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A NOTE ON SYMMETRIES OF INVARIANT SETS WITH COMPACT GROUP ACTIONS

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ABSTRACT. We investigate the symmetries of the asymptotic dynamics of a map equivariant under a compact Lie group Γ. Let Γ^0 denote the connected component of the identity in Γ and let $\omega_f(x_0)$ denote the ω-limit set of the point x_0 under the map f. Assume that $\omega_f(x_0)$ contains a point of trivial isotropy and is not a relative periodic orbit (these are mild assumptions on the dynamics). Melbourne [14] shows that under these assumptions and when Γ^0 is abelian, then generically (in the C^∞ topology) the symmetry group of $\omega_f(x_0)$ contains Γ^0. We show under the same assumptions on the dynamics but without the assumption that Γ^0 is abelian that it is possible to construct a family of perturbations such that for a residual subset of perturbations (in the C^0 topology) the resulting ω-limit point set of x_0 has at least Γ^0 symmetry. Our argument does not extend directly to the C^1 topology.

1. Introduction

Chossat and Golubitsky [4] showed numerically that the dynamics of equivariant mappings can produce attractors that are symmetric and that the symmetry of attractors can change as parameters in the mapping are varied. These symmetries of attractors lead to visually striking pictures [10] and to the existence of *patterns on average* in systems of PDEs [5] and in experiments [12, 16]. If the attractor has a sufficiently nice ergodic measure (a Sinai–Bowen–Ruelle measure), then it can be proved [6] that the time-average

$$U(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T u(x,t)\, dt\,,$$

where $u(x,t)$ is either a solution to a PDE or a time-series in an experiment, is invariant under the symmetries of the attractor A defined by the ω-limit point

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set. That is, if σ is a symmetry of the system and $\sigma A = A$, then

$$U(\sigma x) = U(x).$$

Thus, the time-average can be symmetric even though this symmetry appears at no fixed instant of time in the solution; that is, $u(\sigma x, t) \neq u(x, t)$ for all time t.

The existence of patterns on average motivates the question of which symmetry groups can be the symmetry group of an attractor. In [15] it was shown, perhaps surprisingly, that for finite groups there are restrictions on the possible symmetries. We will be more precise. Let $\Gamma \subset \boldsymbol{O(n)}$ be a finite group and let $\Sigma \subset \Gamma$ be a subgroup. We say that Σ is *admissible* (as a subgroup of Γ acting on $\mathbb{R}^n$) if there exists a Γ-equivariant mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ with an attractor A having symmetry Σ. The *symmetry group* of a set A is

$$\Sigma(A) = \{\sigma \in \Gamma\colon \sigma A = A\}. \tag{1.1}$$

The results in [15] show that restrictions to admissibility are governed by reflections across hyperplanes in Γ. More precisely, suppose that $N \subset \Gamma$ is a subgroup. Let

$$R(\Sigma) = \{\tau \in \Gamma - N\colon \ \dim \operatorname{Fix}(\tau) = n - 1\},$$

that is, $R(\Sigma)$ consists of reflections across hyperplanes that are in Γ but not in N. Now form the union of hyperplanes

$$L(\Sigma) = \bigcup_{\tau \in R(\Sigma)} \operatorname{Fix}(\tau).$$

It is clear that symmetries in N leave $L(\Sigma)$ invariant (since $\gamma \operatorname{Fix}(\tau) = \operatorname{Fix}(\gamma^{-1}\tau\gamma)$ and $\gamma^{-1}\tau\gamma$ is a reflection across a hyperplane when τ is). Hence these symmetries permute the connected components in the complement of $L(\Sigma)$.

We state the result of [15]. If $\Sigma \subset \Gamma$ is admissible, then there exists a normal subgroup $N \subset \Sigma$ satisfying

(a) Σ/N is cyclic, and

(b) N fixes a connected component of $\mathbb{R}^n - L(\Sigma)$.

It follows easily, for example, that hexagonally equivariant mappings (D_6 symmetry) cannot have attractors with precisely triangular (D_3) symmetry. Previously, King and Stewart [13] showed that cyclic subgroups are admissible for planar mappings with dihedral symmetry. More recently, Ashwin and Melbourne [1] have shown that these conditions are both necessary and sufficient by explicitly constructing appropriate mappings and attractors when (a) and (b) are satisfied.

