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SEMI–IMPLICIT DISCRETIZATION OF ABSTRACT EVOLUTION EQUATIONS

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ABSTRACT. Abstract evolution equations in an arbitrary Banach space X, like $\dot{u} = f(t,u,u)$, $t \in (0,T]$, subject to the initial value $u(0) = u_0$, are discretized by a semi-implicit version of the Euler method. The basic assumptions being that $f(t,\cdot,v)$ is one-sided Lipschitz, $\mathcal{R}(I-hf(t,\cdot,v)) = X$ for $h > 0$ sufficiently small, and $f(t,u,\cdot)$ is Lipschitz continuous, we show that the iterative scheme $u_{n+1} = u_n + hf((n+1)\Delta t, u_{n+1}, u_n)$, $n = 0,1,\ldots,N-1$, $\Delta t = T/N$, is stable and consistent, and hence convergent. Applications to systems of evolutionary PDEs are presented and the computational advantages of the semi-implicit method are pointed out.

1. Introduction

Discretizing first in time evolutionary problems (Rothe method, see [3, 7, 8], e.g.), presents the advantage of retaining a great flexibility in the choice of the subsequent space discretization algorithms. One of the most popular methods of this type is the implicit Euler method. In this paper, we propose and analyze a *semi-implicit* version of such a scheme, for a broad class of nonlinear abstract evolution problems in an arbitrary Banach space. Stability, consistency, and convergence are studied and precise error estimates obtained.

Applications are given to semilinear parabolic (integro-) differential systems, in both reflexive and non-reflexive spaces, for the purpose of illustration. The main computational advantages in these cases are: (a) *linearization* at each time step; (b) *decoupling* of systems into *independent stationary subsystems*, thus allowing for *parallel implementation*; (c) handling *local* instead of global (e.g. integral) space operators.

In order to contain the length of the paper, below we give only the main results, leaving all details to a future publication [11].

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2. Convergence analysis

Consider the abstract evolution problem

$$\dot{u} = f(t,u,u)\,,\ t \in (0,T\,]\,; \quad u(0) = u_0\,, \tag{1}$$

where $f(t,\cdot,v)\colon (D_t \subseteq X) \to X$, X being a real or complex Banach space, $t \in (0,T\,]$, $v \in D$, with $\bigcup_{t\in(0,T\,]} D_t \subseteq D \subseteq X$, satisfies the *one-sided Lipschitz condition*

$$\big\|u_1 - u_2 - h\big(f(t,u_1,v) - f(t,u_2,v)\big)\big\| \geq (1-hK_1)\|u_1-u_2\|\,, \tag{2}$$

$\forall\, u_1,\, u_2 \in D_t$, with $hK_1 < 1$, $K_1 \in \mathbb{R}$, and $f(t,u,\cdot)\colon (D \subseteq X) \to X$ satisfies a *classical* Lipschitz condition, $\big\|f(t,u,v_1) - f(t,u,v_2)\big\| \leq K_2\|v_1 - v_2\|$, uniformly in t,u. Moreover, suppose that, for every $h > 0$ sufficiently small, $\mathcal{R}\big(I - hf(t,\cdot,v)\big) = X$, that is the equation $u = hf(t,u,v) + b$ has a (unique) solution u in D_t, for each fixed $t \in (0,T\,]$, $v \in D$, and $b \in X$. Recall that, when f is (strongly) *dissipative* in u, i.e., $K_1 \leq 0$, then the solvability of the previous equation for $h = 1$ suffices (cf. [14]).

We assume that problem (1) has a (unique) "strict" solution, that is that $u(t) \in D_t$ for every $t \in (0,T\,]$, $u \in C^1([0,T\,];X)$, and $u(t)$ solves (1). Our algorithm will approximate such a solution in the C^0–norm. For $n = 0,1,\ldots,N$, we set $t_n = nh$, $h = \Delta t = T/N$, and then consider the (ideal) iterative scheme

$$u_{n+1} = u_n + hf(t_{n+1},u_{n+1},u_n)\,. \tag{3}$$

In order to take into account all relevant (and unavoidable) errors, we analyze the *perturbed* scheme, for $n = 0,1,\ldots,N-1$,

$$\begin{aligned} v_{n+1} &= \tilde{u}_n + hf(t_{n+1},v_{n+1},\tilde{u}_n) + \delta_{n+1}\,, \\ \tilde{u}_{n+1} &= v_{n+1} + \sigma_{n+1}\,, \quad \tilde{u}_{n+1} \in D\,. \end{aligned} \tag{4}$$

The scheme in (4) (and hence that in (3)) is well-defined since the first equation has a unique solution in $D_{t_{n+1}}$, in view of the assumptions made on the operator $f(t,\cdot,v)$ for $t > 0$ and $v \in D$. In (4), $\tilde{u}_0 = u_0 + \delta_0 \in D$, δ_0 denoting the error on the inital data, σ_{n+1} represents the overall error made in solving numerically the first equation, and δ_{n+1} takes into account, basically, the local truncation error. Note that when $f(t,\cdot,v)$ is discontinuous, the term σ_{n+1} *cannot* be embodied in δ_{n+1}. This is the case of evolutionary *partial* differential equations, in contrast to that of *ordinary* differential equations.

