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THE ACCURACY OF DAFERMOS' METHOD FOR NONLINEAR HYPERBOLIC EQUATIONS

by G. W. HEDSTROM

Dafermos' method [1] for the problem

$$u_t + (f(u))_x = 0 \quad (-\infty < x < \infty, t > 0)\,, \tag{1}$$

$$u(x, 0) = g(x), \tag{2}$$

consists of the replacement of f by a piecewiselinear function f_h and g by a piecewise-constant function g_h. Dafermos did no numerical experiments but proved under quite general conditions that there exist $f_h \to f$ and $g_h \to g$ such that the solution of

$$u_t + (f_h(u))_x = 0 \quad (-\infty < x < \infty, t > 0), \tag{3}$$

$$u(x, 0) = g_h(x), \tag{4}$$

tends to a weak solution of (1), (2) which satisfies an entropy condition.

In the paper [2] we reported on numerical experiments for Dafermos' method for a scalar equation (1) and for a 2×2 system. We found that the method gives very good approximations in short computer time, although the programming is difficult; only a fanatic would program Dafermos' method for a nontrivial 3×3 system. In this paper we present a modification of Dafermos' method for the scalar equation (1). This modified version has the advantage of being easier to program than the genuine Dafermos method. Further, numerical experiments indicate that under natural smoothness conditions on f and g the modified Dafermos' method gives $O(h^2)$ approximations to shocks in the solution of (1), (2), where h is a certain mesh parameter. We are able to prove this in a special case.

We first describe the solution of (3), (4) and then present our modified version. We suppose for simplicity that $f \in C^2$ and that $f'' > 0$. We also suppose that the graph of f_h is a polygonal line with vertices at some points $(u_j, f(u_j))$. If x_0 is a point of discontinuity of g_h with $u_L = g_h(x_0-)$ and $u_R = g_h(x_0+)$, then in a neighbourhood of $(x, t) = (x_0, 0)$ we have to solve a Riemann problem. There are two cases. If $u_L > u_R$, then we have a shock

$$x = x_0 + mt\,,$$

where

$$m = (f(u_R) - f(u_L))/(u_R - u_L)\,.$$

If $u_L < u_R$, then we have a discrete rarefaction wave (actually, a sequence of contact discontinuities),

$$x = x_0 + m_j t \qquad (j = 1, 2, \ldots, n),$$

where

$$m_j = (f_h(u_j) - f_h(u_{j-1}))/(u_j - u_{j-1})$$

and the $u_j (j = 1, 2, \ldots, n - 1)$ are the u-coordinates of the vertices of f_h between u_L and u_R,

$$u_0 = u_L < u_1 < u_2 < \ldots < u_n = u_R .$$

For

$$m_j < (x - x_0)/t < m_{j+1} \qquad (j = 1, 2, \ldots, n - 1)$$

the solution u is given by $u = u_j$. Thus, the solution of (3), (4) is determined by a set of lines. When two lines meet, we have a new Riemann problem, producing more lines.

In our modified version we emphasize these lines of discontinuity, choosing to mimic the geometry of the solution of (1) at the expense of giving up the special form (3). We shall not go into detail, but we remark that our version may be locally put in the form (3). From a numerical point of view the beauty of Dafermos' method lies in the geometry: A shock in the solution of (1), (2) is approximated by a broken-line shock in the solution of (3), (4), and in regions where the solution of (1), (2) is smooth, the lines of discontinuity in the solution of (3), (4) approximate the characteristics of (1), (2). It is a problem, though, how g_h is to be obtained from g once f_h has been defined. Presumably, the points of discontinuity of g_h should be chosen by looking at a local inverse function to g.

