Tz)(t)$ for $t \in J$. It follows from the weak convergence of $\{v_{n_k}(t)\}$ to $v(t)$ in $L_1(J)$ that the subsequence $\{h_{n_k}(t)\}$ of the sequence $\{h_n(t)\}$, i.e. for $t \in J$

$$h_{n_k}(t) = - \int_t^\infty [f(s,(z_{n_k})_s + x_s) - f(s,x_s)]ds - \int_t^\infty v_{n_k}(s)ds$$

converges to $h(t)$ a.e. on J. However, the functions $h_{n_k}(t)$ belong to the compact set TB_u. Therefore, there exists a subsequence of the sequence $\{h_{n_k}(t)\}$ which converges to a function $\bar{h}(t)$ uniformly on J. It means that $\bar{h}(t) = h(t) \in (Tz)(t)$ a.e. on J. With this we end the proof of the upper semicontinuity of the operator T.

Consider now B_u. Let be $z(t) \in B_u$, $\xi(t) \in (Tz)(t)$. Then from (*), (F) and γ) from (H_4) we get for $0 < c < \frac{1-\alpha}{2}$ the existence of such $u > 0$ that

$$|\xi(t)| \leq \alpha u + (\rho + u)c < (\alpha + 2c)u < u$$

Thus $\xi(t) \in B_u$ and $TB_u \subset B_u$. We have already proved that TB_u and also $(Tz)(t)$, $z(t) \in B_u$, are compact and, therefore, also closed. From the hypotheses (H_1), (H_3) it follows that $M(z(t))$ is nonempty and convex, therefore, $(Tz)(t)$ is also nonempty and convex. Thus T maps B_u in $cf(B_u)$.

Lemma 5. Let be satisfied (F), (H_1) - (H_4). Let B, B_u be as in Lemma 4. Let $y : R \to R^n$ be a bounded solution of (2) on J. Let $\varphi \in C$ be given. Then the operator T_1 defined on B by the relations: for $z(t) \in B$ it is

$$(T_1 z)(t) = \{-\int_t^{\infty}[f(s,z_s+y_s) - f(s,y_s)]ds + \int_t^{\infty}v(s)ds,\ v(t) \in M(y(t))\},\ t \in J$$

$$(T_1 z)_0 = \{\varphi(t) - \varphi(0) - \int_0^{\infty}[f(s,z_s+y_s) - f(s,y_s)]ds + \int_0^{\infty}v(s)ds\},\ t \le 0$$

maps $B \to 2^B$, is compact and upper semicontinuous in B and there exists such $u \in J$ that T maps B_u in $cf(B_u)$.

The proof of this Lemma can be made in the same way as the proof of Lemma 4.

From Lemma 4 and Lemma 5 follows

Theorem 4. Let be satisfied (F), (H_1)-(H_4). Then between the set of all bounded solutions of (1) and the set of all bounded solutions of (2) there is the asymptotic equivalence. Moreover, if

(11) $\quad tL(t) \in L_1(J),\ tG_0(t,c) \in L_1(J)$ for each $c \ge 0$

then there is p-integral equivalence, $p \ge 1$, between the above mentioned sets of bounded solutions of (1) and of (2).

Proof. Let be $x(t)$ a bounded solution of (1) on J and let $\varphi \in C$ be given. Then by Lemma 4 there exists a ball $B_u \subset B$ such that T maps B_u into $cf(B_u)$, T is upper semicontinuous and TB_u compact. Thus by Fan fixed point theorem T has a fixed point $z(t) \in B_u$, i.e. there exists $v(t) \in M(z(t) + x(t))$ such that

$$z(t) = -\int_t^{\infty}[f(s,z_s+x_s) - f(s,x_s)]ds - \int_t^{\infty}v(s)ds,\quad t \in J$$

$$(z)_0(t) = \varphi(t) - \varphi(0) - \int_0^{\infty}[f(s,z_s+x_s)-f(s,x_s)]ds - \int_0^{\infty}v(s)ds, t\le 0$$

Evidently, $\lim z(t) = \lim(y(t) - x(t)) = 0$ as $t \to \infty$ and $y(t) = x(t) + z(t)$ is a bounded solution of (2). Moreover, if (11) is satisfied, we get

$$|z(t)| \le \sup_{\langle 0,\|z\|\rangle} w(r) \int_t^{\infty} L(s)ds + \int_t^{\infty} G_0(s,\|x\| + \|z\|)ds$$

Thus by Lemma 2 from [3] $z(t) \in L_p(J)$, $p \ge 1$.

Let now $y(t)$ be a bounded solution of (2) and let $\varphi \in C$ be given. Then by Lemma 5 there exists a ball $B_u \subset B$ such that the operator T_1 maps B_u into $cf(B_u)$, T_1 is upper semicontinuous and T_1B_u is compact. Thus Fan fixed point theorem gives the existence of a fixed point of T_1 in B_u, i.e. there exists $v(t) \in M(y(t))$ such that

$$z(t) = -\int_t^{\infty}[f(s,z_s + y_s) - f(s,y_s)]ds + \int_t^{\infty} v(s)ds, \quad t \in J$$

$$(z)_0(t) = \varphi(t) - \varphi(0) - \int_0^{\infty}[f(s,z_s + y_s) - f(s,y_s)]ds + \int_0^{\infty} v(s)ds, \quad t \le 0$$

Evidently, $\lim z(t) = \lim(x(t) - y(t)) = 0$ as $t \to \infty$ and $x(t) = y(t) + z(t)$ is a bounded solution of (1). Moreover, if (11) is satisfied, then

$$|z(t)| \le \sup_{\langle 0,\|z\|\rangle} w(r) \int_t^{\infty} L(s)ds + \int_t^{\infty} G_0(s,\|y\|)ds$$

which by Lemma 2 from [3] means that $z(t) \in L_p(J)$, $p \ge 1$.

References

[1] AUBIN,J.P., CELLINA,A., *Differential Inclusions*, Springer Verlag, 1984, A Series of Comprehensive Studies in Mathematics.

[2] DUNFORD,N., SCHWARTZ,J.T., *Linear Operators*, General Theory, 1958, Interscience Publishers, New York, London.

[3] HAŠČÁK,A., ŠVEC,M., *Integral equivalence of two systems of differential equations*, Czech. Math. J., 32 (107), 1982, 423-436.