Comparing (3) and (4), one can find the *stability* estimate (cf. [11]

$$\max_{0\leq n\leq N} \|\tilde{u}_n - u_n\| \leq \max\left\{1,\rho^N\right\}\Big(\|\delta_0\| + (1-hK_1)^{-1}\sum_{k=1}^{N}\|\delta_k\| + \sum_{k=1}^{N}\|\sigma_k\|\Big), \tag{5}$$

where $hK_1 < 1$, and

$$\rho^N := \left(\frac{1+K_2T/N}{1-K_1T/N}\right)^N \sim \exp\left\{(K_1+K_2)T\right\}, \qquad N \to \infty\ (h \to 0)\,. \tag{6}$$

Observe that the solution $u(t)$ to problem (1) solves an equation like

$$u(t_{n+1}) = u(t_n) + hf\big(t_{n+1}\,,\ u(t_{n+1}), u(t_n)\big) + \omega_{n+1}(h)\,, \tag{7}$$

for a suitable choice of $\omega_{n+1}(h)$, that is $u(t_n)$ solves scheme (4) with $\delta_0 = 0$, $\delta_{n+1} = \omega_{n+1}(h)$, and $\sigma_{n+1} = 0$, for $n = 0, 1, \ldots, N-1$. The stability estimate in (5) requires estimating $\sum\limits_{k=1}^{N} \big\|\omega_{n+1}(h)\big\|$. Using the regularity assumption $u \in C^1([0,T];X)$, one obtains:

$$\sum_{k=1}^{N} \big\|\omega_{n+1}(h)\big\| \leq T\big[\operatorname{osc}(\dot{u};h) + K_2 \operatorname{osc}(u;h)\big]\ , \tag{8}$$

where $\operatorname{osc}(w;h) := \sup\big\{\|w(t_1) - w(t_2)\| : t_1, t_2 \in [0,T], |t_1 - t_2| \leq h\big\}$, and hence $\operatorname{osc}(\dot{u};h) = o(1)$ and $\operatorname{osc}(u;h) = O(h)$. Therefore $\max\limits_{0 \leq n \leq N} \big\|u_n - u(t_n)\big\| \to 0$, as $h \to 0$, which shows what is usually termed *convergence* of the algorithm. When $\dot{u}$ is Lipschitz continuous, the method turns out to be of the *first order*.

More generally, we are interested in estimating the convergence of the algorithm when perturbation terms are introduced implementing it in practice. Therefore, we consider the scheme in (4) with $\delta_k = 0$, $k = 1, 2, \ldots, N$, and, resorting again to (5), we get

$$\begin{aligned}\max_{0 \leq n \leq N} \big\|\tilde{u}_n - u(t_n)\big\| \leq \max\{1, \rho^N\}\Big\{\|\delta_0\| + \sum_{k=1}^{N} \|\sigma_k\| \\ + T(1 - hK_1)^{-1}[\operatorname{osc}(\dot{u};h) + K_2 \operatorname{osc}(u;h)]\Big\}\ .\end{aligned} \tag{9}$$

Such an estimate has been obtained by triangle inequality, using the stability and consistency estimates in (5) and (8), according to a Lax-type equivalence theorem (cf. [1], [11]). We stress again that δ_0 represents the error affecting the initial data, and σ_k the overall error made in solving the k-th equation of the scheme. Note that, when for instance equation (1) represents an evolutionary partial (integro-) differential equation, σ_k embodies the space-discretization errors as well as the errors (if any) on the boundary data.

3. Examples

In this Section, we conclude with some examples, for the purpose of illustrating the method outlined in §2.

(**A**) *Systems of ordinary differential equations.*

When $X = \mathbb{R}^m$, equation (1) represents an m-dimensional system of ODEs. Assuming that f satisfies all the relevant hypotheses in §2, the semi-implicit method includes the so-called "decoupled implicit Euler method" recently studied in [9]. In this case, one writes a given m-dimensional system $\dot{\boldsymbol{u}} = \boldsymbol{F}(t, \boldsymbol{u})$, $\boldsymbol{u}(0) = \boldsymbol{u}_0$ as $\dot{\boldsymbol{u}} = \boldsymbol{f}(t, \boldsymbol{u}, \boldsymbol{u})$, $\boldsymbol{u}(0) = \boldsymbol{u}_0$, where

$$\boldsymbol{f}(t, \boldsymbol{u}, \boldsymbol{v}) = \big(\boldsymbol{F}_1(t, \boldsymbol{u}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r), \ldots, \boldsymbol{F}_r(t, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{r-1}, \boldsymbol{u}_r)\big)\ , \tag{10}$$

$\boldsymbol{F}_k$, $\boldsymbol{u}_k$, $\boldsymbol{v}_k$ being m_k-dimensional vectors, $\sum_{k=1}^{r} m_k = m$. This approach leads to handling, at each time-step, r *independent* nonlinear *algebraic* systems, which can be solved by *parallel* implementation.