The results just mentioned apply to the class of all continuous Γ-equivariant mappings. It is easy to show that the admissible subgroups for diffeomorphisms and flows are more restrictive than the admissible subgroups for the class of all maps. (For example, attractors for planar diffeomorphisms with D_m $(m \geq 3)$ symmetry cannot be D_m symmetric, but the pictures in [10] and the results in [1] show that such attractors are easily found for noninvertible mappings.) The classification of admissibility for diffeomorphisms and flows for subgroups of a finite group Γ have been carried out in [11].

Rather less is known concerning admissibility for continuous compact Lie groups; yet this is also an important issue for patterns on average in applications. For example in [12] Gollub *et al.* have performed the Faraday experiment in a circular container obtaining a beautiful target pattern in the time-average. In [16] a rotating convection experiment was also performed in a circular container and the results were less conclusive. In the time-average, a circularly symmetric target pattern appeared near the boundary of the cylinder; but the time-average (experimentally performed over a finite time period, of course) was disordered near the center of the disk. Finally, turbulent Taylor vortices (a turbulent state with much fine scale structure but having the apparent symmetry of Taylor vortices) occurs in a system which is usually idealized to have $\boldsymbol{O}(\mathbf{2}) \times \boldsymbol{SO}(\mathbf{2})$ symmetry (by assuming periodic boundary conditions in the axial direction).

There are several results concerning attractors in systems with compact symmetry groups. Field [9] has shown that when an attractor is a relative periodic orbit (that is, when the corresponding trajectory of the induced map on orbit space is periodic) the symmetry of the attractor must be a maximal abelian subgroup of Γ and have the form $\boldsymbol{T}^m \times \mathbb{Z}_p$. Moreover, the dynamics on the attractor must be quasi-periodic. Recently, Melbourne [14] has shown that when the attractor is *sufficiently* chaotic (that is, when the attractor contains periodic orbits consisting of points with trivial isotropy) and when Γ^0, the connected component of the identity in Γ, is abelian, then a family of perturbations can be constructed such that for a residual set of parameters (indeed a measure one or prevalent subset of parameters) the associated attractors have symmetry groups containing Γ^0. Moreover, Melbourne's methods should work in the category of smooth maps and in the category of diffeomorphisms.

In this note we generalize Melbourne's method to prove a similar result when Γ^0 is nonabelian. We also construct a family of perturbations that yields attractors with at least Γ^0 symmetry for a residual set of parameters and we need only assume that the attractor contains a point of trivial isotropy. However, our method will work only in the category of continuous mappings and will not yield measure one or prevalence type results. Nevertheless, this result does support the idea that generally chaotic attractors consisting (mostly) of asymmetric points will have symmetry groups containing Γ^0.

2. Symmetries of ω-limit sets

Throughout this section, we shall assume that $\Gamma \subset \boldsymbol{O}(\boldsymbol{n})$ is a (nonfinite) compact Lie group acting on $\mathbb{R}^n$. Note that if $X \subset \mathbb{R}^n$ is closed, then the symmetry group $\Sigma(X)$ is a closed Lie subgroup of Γ.

Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a continuous Γ-equivariant mapping. Let $\mathbb{R}^n/\Gamma$ denote the orbit space of Γ and let $\pi: \mathbb{R}^n \to \mathbb{R}^n/\Gamma$ denote the orbit projection. Let $\hat{f}: \mathbb{R}^n/\Gamma \to \mathbb{R}^n/\Gamma$ denote the map induced by f on the orbit space.

Given $x_0 \in \mathbb{R}^n$, we let $\omega_f(x_0)$ denote the ω-limit point set for f with initial point x_0. Note that $\omega_{\hat{f}}(\pi(x_0)) = \pi(\omega_f(x_0))$. Since f is Γ-equivariant, $\omega_f(\gamma x_0) = \gamma\omega_f(x_0)$ for all $\gamma \in \Gamma$ and hence $\omega_{\hat{f}}(\pi(x_0)) = \pi(\omega_f(\gamma x_0))$. Our interest lies in perturbing f so as to maximize the size of the symmetry group $\Sigma(\omega_f(x_0))$ without changing the projection $\pi(\omega_f(x_0))$. In particular, we show that under appropriate conditions on f and x_0, it is generically true that $\Sigma(\omega_f(x_0)) \supset \Gamma^0$.

