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# ACTIVE SCALARS AND THE EULER EQUATIONS

PETER CONSTANTIN

ABSTRACT. We describe geometric and analytic constraints for blow up in inviscid, incompressible fluid equations.

## 1. Introduction

The three dimensional incompressible Euler equations can be written as a quadratic evolution equation for a divergence-free vector, the vorticity $\omega$. Because of a quadratic stretching mechanism some solutions of the Euler equation could, in principle, evolve from smooth initial data to a singularity in finite time. The numerical evidence ([1], [2], [3], [4], [5]) is suggestive, but inconclusive. An important result ([6]) states that no singularities can occur before the magnitude of the vorticity becomes infinite. In the two dimensional case the vorticity vector does not change direction and its magnitude does not grow — no singularities can form from smooth initial data. In three dimensions, the magnitude $|\omega|$ obeys

$$D_t\,|\omega| = \alpha|\omega|$$

($D_t$ is the material derivative — time derivative along particle trajectories). The stretching factor $\alpha$ has an integral representation ([7], [8]):

$$\alpha(x) = \frac{3}{4\pi} P{\cdot}V{\cdot} \int D(\hat{y}, \xi(x+y), \xi(x))\,|\omega(x+y)|\,\frac{dy}{|y|^3}$$

in terms of the magnitude of vorticity. The geometric integrand $D$ depends on the direction of vorticity $\xi$

$$\xi(x,t) := \frac{\omega(x,t)}{|\omega(x,t)|}\,,$$

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$\hat{y} = \dfrac{y}{|y|}$, and is given by

$$D(e_1, e_2, e_3) = (e_1 \cdot e_3)\big(\mathrm{Det}(e_1, e_2, e_3)\big)\,.$$

For any three unit vectors $e_1, e_2, e_3$. $\mathrm{Det}(\cdots)$ means the determinant of the matrix with columns $\cdots$. It is easy to show that the contribution to $\alpha$ coming from $|y| > L$ are bounded uniformly *a priori* in terms of the kinetic energy of the initial velocity and $L$. Thus blow-up is decided by the local interactions. The importance of the formula lies in the fact that the geometric integrand $D$ vanishes identically if any of the vectors in it are parallel or anti-parallel. Therefore local alignment (or anti-alignment) of vorticity depletes the nonlinearity. This is a generalization of the two dimensional situation where there is no stretching of vorticity. In three dimensions, the more coherent the vorticity field, the more stringent are the conditions for blow-up. The depletion of nonlinearity due to alignment can be used to devise tests for blow-up which take into account geometric information ([9]).

Active scalars are solutions of certain nonlinear advection — diffusion equations. There is a great variety of such equations — enough to provide prototypical examples for much of two dimensional incompressible fluid mechanics. Active scalars provide a convenient class of examples for the inviscid generation of small scales ([7], [8]). One particular such active scalar equation ([10], [11]) has been investigated numerically and shown ([11], [12]) to exhibit sharp fronts, suggesting finite time singularities. In this paper we explain the analogies between the three dimensional incompressible Euler equations and active scalars.

## 2. The Euler Equations

The Euler equations can be written as equations of evolution for $\omega$, the vorticity of a three dimensional incompressible, inviscid fluid:

$$(\partial_t + u \cdot \nabla)\,\omega = S\omega\,. \tag{1}$$

The divergence-free velocity of the fluid $u$ is determined by omega through the Biot–Savart law:

$$u(x) = -\frac{1}{4\pi}\int \left(\nabla \frac{1}{|y|}\right) \times \omega(x+y)\,dy\,.$$

(The integral is over the whole three dimensional space.) The strain matrix $S$, which is the symmetric part of the gradient of velocity, is given in terms of omega by:

$$S(x) = \frac{3}{4\pi} P{\cdot}V{\cdot} \int M\big(\hat{y}, \omega(x+y)\big)\frac{dy}{|y|^3}\,. \tag{2}$$

In (2)

$$\hat{y} = \frac{y}{|y|}$$

and the matrix $M$ is a function of two variables, the first a unit vector, the second a vector, and is given by the formula:

$$M(\hat{y}, \omega) = \frac{1}{2}\left[\, \hat{y} \otimes (\hat{y} \times \omega) + (\hat{y} \times \omega) \otimes \hat{y} \,\right].$$

The matrix $M$ is traceless and symmetric; its mean on the unit sphere is zero when the second variable $\omega$ is held fixed and $M$ is viewed as a function of $\hat{y}$ alone.