In the next two examples, the right-hand side of (1) is of the form

$$f(t,u,v) = Au + B(t,v) \ , \tag{11}$$

that is equation (1) is *semilinear*, and hence the stationary equations in (4) will be *linear*.

(**B**) *(Integro-) differential reaction-diffusion systems in* L^2.

Consider the system

$$\begin{gathered}\frac{\partial w_i}{\partial t} = M_i \Delta w_i + g_i(t,x,\boldsymbol{w}) + \int_\Omega K_i\big(t,x,\xi;\boldsymbol{w}(\xi,t)\big)d\xi, \ x \in \Omega \subset \mathbb{R}^d, \ 0 < t \leq T, \\ \boldsymbol{w}(x,0) = \boldsymbol{w}_0(x) \ , \ x \in \Omega \ ; \ \ w_i(x,t) = \phi_i(x,t) \ \text{ on } \partial\Omega \ \text{ if } \ M_i > 0 \ ,\end{gathered} \tag{12}$$

for the m-dimensional vector $\boldsymbol{w} = (w_1, \ldots, w_m)$, $\mathbf{M} = \operatorname{diag}(M_1, \ldots, M_m)$ with $M_i \geq 0$. Systems like that in (12) arise, e.g., from problems in epidemics and combustion theory. They include, as special cases, the classical reaction-diffusion systems ($\boldsymbol{K} \equiv \mathbf{0}$), purely nonlocal reactions ($\boldsymbol{g} \equiv \mathbf{0}$), as well as certain degenerate problems [some diagonal entries of $\mathbf{M}$ are zero], namely PDEs–ODEs couplings, cf. [4].

Assume that $\boldsymbol{g}(t,x,\cdot)$ is Lipschitz continuous from $\mathbb{R}^m$ into $\mathbb{R}^m$, uniformly in (t,x), with constant C_g, and $g_i(t,\cdot,0) \in L^2(\Omega)$ for $i = 1,2,\ldots,m$. Moreover, let $\boldsymbol{K}(t,x,\xi;\cdot)$ be Lipschitz continuous from $\mathbb{R}^m$ into $\mathbb{R}^m$, uniformly in t, with a Lipschitz constant $C(x,\xi) \in L^2(\Omega \times \Omega)$. Then, problem (12) can be recast into the abstract form (1), (11), with $X = \big(L^2(\Omega)\big)^m$, $A = \mathbf{M}\Delta$, $D_t = \{u \in \big(L^2(\Omega)\big)^m : u_i \in H^2(\Omega),\ u_i(x) = \phi_i(x,t) \text{ on } \partial\Omega \text{ if } M_i > 0\}$; A is monotone and maximal (for suitable boundary data) for each fixed $t > 0$, $B(t,\cdot)$ is the sum of the substitution [or Nemickii] operator associated to $\boldsymbol{g}$ *and* the Uryshon operator associated to $\boldsymbol{K}$, cf., e.g., [13]. It turns out that $D = X$, $B(t,\cdot)$ maps $\big(L^2(\overline{\Omega})\big)^m$ into $\big(L^2(\overline{\Omega})\big)^m$, and is Lipschitz continuous uniformly in t, with constant $C_g + \big\|C(\cdot,\cdot)\big\|_{L^2(\Omega\times\Omega)}$. Therefore, when the abstract problem has a solution $u \in C^1\big([0,T];X\big)$, the method described in §2 converges to it, the estimate (9) holding. The required regularity is guaranteed, e.g., when $\phi_i(x,t) \equiv 0$ on $\partial\Omega$ and $\boldsymbol{w}_0 \in D_t \equiv D(A) = \big\{u \in \big(L^2(\Omega)\big)^m\colon u_i \in H^2(\Omega) \cap H_0^1(\Omega) \text{ if } M_i > 0\big\}$, without further regularity assumptions on $\boldsymbol{g}$, since X is *reflexive*, cf. [12].