Our first task is to describe the class of perturbations of f we shall use. We require that these perturbations induce the map $\hat{f}$ on the orbit space. Indeed, our perturbations of f will be compositions of f with maps which cover the identity map on the orbit space. For our present purposes, it suffices to work with a rather simple class of perturbations of this type. We discuss the general problem of characterizing maps covering the identity map on orbit space in Section (3).

Let Γ act on Γ^0 by inner automorphism. That is, we define

$$\gamma \cdot g = \gamma g \gamma^{-1}$$

for all $g \in \Gamma^0$ and $\gamma \in \Gamma$. Let $\mathcal{X}$ denote the space of Γ-equivariant maps $\eta: \mathbb{R}^n \to \Gamma^0$. Observe that η is Γ-equivariant if

$$\eta(\gamma x)\gamma = \gamma\eta(x) \tag{2.1}$$

for all $\gamma \in \Gamma$ and $x \in \mathbb{R}^n$. We note that $\mathcal{X}$ is a group. In particular, the constant map $e: \mathbb{R}^n \to \Gamma^0$ defined by mapping points to the identity element $e \in \Gamma$ is continuous and Γ-equivariant. Moreover, if $\eta, \nu: \mathbb{R}^n \to \Gamma^0$ are continuous Γ-equivariant maps then so is the product $\nu\eta$ defined by

$$(\nu\eta)(x) = \nu(x)\,\eta(x)$$

for all $x \in \mathbb{R}^n$.

Let T be a compact Γ–invariant neighborhood of $\omega_f(x_0)$. We say that a map $\eta \in \mathcal{X}$ has *support in* T when $\eta = e$ off T. Let $\mathcal{X}(T)$ denote the subspace of $\mathcal{X}$ consisting of mappings supported in T. Note that $\mathcal{X}(T)$ is a subgroup of $\mathcal{X}$. Give $\mathcal{X}$ the usual C^0–topology.

LEMMA 2.1. *The space $\mathcal{X}(T)$ is a Baire space.*

Proof. The space $\mathcal{X}(T)$ is a closed subset of $\mathcal{X}$ and, in the C^0–topology, the space $\mathcal{X}$ is a complete metric space. Hence $\mathcal{X}(T)$ is a complete metric space and a Baire space. □

Given an equivariant mapping $\eta \in \mathcal{X}(T)$, we think of $f_\eta(x) = \eta(x)f(x)$ as a perturbation of f. Note that (2.1) implies that f_η is Γ–equivariant and that the maps f and f_η induce the same mapping on space of Γ orbits. That is, $\hat{f} = \hat{f}_\eta$, for all $\eta \in \mathcal{X}(T)$. In particular, $\pi\big(\omega_{f_\eta}(x_0)\big) = \pi\big(\omega_f(x_0)\big)$. The differences in the dynamics of f and f_η are related to how points are moved along group orbits.

Our main result states that for a residual subset of $\eta \in \mathcal{X}(T)$, the perturbations f_η have ω-limit sets that are at least Γ^0–invariant.

THEOREM 2.2. *Let f be Γ–equivariant and let $x_0 \in \mathbb{R}^n$. Assume that*

(a) *$f^m(x_0)$ is of trivial isotropy for all $m \geq 0$, and*

(b) *$\omega_f(x_0)$ is not a relative periodic orbit.*

Then there is a residual set $\mathcal{F} \subset \mathcal{X}(T)$ such that if $\eta \in \mathcal{F}$, then $\Gamma^0 \subset \Sigma\big(\omega_{f_\eta}(x_0)\big)$.