The strain matrix and the vorticity balance each other in $L^2$. More precisely, the gradient matrix which can be decomposed as

$$\nabla u = S + \frac{1}{2}\omega \times \; .$$

satisfies

$$\int |\nabla u|^2 dx = \int |\omega|^2 dx = 2 \int \operatorname{Tr} S^2 dx\,.$$

The right hand side of (1) is quadratic in $\omega$ and, in view of the balance of the $L^2$ norms of $S$ and $\omega$ shown above, it seems possible that finite time singularities might occur in the vorticity. Moreover, by dimensional analysis, $\omega \sim \dfrac{1}{T-t}$. The basic Beale–Kato–Majda result [6] states that no singularities can occur in the solution of (1) before the time integral of the maximum modulus of vorticity diverges: If

$$\int_0^T \big\|\omega(\cdot, t)\big\|_{L^\infty} dt < \infty\,,$$

and if the initial vorticity is smooth and localized, then so is the solution up to time $T$. Thus, the vorticity itself needs to become infinite, at a fast enough rate. Then the $L^\infty$ norm of the vorticity defines a frequency: it is a maximal instantaneous rate of rotation of the fluid particles. The Euler equations conserve kinetic energy:

$$\int \big|u(x,t)\big|^2 dx = \int \big|u(x,0)\big|^2 dx\,.$$

The vorticity defines length scales, in a natural fashion. Denote by $|\omega|_\mu$ the Hölder seminorm

$$|\omega|_\mu := \sup_{0<|x-y|<L} \frac{\big|\omega(x) - \omega(y)\big|}{|x-y|^\mu}$$

for $0 < \mu \leq 1$. We can associate to this seminorm in a natural way a quantity with dimension of length given by

$$\ell_\mu(t) := \min\left\{L\,;\ \left(\frac{|\omega(\cdot,t)|_\mu}{\|u(\cdot,t)\|_{L^2}}\right)^{\frac{-2}{2\mu+5}}\right\}.$$

The $L$ appearing in the definitions of $|\omega|_\mu$ and $\ell_\mu$ is a fixed reference length scale. One can prove the following

**THEOREM 2.1.** *Assume that the initial vorticity $\omega_0$ is smooth and compactly supported. Then the solution of the Euler equation corresponding to $\omega_0$ is smooth ($C_0^\infty$) on the time interval $0 \leq t \leq T$ if and only if*

$$\int_0^T \left(\ell_\mu(t)\right)^{-\frac{5}{2}} dt \ < \ \infty$$

*holds for some $\mu$, $0 < \mu \leq 1$.*

Therefore blow-up cannot occur in omega without the development of infinite spatial gradients, or in other words, without the formation of small scales. Thus, if blow-up occurs, then the vorticity magnitude must become infinite fast enough and the gradient of vorticity must grow fast, too. In order to go farther we study the stretching rate $\alpha$

$$\alpha(x) := S(x)\xi(x)\cdot\xi(x) \tag{3}$$

and the direction of the vorticity $\xi$:

$$\xi(x) := \frac{\omega(x)}{|\omega(x)|}\,. \tag{4}$$

The rotational region $\{x : |\omega(x)| > 0\}$ is material (carried by the fluid). Both $\alpha$ and $\xi$ are defined in it. They play a crucial role; $\alpha$ is simply the material derivative of the logarithm of the vorticity magnitude:

$$(\partial_t + u\cdot\nabla)|\omega| = \alpha\,|\omega|\,. \tag{5}$$

From (2) we deduce the integral representation:

$$\alpha(x) = \frac{3}{4\pi}P{\cdot}V{\cdot}\int D\big(\hat{y},\xi(x+y),\xi(x)\big)\,\big|\omega(x+y)\big|\frac{dy}{|y|^3}\,, \tag{6}$$

where $D$ is

$$D(e_1,e_2,e_3) = (e_1\cdot e_3)\big(\mathrm{Det}(e_1,e_2,e_3)\big) \tag{7}$$

The Det in $D$ is the determinant of the matrix whose columns are the three unit column vectors $e_1, e_2$ and $e_3$. In (6) $D$ is computed with $e_1 = \hat{y}$, $e_2 = \xi(x+y)$ and $e_3 = \xi(x)$. The geometric significance of $D$ is clear: it is proportional to the volume of the prism of edges equal to $\hat{y}$, $\xi(x+y)$, $\xi(x)$. In particular it depends on $\xi(x+y)$ only through $P^{\perp}_{\xi(x)}\xi(x+y)$, the projection of $\xi(x+y)$ orthogonal to $\xi(x)$. Thus

$$\left|D(\hat{y}, \xi(x+y), \xi(x))\right| \leq \left|P^{\perp}_{\xi(x)}\xi(x+y)\right|. \tag{8}$$

If $\cos\phi = \xi(x+y)\cdot\xi(x)$ then the inequality above is simply

$$|D| \leq |\sin\phi|.$$

Note that $|\sin\phi| \leq 2|\sin(\frac{\phi}{2})| = |\xi(x+y) - \xi(x)|$. It is important to make the distinction between the half-angle and the angle: obviously $D = 0$ if $\xi(x+y) = -\xi(x)$ but $|\xi(x+y) - \xi(x)| = 2$ in that case. Thus the integrand (7) vanishes identically not only in the parallel but also in the antiparallel case, a fact of physical importance. It is clear that

$$\int_0^T \|\alpha(\cdot, t)\|_{L^\infty} dt < \infty \tag{9}$$

is sufficient for regularity on the time interval $[0, T]$. (Because of the Beale–Kato–Majda result and (5).) It is also obvious that $|\alpha(x)| \leq \sqrt{\mathrm{Tr}\, S^2(x)}$. Now we will investigate the effect of geometry on the stretching factor $\alpha$. Let us consider a situation in which

$$\left|P^{\perp}_{\xi(x)}(\xi(x+y))\right| \leq \frac{|y|}{R} \tag{10}$$

for $|y| \leq L$, $|\omega| \geq M > 0$. This can happen even if the function $\xi$ is *not* locally Lipschitz, for instance if two antiparallel vortex lines osculate. The inequality is assumed if both $x$ and $x+y$ are in the rotational region and the values of the vorticity exceeds a reference value $M$; $R = R(t)$ may be a function of time, $L$ is a fixed length unit. (I avoid taking $L = 1$ so that the statements will be dimensionally balanced.) Let us denote by

$$\sup_x \frac{1}{L} \int_{|y|\leq L} \left|\omega(x+y)\right| \frac{dy}{|y|^2} := \Omega_3(t).$$

Here is an example of a statement which can be proved exploiting the inequality (10):

**THEOREM 2.2.** *Assume that the initial vorticity $\omega_0$ is smooth and compactly supported. Assume that the solution of the Euler equation satisfies*

$$\int_0^T \Omega_3(t) \frac{L}{R(t)}\, dt < \infty.$$

*Then the solution is smooth on for $0 \le t \le T$.*

As a corollary we have that, if $R$ is bounded below, then instead the $L^\infty$ blow-up condition we have a much weaker Morrey–Campanato condition.

