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In: Miloš Ráb and Jaromír Vosmanský (eds.): Proceedings of Equadiff III, 3rd Czechoslovak Conference on Differential Equations and Their Applications. Brno, Czechoslovakia, August 28 - September 1, 1972. Univ. J. E. Purkyně - Přírodovědecká fakulta, Brno, 1973. Folia Facultatis Scientiarum Naturalium Universitatis Purkynianae Brunensis. Seria Monographia, Tomus I. pp. 211--216.

Persistent URL: http://dml.cz/dmlcz/700092

# VARIATIONAL METHODS IN MATHEMATICAL THEORY OF VISCOELASTICITY

by J. BRILLA

## 1. INTRODUCTION

We shall deal with the differential equation

$$K_{ijkl}(D)\, w_{,ijkl} = L(D)\, q \quad \text{in } \Omega, \tag{1.1}$$

where

$$K_{ijkl}(D) = \sum_{\nu=0}^{p} K_{ijkl}^{(\nu)} D^{\nu}, \tag{1.2}$$

$$L(D) = \sum_{\mu=0}^{s} L_{\mu} D^{\mu}, \tag{1.3}$$

are polynomials in $D = \dfrac{\partial}{\partial t}$. We use the usual indicial notation. Latin subscripts have the range of integers 1, 2 and summation over repeated Latin subscripts is implied. Subscripts preceded by a comma indicate differentiation with respect to corresponding Cartesian spatial coordinates.

We shall consider following boundary conditions

$$w = \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.4}$$

or

$$w = 0, \quad K_{ijkl}(D)\, w_{,ij} \nu_{kn} \nu_{en} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.5}$$

where $\nu_{kn} = \cos(x_k, n)$ and $n$ is the outward normal to $\partial\Omega$. The initial conditions are

$$\frac{\partial^{\varkappa} w}{\partial t^{\varkappa}} = 0 \qquad (\varkappa = 0, 1, 2, \ldots, p-1). \tag{1.6}$$

We assume $\Omega$ is bounded domain in $E_2$ with Lipschitzian boundary $\partial\Omega$.

The equation (1.1) is a differential equation of a viscoelastic plate of a material of the differential type. In the case of real materials it holds

$$K_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \geqq 0 \tag{1.7}$$

for arbitrary values of $\varepsilon_{ij}$, where equality occurs if, and only if, $\varepsilon_{ij} = 0$ for all $i, j$. Further, the coefficients $K_{ijkl}$ are symmetric

$$K_{ijkl} = K_{jikl} = K_{ijlk} = K_{klij} \tag{1.8}$$

and polynomials (1.2–3) have real negative roots.

Simultaneously we shall consider the integrodifferential equation

$$\int_0^t G_{ijkl}(t-\tau) \frac{\partial}{\partial \tau} w_{,ijkl}(\tau)\, d\tau = q \tag{1.9}$$

with boundary conditions (1.4), (1.5) respectively and the first initial condition (1.6). This is the equation of a viscoelastic plate of a material of the integral type. In the real case it holds

$$G_{ijkl}(\tau)\, \varepsilon_{ij}\varepsilon_{kl} \geqq 0 \tag{1.10}$$

and $G_{ijkl}$ are symmetric like $K_{ijkl}$.

## 2. GENERALIZED POTENTIAL ENERGY

Now we shall assume that $q(x, t)$ belongs to the class of slowly increasing functions $U(x, t)$, which fulfil in $\Omega$ for $t > 0$ for each $\delta > 0$ the conditions

$$|U(x, t)| < M(\delta)\, e^{\delta t}, \tag{2.1}$$

where $M(\delta)$ depends on $U$ but does not depend on $x$.

Applying Laplace transform to (1.1) and (1.9) one obtains

$$K_{ijkl}(p)\, \tilde{w}_{,ijkl} = L(p)\, \tilde{q} \tag{2.2}$$

and

$$pG_{ijkl}(p)\, \tilde{w}_{,ijkl} = \tilde{q}. \tag{2.3}$$

Thus in the form of Laplace transform both equations (2.2) and (2.3) are of the same type. Exact solutions of the corresponding boundary value problems are as usual transcendental functions of the transform parameter $p$ [1–3] involving great difficulties in inverse transform. Therefore it appears convenient to apply variational methods.