Remarks 2.3. (1) Assumption (a) is satisfied if $\omega_f(x_0)$ contains a point of trivial isotropy. On the other hand, if $f^m(x_0)$ is of trivial isotropy for all $m \geq 0$, it need not follow that $\omega_f(x_0)$ contains any points of trivial isotropy. Of course, if f is invertible then all points of the sequence $f^m(x_0)$ will have trivial isotropy provided only that the initial point x_0 has trivial isotropy.

(2) We may interpret the theorem in two ways. First of all, think of x_0 as a 'generic' point and note that in most applications x_0 will have trivial isotropy. Assume that f is invertible. The conclusion of the theorem implies that, on average, we expect to see at least Γ^0 symmetry. In this situation, it will be the case that in our proof of Theorem 2.2, we typically do not perturb the dynamics *on* $\omega_f(x_0)$ but rather perturb the sequence $f^m(x_0)$. On the other hand, suppose that $x_0 \in \omega_f(x_0)$. The theorem now makes an assertion about how we can modify the dynamics of f on $\omega_f(x_0)$ to achieve symmetry Γ^0.

(3) Suppose that f is smooth. We do not claim in our theorem to have found a residual set of *smooth* perturbations of f that yield symmetry Γ^0. Indeed, we

are unable to show that we can obtain symmetry Γ^0 using C^0–small smooth perturbations. Under the assumption that Γ^0 is abelian, Melbourne [14] has proved a similar result for smooth mappings which uses perturbations η that are constant (and not compactly supported). Under rather restrictive conditions on dynamics and maps, Ashwin, Stewart and Chossat [2] obtain perturbation results that apply when Γ^0 is not abelian.

2.1 Technical preliminaries.

We collect together a number of elementary results and definitions that we need for the proof of Theorem 2.2.

LEMMA 2.4. *Let τ be in Γ^0, let V be a neighborhood of e, and let U be a neighborhood of τ. Then there exists $\gamma \in V$ such that $\gamma^q \in U$ for infinitely many integers $q > 0$.*

Proof. Let $T \subset \Gamma^0$ be a maximal torus which contains τ [3]. Choose a topological generator $\gamma \in V$ for T. Clearly, $\gamma^q \in U$ for infinitely many values of q. □

Given x_0 and f as in the statement of Theorem 2.2, define

$$W_f(x_0) = \bigcup_{\gamma \in \Gamma^0} \gamma\omega_f(x_0).$$

LEMMA 2.5. *Suppose $x_0 \in \mathbb{R}^n$, $\eta \in \mathcal{X}(T)$ and $\omega_f(x_0)$ is not a relative periodic orbit. Then*

(α) $\omega_{f_\eta}(x_0) \subset W_f(x_0)$,

(β) *If $x \in W_f(x_0)$, then there exist $y \in \omega_{f_\eta}$ and $\sigma \in \Gamma^0$ such that $\sigma y = x$, and*

(γ) *ω_{f_η} is not a relative periodic orbit.*

Proof. For all $m \geq 0$, $f^m(x_0)$ and $f^m_\eta(x_0)$ lie on the same Γ^0–orbit. It follows easily that $\omega_{f_\eta}(x_0) \subset W_f(x_0)$, proving ($\alpha$). Statement ($\beta$) is an immediate consequence of (α). Finally, (γ) follows since f, f_η induce the same map $\hat{f}$ on orbit space and $\omega_f(x_0)$ is a relative periodic orbit if and only if $\pi(\omega_f(x_0))$ is a periodic orbit of $\hat{f}$. □

Finally, we give a precise description of the C^0 metric structure on $\mathcal{X}(T)$, based on [8, §3]. We begin by showing that there is a metric on Γ that is both left and right invariant. To see this, observe that

$$(\tau, \gamma) \cdot g = \tau g \gamma^{-1},$$

where $(\tau,\gamma)\in\Gamma\times\Gamma$, $g\in\Gamma$, defines an action of $\Gamma\times\Gamma$ on Γ. (Note that this action restricted to the diagonal subgroup is just the action of Γ on Γ^0 by inner automorphism. This point will be used in subsequent discussion.) As a result we may choose a $\Gamma\times\Gamma$–invariant Riemannian metric on Γ; let d_Γ denote the corresponding distance function on Γ. It follows from the $\Gamma\times\Gamma$–invariance of d_Γ that for all $\gamma,x,y\in\Gamma$ we have

$$d_\Gamma(\gamma x,\gamma y)=d_\Gamma(x,y)=d_\Gamma(x\gamma,y\gamma)\,. \tag{2.2}$$