If we assume that the vorticity direction is locally Lipschitz continuous in regions where the vorticity magnitude exceeds a reference value — a stronger assumption than (10) — i.e., that

$$\left|\xi(x+y) - \xi(x)\right| \le \frac{|y|}{\rho} \tag{11}$$

for $|y| \le L$, $|\omega| \ge M$, then one can relax the assumption on $|\omega|$. We denote by

$$U(t) := \sup_x \left|u(x,t)\right|,$$

the $L^\infty$ norm of the velocity, and by

$$N_1(t) := L^{-3} \sup_x \int_{|y|\le L} \left|\omega(x+y)\right| dy$$

the $L^1_{\text{loc}}$ norm of $\omega$. One can prove:

**THEOREM 2.3.** *Assume that the initial vorticity, $\omega_0$ is smooth and compactly supported. Assume that the corresponding solution of the Euler satisfies:*

$$\int_0^T N_1(s)\left(\frac{L}{\rho(s)}\right)^3 ds < \infty,$$

*and*

$$\int_0^T \frac{U(s)}{\rho(s)}\, ds < \infty.$$

*Then*

$$\sup_{0\le t\le T} \frac{\left\|\omega(\cdot,t)\right\|_{L^\infty}}{N_1(t)} < \infty.$$

In particular, if the velocity and the $L^1_{\text{loc}}$ norm of vorticity are bounded on the time interval $[0,T]$ and $\rho$ is bounded away from $0$, then no singularities can arise. The proofs and other related blow-up tests will be presented in [9].

One can prove a full array of similar results. They interpolate between the Beale–Kato–Majda result, which requires the highest norm of vorticity magnitude but no assumption on the vorticity direction and the last result which requires the highest norm on the vorticity direction but only $L^1_{\text{loc}}$ norm of the magnitude. One can prove also results involving Hölder norms of the direction, so that one can evaluate the likelihood of cusps in vortex lines. All these results have the common feature that the more one is willing to assume regarding the geometric structure of the vorticity direction in regions of intense vorticity, the more stringent is the condition that the vorticity magnitude must satisfy to blow-up. The results above seem to indicate that there is a direct relationship between the complexity of the vortex line field and the likelihood of inviscid blow-up. From the discussion above we see that the direction of vorticity is very important. Its time evolution is given by

$$D_t\xi = P_\xi^\perp(S\xi), \tag{12}$$

where we denoted

$$D_t := \partial_t + u\cdot\nabla \tag{13}$$

the convective derivative, and

$$P_\xi^\perp(S\xi) = (S-\alpha)\xi\,. \tag{14}$$

Consider a Frenet moving frame formed with the unit vectors

$$\xi_1 = \xi\,,$$
$$\xi_2 = \frac{\xi\cdot\nabla\xi}{|\xi\cdot\nabla\xi|},$$

and

$$\xi_3 = \xi_1 \times \xi_2\,.$$

The Frenet equations are

$$\xi\cdot\nabla\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} = \begin{pmatrix}0 & \kappa_1 & 0\\ -\kappa_1 & 0 & \kappa_2\\ 0 & -\kappa_2 & 0\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}$$

where $\kappa_1$ is the curvature of the vortex line

$$\kappa_1 = |\xi\cdot\nabla\xi_1|$$

and $\kappa_2$ the torsion. (Note that $\xi\cdot\nabla$ is the derivative with respect to arclength along the vortex line.) Alignment of the direction of vorticity with any eigenvector of the strain matrix is equivalent to $\kappa_1 = 0$. This implies, at that instance of time, that the direction of vorticity satisfies $D_t(\xi_1) = 0$. The time evolution of the Frenet frame is computed using the important commutation relation

$$[D_t\,,\,\xi\cdot\nabla] = -\alpha\xi\cdot\nabla \tag{15}$$

which is a consequence of the fact that vortex lines are material. The evolution of the Frenet frame is given by:

$$D_t\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} = \begin{pmatrix}0 & \beta & \gamma\\ -\beta & 0 & \sigma\\ -\gamma & -\sigma & 0\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}$$

where

$$\beta := S\xi_1 \cdot \xi_2\,, \tag{16}$$

$$\gamma := S\xi_1 \cdot \xi_3\,, \tag{17}$$

and

$$\sigma := \frac{1}{\kappa_1}(\kappa_2\beta + \xi \cdot \nabla\gamma)\,. \tag{18}$$