Making double use of the Green formula it is easy to prove that for both types of boundary conditions (1.4) or (1.5) it holds

$$(K\tilde{w}, \tilde{w})_\Omega = \iint_\Omega K_{ijkl}(p)\, \tilde{w}_{,ijkl}\tilde{w}\, d\Omega = \iint_\Omega K_{ijkl}(p)\, \tilde{w}_{,ij}\tilde{w}_{,kl}\, d\Omega, \tag{2.4}$$

where $p$ is considered as a parameter.

Hence for each real $p$ the operator $K_{ijkl}(p)$ is selfadjoint. We have proved [4] that for each positive real $p$

$$(K\tilde{w}, \tilde{w})_\Omega \geqq \gamma^2 \|\tilde{w}\|^2, \tag{2.5}$$

where

$$\|\tilde{w}\| = \|\tilde{w}(.,p)\| = \left(\iint \tilde{w}^2\, d\Omega\right)^{1/2}. \tag{2.6}$$

Thus $\|\tilde{w}\|_\Omega$ is a function of $p$ and has for each fixed real $p$ the property of the norm. It is obvious that $|(K\tilde{w}, \tilde{w})|$ can be equal to zero if, and only if $\operatorname{Re} p < 0$. Thus we have

**Theorem I.** *If* $L\tilde{q} \in L_2(p)$ *for* Re $p > 0$ *then there exists a weak solution* $\tilde{w}(p) \in W_2^2(p)$ *of the equation* (2.2) *with corresponding boundary conditions. This solution minimizes the functional*

$$2V(\tilde{w}) = (K\tilde{w}, \tilde{w})_\Omega - (L\tilde{q}, \tilde{w})_\Omega - (\tilde{w}, L\tilde{q})_\Omega =$$
$$= \iint_\Omega K_{ijkl}(p)\, \tilde{w},_{ij}\tilde{w},_{kl}\, d\Omega - 2 \iint_\Omega L\tilde{q}\tilde{w}\, d\Omega \tag{2.7}$$

*for each positive real value of* $p$.

Then the solution can be sougt in the form

$$\tilde{w}_n = \tilde{a}_\alpha \varphi_\alpha \quad (\alpha = 1, 2, \dots, n), \tag{2.8}$$

where $\varphi_\alpha$ are the first $n$ terms of a sequence of coordinate functions $\{\varphi_\alpha\}$ complete in the domain under consideration.

Then, inserting (2.8) into (2.7) we arrive at

$$2V(\tilde{w}) = \tilde{a}_\alpha \tilde{a}_\beta (K\varphi_\alpha, \varphi_\beta) - 2\tilde{a}_\beta (L\tilde{q}, \varphi_\beta) \quad (\alpha, \beta = 1, 2, \dots, n) \tag{2.9}$$

which can be minimized by determining the coefficients $\tilde{a}_\alpha$ from the system

$$\frac{\partial V}{\partial \tilde{a}_\beta} = \tilde{a}_\alpha (K\varphi_\alpha, \varphi_\beta) - (L\tilde{q}, \varphi_\beta) = 0. \tag{2.10}$$

Thus $\tilde{a}_\alpha$ are given by the formula

$$\tilde{a}_\alpha = \frac{F_{\alpha\beta}(L\tilde{q}, \varphi_\beta)}{|(K\varphi_\alpha, \varphi_\beta)|}, \tag{2.11}$$

where

$$(K\varphi_\alpha, \varphi_\beta) = \iint_\Omega K_{ijkl}(p)\, \varphi_{\alpha,ij}\varphi_{\beta,kl}\, d\Omega \tag{2.12}$$

and $F_{\alpha\beta}$ is the adjoint matrix and $|(K\varphi_\alpha, \varphi_\beta)|$ the determinant. As the operators $K$ and $L$ are polynomials in $p$, $\tilde{a}_\alpha$ are rational functions of the transform parameter $p$ and the inverse transform can be achieved by the method of decomposition into partial fractions.