The explicit C^0 metric on $\mathcal{X}(T)$ that we use is defined by

$$d_X(\eta,\mu)=\sup_{x\in T} d_\Gamma\big(\eta(x),\mu(x)\big)\,,$$

for $\eta,\mu\in\mathcal{X}(T)$. It follows from (2.2) that

$$d_X(\nu\eta,\eta)=d_X(\nu,e)\,, \tag{2.3}$$

for $\nu,\eta\in\mathcal{X}(T)$.

2.2 Proof of Theorem 2.2.

We start by defining the set $\mathcal{F}\subset\mathcal{X}(T)$. Let $\{x_i\}_{i=1}^\infty$ be a countable dense subset of $W_f(x_0)$ and let $B^*_{1/m}(x_i)$ denote the punctured disk neighborhood of radius $1/m$ about x_i. In particular, this neighborhood does not include x_i. Define

$$\mathcal{F}_{i,m}=\{\eta\in\mathcal{X}(T)\mid f^j_\eta(x_0\in B^*_{1/m}(x_i)\quad\text{for some positive integer } j\}$$

and set

$$\mathcal{F}=\bigcap_{i,m}\mathcal{F}_{i,m}\,.$$

Since $x_i\notin B^*_{1/m}(x_i)$, it follows that if $\eta\in\mathcal{F}$ then $\omega_{f_\eta}(x_0)=W_f(x_0)$. We shall show that each $\mathcal{F}_{i,m}$ is an open dense subset of $\mathcal{X}(T)$. It then follows from Lemma 2.1 that $\mathcal{F}$ is residual. Since $\mathcal{F}_{i,m}$ is obviously open, we need only prove density.

P r o o f o f d e n s i t y. Let $\eta\in\mathcal{X}(T)$ and $\varepsilon>0$ be given. Fix x_i and m. It follows from (2.3) that it suffices to construct $\nu\in\mathcal{X}(T)$ such that $\nu\eta\in\mathcal{F}_{i,m}$ and $d_X(\nu,e)<\varepsilon$.

Let V be the open ε-disk centered at the identity in Γ^0. It follows from (2.2) that V is invariant under inner automorphisms of Γ. We will perturb f_η

to $f_{\nu\eta} = \nu f_\eta$, where $\nu \in \mathcal{X}(T)$ is smooth and $\nu(\mathbb{R}^n) \subset V$. Since ν takes values in V, it will follow that $d_X(\nu, e) < \varepsilon$.

Since $x_i \in W_f(x_0)$, it follows from Lemma 2.5 (β), that there exists $z \in \omega_{f_\eta}$ such that $\tau z = x_i$ for some $\tau \in \Gamma^0$. Take U to be a sufficiently small neighborhood of τ so that $\delta z \in B_{1/2m}(x_i)$ whenever $\delta \in U$. Now choose S to be a slice at z and let $\mathcal{S} = \Gamma S$.

It follows from Lemma 2.4, that there exists $\gamma \in V$ such that $\gamma^q \in U$ for some integer $q > 0$. Hence $\gamma^q z \in B_{1/2m}(x_i)$. Since $z \in \omega_{f_\eta}$, there is an integer N such that

$$d_X\big(f_\eta^N(x_0), z\big) \leq \frac{1}{2m}$$

and

$$\operatorname{card}\big(\{x_0, f_\eta(x_0), \ldots, f_\eta^{N-1}(x_0)\} \cap \mathcal{S}\big) \geq q.$$