(**C**) *An example in a non-reflexive space.*

Consider the following initial-boundary value problem for the system

$$\begin{gathered}\frac{\partial w_i}{\partial t} = L_i(x)w_i - c_i(x)w_i + g_i(t,x,\boldsymbol{w}), \quad x \in \Omega \subset \mathbb{R}^d, \ 0 < t \leq T \ , \\ \boldsymbol{w}(x,0) = \boldsymbol{w}_0(x) \ , \ x \in \Omega \ ; \ \ w_i(x,t) = 0 \ \text{ on } \partial\Omega \ \text{ if } \ L_i \neq 0 \ .\end{gathered} \tag{13}$$

In (13), $\boldsymbol{w} \in \mathbb{R}^m$, and $L_i(x) = a_{jk}^{(i)}\partial^2_{x_j x_k} + b_j^{(i)}\partial_{x_j}$ denotes, for each $i = 1, 2, \ldots, m$, a linear strictly elliptic operator, $a_{jk}^{(i)}, b_j^{(i)}, c_i \in C^0(\overline{\Omega})$, $c_i \geq 0$. Equations in (13) are parabolic *unless* the operator L_j is replaced by 0 for some j; in such a case, the corresponding equation reduces to an ODE and *no* boundary data are imposed on w_j. Moreover, assume that $\boldsymbol{g}$ is Lipschitz continuous with respect to $\boldsymbol{w}$, uniformly in (x,t), $\boldsymbol{g} \in C^1([0,T] \times \Omega \times \mathbb{R}^m)$, and that $g_i(t,x,0) = 0$ for $x \in \partial\Omega$. Problems like that in (13), namely involving systems of parabolic-ordinary differential equations, are encountered in several applications (see [4, Ch. 8], for instance).

Again, problem (13) can be recast into the abstract form (1), (11), in the *non-reflexive* space $X = \left(C_0^0(\overline{\Omega})\right)^m$, with $A = \mathrm{diag}(L_1 - c_1, \ldots, L_m - c_m)$, $D(A) = \left\{u \in \left(W^{2,p}(\Omega)\right)^m \ \forall p < \infty\colon L_i u_i \in C^0(\overline{\Omega}),\ u_i = 0, \text{on } \partial\Omega \text{ for } L_i \neq 0\right\}$, $B(t,\cdot)$ is the substitution operator associated to $\boldsymbol{g}$, and $D = X$. It follows that $B(t,\cdot)$ maps X into X, and is Lipschitz continuous uniformly in t. The operator A turns out to be maximal dissipative on X, indeed strongly dissipative when $c_i(x) \geq c > 0$ for all i's. However, X being non-reflexive, the regularity result used in Example (B) cannot be invoked to ensure the existence of a $C^1([0,T];X)$–solution. Nevertheless, the C^1–regularity assumed for $\boldsymbol{g}$ implies that of $B(\cdot,\cdot)$ (cf. [13], e.g.), and hence, a $C^1([0,T];X)$–solution to (1), (11) *does* exist by a classical result in semigroup theory (cf. [5, Theorem 1.5, p.187]). Therefore, the semi-implicit Euler method introduced in §2 converges and the estimate (9) holds.

The algorithm (3) becomes, for the problem in Example (B),

$$\begin{aligned} u_{i,n+1}(x) &= u_{i,n}(x) + hM_i\Delta u_{i,n+1}(x) + hg_i\big(t_{n+1}, x, u_{1,n}(x), \ldots, u_{m,n}(x)\big) \\ &\quad + h\int_\Omega K_i\big(t_{n+1}, x, \xi; u_{1,n}(\xi), \ldots, u_{m,n}(\xi)\big)d\xi\,, \quad x \in \Omega\,, \end{aligned} \tag{14}$$

for $i = 1, 2, \ldots, m$ and $n = 0, 1, \ldots, N-1$, with the boundary conditions $u_i(x) = \phi_i(x, t_{n+1})$, $x \in \partial\Omega$, if $M_i > 0$. Note that (14) represents, *at each* time-step, a system of (linear) inhomogeneous *independent* Helmholtz equations. If $M_j = 0$ for some j, the j-th equation in (14) yields explicitly $u_{j,n+1}(x)$. Clearly, from the computational standpoint, *all* advantages mentioned in §1 have been obtained, namely *linearization*, *decoupling*, and removal of the nonlocal operator from the implicit part.

In numerically solving each linear Helmholtz equation in (14), one could adopt, for instance, a *probabilistic* method, particularly convenient (if not the only feasible, in practice), when the space dimension is high. The direct application of a probabilistic method to the original semilinear system would be much more difficult (cf. [10], e.g.).

Similar advantages are obtained in problem (C), the main difference being that the solution's components in this case is approximated (on the nodes t_n) in the sup-norm, instead of in the L^2–norm.

We confined ourselves to reaction-diffusion systems for the purpose of illustration. The semi-implicit Euler method however could be applied to many other instances, such as systems of semilinear *hyperbolic* equations (cf. [2, 6], e.g., for the implicit Euler method in the abstract hyperbolic case).

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