SHAPE OPTIMIZATION IN CONTACT PROBLEMS

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Section D

Introduction

Optimization of design is a fundamental goal of every engineer who strives to create a component, device or system to meet a need. Designers seek to evolve the "best" design in terms of weight, costs, reliability.This objective is very important especially nowadays because of resource scarcity, increasing costs of materials, energy, etc.

As the analytical solutions of such problems are not available in general,the development of efficient numerical methods for practical applications was necessary.An important progress was made in last 20 years together with development of finite element method and mathematical programming techniques with special reference to the structural design problems.A great number of papers has been already written on this topic.Representative survey of latter results can be found in [1]. Optimal shape design from mathematical point of view has been widely studied by french school of applied mathematics ([2],[3]). In most of papers the case when the physical system is described by elliptic equations is presented.On the other hand there is a lot of problems in mechanics,whose variational forms are inequalities ([4],[5]) so that the study of optimal shapes and their approximations for this kind of state problem is natural.

The present contribution is concerned with the existence and the approximation of a solution to the contour design problem for a planar deformable body unilaterally supported by a rigid foundation.We shall discuss the following physical situations:

1° the body is elastic, no friction;

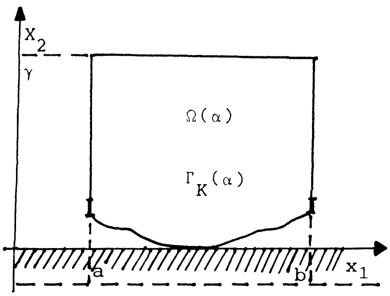
2° the body is elastic, the influence of friction between it and the foundation is taken into account;

3° the body is elastic-perfectly plastic,obeying Henky's law, no friction.

In all these cases the total potential energy of the body will be minimized. In the engineering literature,design problems for elastic bodies in contact are discussed ([6],[7]) but a rigorous analysis of these problems remains to be undertaken.

1. Setting of the problem

First we shall discuss 1° and 2°. Let us assume a plane elastic body $\Omega \subset R^2$ (see fig.1) unilaterally supported by a rigid foundation $\{(x_1,x_2)\,|\,x_2 \le 0\}$ and subjected to a body force $F = (F_1, F_2)$ and to surface tractions $P = (P_1,P_2)$ on a portion of the boundary.



Let the boundary $\partial\Omega$ be decomposed as follows

$$\partial\Omega = \overline{\Gamma}_u \cup \overline{\Gamma}_P \cup \overline{\Gamma}_C(\alpha) ,$$

with Γ_u, Γ_P and $\Gamma_C(\alpha)$ non-overlapping parts of $\partial\Omega$, Γ_u and $\Gamma_C(\alpha)$ non-empty. Contact part $\overline{\Gamma}_C(\alpha)$ will be given by the graph of a function $\alpha \in U_{ad}$ (will be specified later). As Ω depends on α, we shall write $\Omega(\alpha)$ in what follows. In order to give variational formulation of contact problems we introduce a Hilbert space $V(\alpha)$ of virtual displacements:

$$V(\alpha) = \{v = (v_1,v_2) \in (H^1(\Omega(\alpha)))^2\,|\,v_i = 0 \text{ on } \Gamma_u,\ i = 1,2\}$$

and its closed convex subset $K(\alpha)$ of admissible displacements:

$$K(\alpha) = \{v \in V(\alpha)\,|\,v_2(x_1,\alpha(x_1)) \ge -\alpha(x_1)\ \forall\, x_1 \in \langle a,b\rangle\} .$$

The total potential energy functional $J_\alpha : V(\alpha) \to R^1$ is now given by

$$(1.1)\qquad J_\alpha(v) = 1/2\langle \tau(v),\varepsilon(v)\rangle_\alpha - \langle L,v\rangle_\alpha + j_{\alpha,g}(v) ,$$

where

$$\langle \tau(v),\varepsilon(v)\rangle_\alpha = \int_{\Omega(\alpha)} F_i v_i\,dx + \int_{\Gamma_P} P_i v_i\,ds$$

with $F \in (L^2(\hat{\Omega}))^2$, $\hat{\Omega} \supset \Omega(\alpha)$, $P \in (L^2(\hat{\Gamma}_P))^2$, $\hat{\Gamma}_P \supset \Gamma_P$ $\forall\alpha \in U_{ad}$ and

$$j_{\alpha,g}(v) = g \int_{\Gamma_C(\alpha)} |v_1|\,ds ,$$

where $g \ge 0$ is a given constant, $g = 0$ for frictionless case. Above the standard notations and conventions of elasticity are used.