Let M be the smallest such integer N. It follows from the definition of M that at least q of the iterates $f_\eta^j(x_0)$ lie in $\mathcal{S}$. Suppose

$$\{x_0, f_\eta(x_0), \ldots, f_\eta^{M-1}(x_0)\} \cap \mathcal{S} = \{f_\eta^{j_1}(x_0), f_\eta^{j_2}(x_0), \ldots, f_\eta^{j_p}(x_0)\},$$

where $p \geq q$. Let $z_k = f_\eta^{j_k}(x_0)$. Since we are assuming that all points $f^n(x_0)$ have trivial isotropy, it follows that z_k has trivial isotropy. It follows from Lemma 2.5 (γ) that the orbit of x_0 is not a relative periodic orbit and so the z_k are all lie on different Γ–orbits. Next, take (closed) slices $S_k \subset \mathcal{S}$ about each point z_k for $k = 1, \ldots, p$. Since the points z_k all lie on different Γ–orbits, we can assume that the Γ–orbits of the slices S_k are mutually disjoint. Since z_k has trivial isotropy, each slice S_k consists of points with trivial isotropy.

Define

$$\nu(z_k) = \begin{cases} \gamma, & 1 \leq k \leq q, \\ e, & q < k \leq p. \end{cases}$$

Smoothly extend ν to a map $\nu\colon \bigcup_{k=1}^{p} S_k \to V \subset \Gamma^0$ which is equal to the identity element $e \in \Gamma$ on a neighborhood of the boundary of each slice S_k. Next extend ν to $\mathcal{S}$ by

$$\nu(\gamma x) = \gamma \nu(x) \gamma^{-1}$$

for all $\gamma \in \Gamma$ and $x \in S$. Since all points in the sets S_k have trivial isotropy, this extension is well-defined and Γ–equivariant. Since V is invariant under all inner automorphisms, $\nu(x) \in V$ for all $x \in \mathcal{S}$. Note that ν has support in $\mathcal{S}$ and can be extended smoothly off $\mathcal{S}$ by $\nu(x) = e$.

To complete the proof we show that $f^M_{\nu\eta}(x_0) \in B^*_{1/m}(x_i)$ and hence that $\nu\eta \in \mathcal{F}_{i,m}$. By construction, for $1 \le j \le M$,

$$\nu\big(f^j_\eta(x_0)\big) = \begin{cases} \gamma, & j \in \{j_1, \dots, j_q\}, \\ e, & j \notin \{j_1, \dots, j_q\}. \end{cases}$$

Thus

$$\begin{aligned} f^M_{\nu\eta}(x_0) &= \nu(x_0)\nu\big(f_\eta(x_0)\big) \dots \nu\big(f^{M-1}_\eta(x_0)\big) f^M_\eta(x_0) \\ &= \gamma^q f^M_\eta(x_0). \end{aligned}$$

We have

$$\begin{aligned} d\big(\gamma^q f^M_\eta(x_0), x_i\big) &\le d\big(\gamma^q f^M_\eta(x_0), \gamma^q z\big) + d(\gamma^q z, x_i) \\ &< d\big(f^M_\eta(x_0), z\big) + \frac{1}{2m} \\ &\le \frac{1}{m}, \end{aligned}$$

since $d\big(f^M_\eta(x_0), z\big) \le \frac{1}{2m}$. Hence, $\gamma^q f^M_\eta(x_0) \in B_{1/m}(x_i)$. If necessary, we may perturb ν slightly to ensure that $\gamma^q f^M_\eta(x_0) \ne x_i$ and so obtain $\gamma^q f^M_\eta(x_0) \in B^*_{1/m}(x_i)$. □

3. Perturbing along group orbits

We continue to assume that $\Gamma \subset \boldsymbol{O(n)}$ is a (nonfinite) compact Lie group acting on $\mathbb{R}^n$. Suppose that $f, f' \colon \mathbb{R}^n \to \mathbb{R}^n$ are Γ–equivariant homeomorphisms. Then f, f' induce the same map on the orbit space $\mathbb{R}^n/\Gamma$ if and only if $f' \circ f^{-1}$ covers the identity map on the orbit space. We let $\mathcal{I}_\Gamma$ denote the set of all continuous equivariant maps which cover the identity map on the orbit space and note that $\mathcal{I}_\Gamma$ is a group. Let T be a compact Γ–invariant subset of $\mathbb{R}^n$ and let $\mathcal{I}_\Gamma(T)$ denote the subgroup of $\mathcal{I}_\Gamma$ consisting of homeomorphisms which are equal to the identity map outside of T.