Finally, using the Frenet equations, their evolution equations and the commutation relation (15) one obtains the evolution equation for the curvature and torsion

$$D_t\kappa = -\alpha\kappa + \gamma\kappa^{\perp} + \xi \cdot \nabla\nu\,, \tag{19}$$

where

$$\kappa = \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix},$$

$$\kappa^{\perp} = \begin{pmatrix} -\kappa_2 \\ \kappa_1 \end{pmatrix},$$

and

$$\nu = \begin{pmatrix} \beta \\ \sigma \end{pmatrix}.$$

The minus sign in (19) is remarkable: it describes a mechanism of straightening of the vortex line, as it stretches because of (5). For instance, if

$$\xi \cdot \nabla\hat{\kappa} = 0\,,$$

where

$$\hat{\kappa} = \frac{\kappa}{|\kappa|}\,,$$

then

$$\oint |\kappa|\, ds = \text{constant}\,.$$

The condition $\xi \cdot \nabla\hat{\kappa} = 0$ is invariant in time, i.e., if verified initially, it holds for all time. It is equivalent to the requirement that, along the vortex line

$$\kappa_2 = \lambda\kappa_1$$

with $\lambda$ constant. The fact that the integral is conserved in time is a statement of average diminishing of curvature at the same rate as the arclength element $|\omega|$ of the vortex line increases.

## 3. Active Scalars

Active scalars are solutions of

$$\partial_t \theta + u \cdot \nabla \theta = 0, \tag{20}$$

where:

$$u = \nabla^{\perp} \psi, \tag{21}$$

and

$$\psi = A(\theta), \tag{22}$$

In (21) the symbol $\nabla^{\perp}$ represents the curl operator. In this section the spatial dimension is two, so that

$$\nabla^{\perp} = J \nabla, \tag{23}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In this paper we consider a simple linear operator of the type

$$A(\theta)(x) = \int a(x-y)\,\theta(y)\,dy, \tag{24}$$

where the function $a$ is smooth away from the origin. The strength of the singularity at the origin is important. If the function $a$ is not too singular at the origin, say

$$|x|^{\delta}|a(x)| \leq C$$

for small $|x|$ and $0 \leq \delta \leq 1$, then the equation (20) is well posed in appropriate Sobolev spaces.

Perhaps the most important and familiar example of a two dimensional, incompressible system which can be formulated as an active scalar is that of the Euler equations. The scalar is $\theta = \omega$ with $\omega$ the vorticity; in this case $a(x) = \frac{1}{2\pi}\log(|x|)$. There are quite a number of physically significant active scalar equations. We mention only two other examples: the infinite Prandtl number convection, [13], and a quasi-geostrophic model ([10], [11]) (corresponding to $a = 1/r$ in (24)). These examples illustrate three prototypical behaviors, exponential growth, superexponential growth and blow-up (the last not proven). What determines whether blow-up occurs or not? The best way to understand this is by looking at the equation obeyed by $\nabla^{\perp}\theta$:

$$\left(\partial_t + u \cdot \nabla\right) \nabla^{\perp}\theta = (\nabla u)\nabla^{\perp}\theta. \tag{25}$$

This equation expresses the fact that the tangent vector to the iso-$\theta$ lines is stretched by the strain matrix

$$S(x) = \frac{1}{2}\left(\nabla u + (\nabla u)^{*}\right)$$

and rotated by the the antisymmetric part of the gradient

$$\frac{1}{2}\left(\nabla u - (\nabla u)^*\right) = \frac{1}{2}\omega J\,.$$

Equation (25) is very similar to the equation obeyed by the vorticity in the three dimensional Euler equation.