By a *variational formulation* of the contact problem (with or without friction) we mean the problem

$$(P(\alpha))\ \begin{cases} \text{find } u(\alpha) \in K(\alpha) \text{ such that} \\ J_\alpha(u(\alpha)) \le J_\alpha(v) \quad \forall v \in K(\alpha) . \end{cases}$$

Let

$$U^1_{ad} = \{\alpha \in C^{0,1}(\langle a,b\rangle)\,|\,0 \le \alpha(x_1) \le C_0,\ |\alpha'(x_1)| \le C_1,\ \forall x_1 \in \langle a,b\rangle ,\ \text{meas}\ \Omega(\alpha) = C_2\}$$

if $g = 0$ and

$$U^2_{ad} = \{\alpha \in C^{1,1}(\langle a,b\rangle) \mid 0 \le \alpha(x_1) \le C_0, |\alpha'(x_1)| \le C_1, |\alpha''(x_1)| \le C_2, \forall x_1 \in \langle a,b\rangle, \text{ meas } \Omega(\alpha) = C_3\}$$

if $g > 0$. $C_0, \ldots, C_3$ are positive constants chosen in such a way that $U^j_{ad} \neq \phi$, $j = 1,2$. Now the shape optimization problem reads as follows:

$$(Pj) \begin{cases} \text{find } \alpha^* \in U^j_{ad} \text{ such that} \\ E(\alpha^*) \le E(\alpha) \ \forall \alpha \in U^j_{ad}, \ j = 1,2 , \end{cases}$$

where

$$E(\alpha) = J_\alpha(u(\alpha))$$

and $u(\alpha) \in K(\alpha)$ solves $(P(\alpha))$ on $\Omega(\alpha)$. As the existence of a solution of (Pj), $j = 1,2$ is concerned, it holds

Theorem 1.1 There exists at least one solution α^* of (Pj), $j = 1,2$. For the proof we refer to [8] if $j = 1$ and [9] if $j = 2$.

Let us pass to 3°. We shall assume that the material of the deformable body is elasto-perfectly plastic, obeying Henky's law. It is known that the formulation in terms of stresses is more suitable that in displacements. This motivates the definition of optimal shape design problem.

Let $U_{ad} = U^1_{ad}$ and $\alpha \in U_{ad}$. $K(\alpha)$ will have the same meaning as before. We introduce the following notations:

$$S(\alpha) = \{\tau \in (L^2(\Omega(\alpha)))^4 \mid \tau_{ij} = \tau_{ji} \text{ a.e. in } \Omega(\alpha), \langle \tau, \varepsilon(v)\rangle_\alpha \ge \langle L,v\rangle_\alpha \ \forall v \in K(\alpha)\},$$

$$B(\alpha) = \{\tau \in (L^2(\Omega(\alpha)))^4 \mid f(\tau(x)) \le 1 \text{ a.e. in } \Omega(\alpha)\}$$

and

$$S_\alpha(\tau) = 1/2 \|\tau\|^2_{0,\Omega(\alpha)} ,$$

where $f : R^4 \to R^1$ is convex and continuous,

$$\|\tau\|^2_{0,\Omega(\alpha)} = (\tau,\tau)_{\Omega(\alpha)} = \int_{\Omega(\alpha)} c_{ijkl}\tau_{ij}\tau_{kl}\,dx .$$

c_{ijkl} are coefficients of inverse Hooke's law.

Let $\tilde{U}_{ad} \subseteq U_{ad}$ be the set of all α, satisfying

$$\alpha \in \tilde{U}_{ad} \text{ iff } S(\alpha) \cap B(\alpha) \neq \phi .$$

Next we shall suppose that $\tilde{U}_{ad} \neq \phi$.