This problem of the choice of g_h has led us to the following modification of Dafermos' method. Given $f \in C^2$ with $f'' > 0$ and given g piecewise continuous with g constant on some intervals $(-\infty, a)$ and (b, ∞), we construct g_h as follows without reference to any f_h. The points of discontinuity $x_k (k = 0, 1, \ldots, N)$ of g_h are taken to be the points of discontinuity of g, together with the points jh in the interval $(a - h, b + h)$, where h is a positive number and $j = 0, \pm 1, \ldots$. The value of g_h on (x_{k-1}, x_k) $(k = 1, 2, \ldots, N - 1)$ is defined as

$$u_k = g((x_{k-1} + x_k)/2) ,$$

and

$$g_h(x) = u_0 = g(a - h) \qquad (x < x_0) ,$$
$$g_h(x) = u_{N+1} = g(b + h) \qquad (x > x_N) .$$

If z is a point at which $g(z-) < g(z+)$, we may want to take

$$x_{k_1} = x_{k_1+1} = \ldots = x_{k_2} = z$$

and insert values

$$u_{k_1} < u_{k_1+1} < \ldots < u_{k_2+1} .$$

From the point $(x, t) = (x_k, 0)$ we draw the line

$$x = x_k + m_k t ,$$

where

$$m_k = (f(u_{k+1}) - f(u_k))/(u_{k+1} - u_k) \tag{5}$$

($k = 0, 1, \ldots, N$). Until the first intersection we define u by taking

$$u = u_k \quad (x_{k-1} + m_{k-1}\, t < x < x_k + m_k t, \qquad k = 1, 2, \ldots, N-1)\,, \tag{6}$$

$$u = u_0 \quad (x < x_0 + m_0 t)\,, \tag{7}$$

$$u = u_{N+1} \quad (x > x_N + m_N t)\,. \tag{8}$$

Suppose that the first intersection occurs at $(x, t) = (\xi, \tau)$ and that $u(\xi -, \tau) = u_L$ and $u(\xi +, \tau) = u_R$. From the convexity of f and from the definition (5) of the m_k it follows that $u_L > u_R$, so that a shock is formed starting at (ξ, τ). Near the point (ξ, τ) we define u by

$$u = u_L \quad (x < \xi + m(t - \tau), t > \tau)\,,$$

$$u = u_R \quad (x > \xi + m(t - \tau), t > \tau)\,,$$

where m is given by the Rankine–Hugoniot condition

$$m = (f(u_R) - f(u_L))/(u_R - u_L)\,.$$

Away from (ξ, τ) we simply continue the solution (6), (7), (8). At the next intersection we repeat the process.

Because the number of lines is reduced at each intersection, the lagorithm terminates naturally. The final configuration is either a single line or a collection of diverging lines.

We close with some remarks about accuracy. If g is twice continuously differentiable near x_k, then the line

$$x = x_k + m_k t$$

is a good approximation to the characteristic

$$x = x_k + t f'(g(x_k))\,,$$

because m_k approximates $f'(g(x_k))$ to within $O(h^2)$ as $h \to 0$. Further, numerical experiments indicate that shocks in the solution to (1), (2) are also approximated to within $O(h^2)$, but we have been able to prove that this is true only in the following case.

Theorem. *Let $f \in C^3, f'' > 0$ and let g be of the following special form. Let*

$$g(x) = u_0 = \text{const.}\ (x < 0)\,,$$

$g \in C^2(0, \infty)$, $g(0+) < u_0$, *g nondecreasing on* $(0, \infty)$, *and*

$$g(x) = u_{N+1} = \text{const.}\ (x > b)\,.$$

Then the solution of problem (1), (2) *has a single shock* $x = \varphi(t)$ *starting from the origin in the* (x, t)*-plane, and our modified Dafermos' method also gives a single shock* $x = \varphi_h(t)$. *Furthermore, there exists a constant* C *such that*

$$| \varphi(t) - \varphi_h(t) | \leqq CTh^2 \quad (0 \leqq t \leqq T) .$$

The proof is based on an integration of the Rankine—Hugoniot equation

$$\mathrm{d}x/\mathrm{d}t = (f(g(\alpha) - f(u_0))/(g(\alpha) - u_0), \quad x_0 = 0 ,$$

after changing coordinates in terms of the characteristics

$$x = \alpha + tf'(g(\alpha)) .$$

We make a similar change of variables to integrate the corresponding eqution in the Dafermos' method.

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