The three prototypes can be easily identified in the context of the simplest equation of state (22) – (24). The infinite Prandtl number convection corresponds to $a$ which satisfies

$$|x|^{2+\delta}\left|\nabla\nabla a(x)\right| \le C \tag{26}$$

for small $|x|$ with $\delta = -1$. For the Euler equations $\delta = 0$; the quasi-geostrophic model corresponds to the case $\delta = 1$. If $a$ is homogeneous of order $-\delta$ in a neighborhood of the origin and if $\delta > 0$ then dimensional analysis predicts blow up in finite time. Dimensional analysis may fail to predict correctly if it does not take into account additional structure. Take, for instance

$$a(x_1, x_2) = \frac{1}{x_1}\,.$$

Then not only is there no blow up, but one can integrate (20):

$$\theta(x_1, x_2, t) = \theta_0\big(x_1, x_2 - v(x_1)\,t\big)\,,$$

where $\theta_0$ is the initial $\theta$ and

$$v(x_1) = \int_{-\infty}^{\infty} |\partial_1|\,\theta_0(x_1, y)\,dy\,.$$

The operator $|\partial_1|$ is that of multiplication by $|k_1|$ in Fourier representation.

If (26) holds for negative $\delta$ then the prediction based on dimensional analysis is that of global existence. In this case the prediction is correct: one can prove that the equation has global solutions. The gradient of $\theta$ grows at most exponentially. In all these models, if the spatial $L^\infty$ norm of the gradient of the scalar is time integrable, then it is actually locally bounded in time and so are all higher derivatives. This analogue of the well-known Beale–Kato–Majda estimate [6] can be proved using calculus inequalities. The class of examples (20) – (24) admits a simple Lagrangian description. If

$$q \mapsto X(q, t)$$

denotes the position at time $t$ of a particle which was initially at $q$ then the the diffeomorphism $X$ obeys an equation in function space

$$\frac{d}{dt}X = U_{\theta_0}(X)\,,$$

where the functional $U_{\theta_0}$ is given by

$$U_{\theta_0}(X)(q) = \int a\big(X(q) - X(q')\big)\big\{\theta_0(q'); X(q')\big\}dq'$$

and

$$\{\theta_0; X\} = (J\nabla\theta_0) \cdot \check{\nabla} X\,.$$

Let us return to the case $\delta > 0$, with, for instance $a = 1/r$ and apply in the present context the geometric considerations of the preceding section. The analogue of the vorticity is $\nabla^\perp\theta$; the analogue of the direction of vorticity is

$$\xi = \frac{\nabla^\perp\theta}{|\nabla^\perp\theta|}\,,$$

and the analogue of the stretching factor $\alpha$ is

$$\alpha = \big((\nabla u)\xi\big) \cdot \xi\,.$$

The analogue of (6) follows from (22)–(24):

$$\alpha(x) = P{\cdot}V{\cdot}\int \big(\xi(x)\cdot\hat{y}\big)\big(\xi(x)\cdot\xi(x+y)\big)|\nabla^\perp\theta(x+y)|\frac{dy}{|y|^2}\,. \tag{27}$$

The direction $\xi$ evolves according to

$$D_t\xi = \beta\xi^\perp,$$

where $\xi^\perp = J\xi$ and

$$\beta = (\nabla u)\xi \cdot \xi^\perp,$$

which in view of (22)–(24) implies

$$\beta(x) = P{\cdot}V{\cdot}\int \big(\xi(x)\cdot\hat{y}\big)\big(\xi^\perp(x)\cdot\xi(x+y)\big)|\nabla^\perp\theta(x+y)|\frac{dy}{|y|^2}\,. \tag{28}$$

The Frenet frame equations are

$$\xi\cdot\nabla\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = -\kappa J\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix},$$

where $\kappa$ is the curvature of a level set

$$\kappa = -\big(\xi\cdot\nabla\xi\big)\cdot\xi^\perp$$

and

$$\xi_1 = \xi, \qquad \xi_2 = -\xi^\perp.$$

The Frenet frame evolves according to

$$D_t \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \beta J \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