Shape optimization problem for *elasto-perfectly plastic* body is defined as follows:

$$(P3) \begin{cases} \text{find } \alpha^* \in \tilde{U}_{ad} \text{ such that} \\ E(\alpha^*) \le E(\alpha) \quad \forall \alpha \in \tilde{U}_{ad} , \end{cases}$$

where $E(\alpha) = S_\alpha(\sigma(\alpha))$ and $\sigma(\alpha)$ is a solution of

$$(P(\alpha))_{pl} \quad \sigma(\alpha) \in S(\alpha) \cap B(\alpha) : S_\alpha(\sigma(\alpha)) \le S_\alpha(\tau) \ \forall \tau \in S(\alpha) \cap B(\alpha).$$

It holds:

Theorem 1.2 Let $\tilde{U}_{ad} \neq \phi$. Then (P3) has at least one solution α^*. For the proof see [10].

2. Approximation of optimal shape design problems

Here we deal with the approximation of (P1) only. Let $a = a_0 < a_1 < \ldots < a_N = b$ be a partition of $\langle a,b\rangle$ and

$$U_{ad}^{1h} = \{\alpha \in C(\langle a,b\rangle) \,|\, \alpha|_{a_{i-1}a_i} \in P_1(\langle a_{i-1}a_i\rangle)\} \cap U_{ad}^1 ,$$

where P_1 denotes the set of linear functions. For any $\alpha_h \in U_{ad}^{1h}$ we define

$$\Omega(\alpha_h) = \{(x_1,x_2) \in R^2 |\ a < x_1 < b,\ \alpha_h(x_1) < x_2 < \gamma\} ,$$

i.e. the variable contact part of the boundary is now approximated by a piecewise linear arc $\Gamma_C(\alpha_h)$. By $T_h(\alpha_h)$, $\alpha_h \in U_{ad}^{1h}$ we denote the triangulation of $\Omega(\alpha_h)$. Finally let

$$K_h(\alpha_h) = \{v_h \in (C(\overline{\Omega(\alpha_h)}))^2 | v_h|_{T_i} \in (P_1(T_i))^2 \ \forall T_i \in T_h(\alpha_h) ,$$

$$v_h = 0 \text{ on } \Gamma_u,\ v_{2h}(a_i,\alpha_h(a_i)) \ge -\alpha_h(a_i) \ \forall i\} .$$

The approximation of (P1) is now defined as follows:

$$(P1)_h \begin{cases} \text{find } \alpha_h^* \in U_{ad}^{1h} \text{ such that} \\ E_h(\alpha_h^*) \le E_h(\alpha_h) \ \forall \alpha_h \in U_{ad}^{1h} , \end{cases}$$

where $E_h(\alpha_h) = J_{\alpha_h}(u_h(\alpha_h))$ and $u_h(\alpha_h) \in K_h(\alpha_h)$ is usual Ritz-Galerkin approximation of $(P(\alpha_h))$ on $K_h(\alpha_h)$. Under certain regularity assumptions, concerning the family $\{T_h(\alpha_h)\}$, $h \to 0+$, $\alpha_h \in U_{ad}^{1h}$, the following relation between (P1) and $(P1)_h$ can be established:

Theorem 2.1 Let $\alpha_h^* \in U_{ad}^{1h}$ be a solution of $(P1)_h$ and let $u_h^*(\alpha_h^*)$ solves the approximate state problem on $\Omega(\alpha_h^*)$. Then there exists a subsequence $\{\alpha_{h_j}^*\} \subset \{\alpha_h^*\}$, an element $\alpha^* \in U_{ad}^1$ and $u^*(\alpha^*)$ such that

$$\alpha_{h_j}^* \rightrightarrows \alpha^* \text{ (uniformly) in } C^0(\langle a,b\rangle) \text{ for } h_j \to 0+$$

$$u^*_{h_j}(\alpha^*_{h_j}) \rightarrow u^*(\alpha^*) \text{ (weakly) in } (H^1(G_m(\alpha^*))^2$$

for $h_j \rightarrow 0+$ and for any $m \geq m_0$, where α^* is a solution of (P1), $u^*(\alpha^*) \in K(\alpha^*)$ is the corresponding state on $\Omega(\alpha^*)$ and

$$G_m(\alpha^*) = \{(x_1,x_2) \in R^2 \mid x_1 \in \langle a,b \rangle,\ \alpha^*(x_1) + 1/m < x_2 < \gamma\} .$$

For the proof see [11].

Sensitivity analysis for (P1),(P2),(P3) as well as numerical results can be found in [11], [9] and [10], respectively.

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