In Theorem 2.2, we used a restricted class of elements in $\mathcal{I}_\Gamma$ to study perturbations of f. More specifically, given $\eta \in \mathcal{X}$, we define the map $\tilde{\eta} \in \mathcal{I}_\Gamma$ by $\tilde{\eta}(x) = \eta(x)x$ for all $x \in \mathbb{R}^n$. (Observe that $\tilde{\eta}$ is a homeomorphism with $\tilde{\eta}^{-1}(x) = \eta(x)^{-1}x$.) Also note that $f_\eta = \tilde{\eta} \circ f \in \mathcal{I}_\Gamma(T)$ when $\eta \in \mathcal{X}(T)$.

It is natural to ask the extent to which elements $\eta \in \mathcal{X}$ represent general elements of $\mathcal{I}_\Gamma$. Questions of this type are discussed in some detail in [7, 8] and we only give a brief indication of some of the results and problems.

Suppose that $\eta \in \mathcal{X}$. Since η takes values in Γ^0, it follows that η is homotopic to the constant map e and $\tilde{\eta}$ is isotopic to the identity map of $\mathbb{R}^n$ (through elements of $\mathcal{I}_\Gamma$). Of course, elements of $\mathcal{I}_\Gamma$ need not be isotopic to the identity; indeed, the isotopy question is quite subtle for general Γ–manifolds (see [7]). In any case, from the point of view of perturbation theory, it seems natural to restrict attention to those elements of $\mathcal{I}_\Gamma$ which are isotopic to the identity.

Let T be a compact Γ–invariant domain in $\mathbb{R}^n$ and suppose that all Γ orbits in T have finite isotropy. (More generally, we can ask that all Γ–orbits have the same dimension.) It follows from [8, §3] that if $\varphi \in \mathcal{I}_\Gamma(T)$ is isotopic to the identity then $\varphi = \tilde{\eta}$ for some $\eta \in \mathcal{X}(T)$. Similar results hold if we work within the smooth category.

Matters become significantly more complicated when the dimension of Γ–orbits in T varies. For example, it is shown in [7] that if $\Gamma = \mathrm{SU}(2)$ acts in the standard way on $\mathbb{C}^2 = \mathbb{R}^4$ and T is a disk neighborhood of the origin then $\mathcal{X}(T)$ forms a small subset of the identity component of $\mathcal{I}_\Gamma(T)$. In particular, if $A\colon \mathbb{R}^4 \to \mathbb{R}^4$ is linear and Γ–equivariant and we write $A(x) = \eta(x)\, x$, $x \in \mathbb{R}^4$, then $\eta\colon \mathbb{R}^4 \to \Gamma$ will generally *not* be continuous at $x = 0$.

When we want to allow for variation in the dimension of Γ–orbits, it is appropriate to attempt to represent elements of $\mathcal{I}_\Gamma$ as time-one maps of vector fields which are tangent to Γ–orbits. Let $\mathcal{I}_\Gamma^\infty$ denote the subspace of smooth (that is, C^∞) diffeomorphisms in $\mathcal{I}_\Gamma$. Let $\mathcal{V}^\infty$ denotes the space of all smooth Γ–equivariant vector fields on $\mathbb{R}^n$ which are everywhere tangent to Γ–orbits. Let $\mathcal{V}^\infty(T)$ denote the subspace of $\mathcal{V}^\infty$ consisting of vector fields supported in T. We have a natural map $\exp\colon \mathcal{V}^\infty(T) \to \mathcal{I}_\Gamma^\infty(T)$ defined by mapping $X \in \mathcal{V}^\infty(T)$ to the time-one map of the flow of X. Obviously, all maps in the image of exp are isotopic to the identity map. It is natural to conjecture that $\exp(\mathcal{V}^\infty(T))$ is equal to the identity component of $\mathcal{I}_\Gamma^\infty(T)$. However, as far as we are aware, this conjecture has not yet been resolved though it is known to hold for many representations. We refer the reader to Schwarz [17, §4] for a discussion in the context of compact Γ–manifolds.

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