The commutation relation

$$[D_t, \xi \cdot \nabla] = -\alpha \xi \cdot \nabla \tag{29}$$

holds. Consequently, the time evolution of the curvature is

$$D_t \kappa = -\alpha \kappa - \xi \cdot \nabla \beta .$$

Note the minus sign. When the magnitude $|\nabla^\perp \theta|$ grows $\alpha$ is positive; when $\alpha$ is positive the curvature decreases: the quantity $\kappa |\nabla^\perp \theta|$ obeys a conservation law. Its integral along any closed level set is conserved:

$$\frac{d}{dt} \oint \kappa \, ds = 0 .$$

This follows from

$$D_t\big(\kappa |\nabla^\perp \theta|\big) = -\nabla^\perp \theta \cdot \nabla \beta .$$

The integral of $\kappa |\nabla^\perp \theta|$ on a closed level set of $\theta$ is the rotation number of that level set. The equation above just verifies that this rotation number does not change in time during a smooth evolution. If we integrate the same quantity on a planar region bounded by level sets we obtain also a constant of motion; it is the "sum" of the rotation numbers of the level sets contained in the region:

$$\frac{d}{dt} \int_{\{x; c_1 \le \theta(x,t) \le c_2\}} \kappa |\nabla^\perp \theta| dx = 0 .$$

Although the sign of $\kappa$ is not constant, the conservation of the average product $\kappa |\nabla^\perp \theta|$ indicates a straightening effect, if the arclength blow-up occurs. Numerical evidence ([11]) for the model $a = 1/r$ shows the formation of sharp straight interfaces, consistent with our present theoretical knowledge. Details and proofs of these results will be presented in [12].

## 4. Analogies

We will summarize here the analogies between active scalars ($a = 1/r$ model) and incompressible ideal fluids. In both cases the equation is an advection—stretching equation for a divergence-free function, the "vorticity":

$$D_t \omega = (\nabla u)\, \omega, \quad \text{respectively,} \quad D_t(\tau) = (\nabla u)\tau, \quad \text{with} \quad \tau = \nabla^\perp \theta ,$$

where

$$D_t = u \cdot \nabla \,.$$

The adverting divergence-free velocity $u$ is obtained from the "vorticity" via an integral with a singularity of order $n-1$ where $n$ is the dimension of space. Dimensional analysis suggests that singularities might form in finite time. A Beale–Kato–Majda type result guarantees that singularities can occur only if the magnitude of the "vorticity" diverges. The magnitude $A$ obeys an evolution equation

$$D_t A = \alpha A \,,$$

where

$$A = |\omega|, \quad \text{respectively,} \quad A = |\tau| \,,$$

$$\alpha = \big((\nabla u)\xi\big) \cdot \xi \,,$$

and where $\xi$ is the unit vector in the direction of the "vorticity"

$$\omega = A\xi \,, \qquad \tau = A\xi \,.$$

The solution is smooth if and only if

$$\int_0^T \big\|\alpha(\cdot,t)\big\|_{L^\infty} dt < \infty \,.$$

The stretching factor $\alpha$ has a nontrivial singular integral representation as a Cauchy principal value integral

$$\alpha(x) = P{\cdot}V{\cdot} \int D\big(\hat{y}, \xi(x+y), \xi(y)\big) A(x+y) \frac{dy}{|y|^n}$$

where the geometric factor $D$ cancels not only after spherical average, but also pointwise for certain key configurations. Associated to the "vorticity" there is a line (vortex line, level set) tangent to it. The Frenet equations define curvatures $\kappa$ for this object. The commutation relation

$$[D_t, \xi \cdot \nabla\,] = -\alpha \xi \cdot \nabla$$

holds, and reflects the fact that the line is material. The commutation relation can be used to deduce time evolution equations for the Frenet frame and for the curvatures. These equations have similar structures and seem to indicate a nonlinear balance between the magnitude $A$ and the curvatures $\kappa$ which have profound implications regarding the structure of singularities.

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