Particularly we shall analyze the case when

$$K_{ijkl}(p) = K_{ijkl}^{(0)} + pK_{ijkl}^{(1)}, \qquad L = 1. \tag{2.13}$$

Then the differential equation (2.2) assumes the form

$$(K_{ijkl}^{(0)} + pK_{ijkl}^{(1)})\, \tilde{w},_{ijkl} = \tilde{q} \tag{2.14}$$

and the coefficients of the approximate solution (2.11) are given by

$$a_\alpha = \frac{F_{\alpha\beta}(\tilde{q}, \varphi_\beta)}{|(K\varphi_\alpha, \varphi_\beta)|} = \sum_{\gamma=1}^{n} \frac{A_{\alpha\beta}(p_\gamma)}{p + p_\gamma} (\tilde{q}, \varphi_\beta), \tag{2.15}$$

where $-p_\gamma$ are the roots of the determinantal equation

$$\Delta(p) = |(K\varphi_\alpha, \varphi_\beta)| = 0$$

assumed to be distinct and

$$A_{\alpha\beta}(p) = \frac{F_{\alpha\beta}(-p_\gamma)}{\Delta^{(1)}(-p_\gamma)}, \qquad \Delta^{(1)}(p) = \frac{\mathrm{d}\,\Delta(p)}{\mathrm{d}p}.$$

Then

$$\tilde{w}_n = \sum_{\gamma=1}^{n} \frac{A_{\alpha\beta}(p_\gamma)}{p + p_\gamma} (\tilde{q}, \varphi_\beta)\, \varphi_\alpha \tag{2.16}$$

and the inverse transform is given by the convolutional product

$$w_n = \sum_{\gamma=1}^{n} \varphi_\alpha \int_0^t (q(\tau), \varphi_\beta)\, A_{\alpha\beta}(p_\gamma)\, e^{-p_\gamma(t-\tau)}\, \mathrm{d}\tau. \tag{2.17}$$

In the case when the load $q$ is constant in time

$$q = q(x_1, x_2)\, H(t)$$

and

$$w_n = \sum_{\gamma=1}^{n} \frac{1}{p_\gamma} A_{\alpha\beta}(p_\gamma)\, (q, \varphi_\beta)\, \varphi_\alpha (1 - e^{-p_\gamma t}). \tag{2.18}$$

It is easy to prove that $p_\gamma$ are real and positive, thus the character of the solution is in agreement with what is expected from the physical point of view.

## 3. CONVOLUTIONAL VARIATIONAL PRINCIPLE

Applying the convolutional theorem to the functional (2.7) one obtains

$$2V(w) = \int_0^t [(Kw(\tau), w(t-\tau)) - 2(Lq(\tau), w(t-\tau))]\, \mathrm{d}\tau, \tag{3.1}$$

or

$$2V(w) = \int_0^t \iint_\Omega \Big[ w_{,kl}(t-\tau)\, K_{ijkl}\Big(\frac{\partial}{\partial\tau}\Big) w_{,ij}(\tau) - 2w(t-\tau)\, L\Big(\frac{\partial}{\partial\tau}\Big) q(\tau) \Big]\, \mathrm{d}\Omega\, \mathrm{d}\tau. \tag{3.2}$$

Thus we can formulate the convolutional variational principle.

**Theorem II.** *The first variation* $\delta_1 V$ *of the functional* $V$, *defined by* (3.1) *or* (3.2) *vanishes if, and only if the differential equation* (1.1) *,boundary conditions* (1.4) *or* (1.5) *and initial conditions* (1.6) *are satisfied.*

To prove this theorem we begin with the variation of $w$ as being given by $w(\tau) +$

$+ \delta_1 w(\tau)$. Then the resulting first variation making use of the commutativity property of the convolutional product can be written as

$$\delta_1 V = \int_0^t \big(Kw(\tau) - Lq(\tau), \delta_1 w(t - \tau)\big)\, \mathrm{d}\tau. \tag{3.3}$$

As $\delta_1 w(t - \tau)$ is arbitrary, according to Titchmarsh theorem on convolutional product $\delta_1 V$ vanishes if, and only if $Kw(\tau) - Lq(\tau) = 0$.

Similarly as in the case of variational principles we can use the convolutional variational principle for the approximate solution of equations of viscoelastic plates.

In agreement with the form of functional (3.1) we shall assume that the right hand side function $Lq$ and the solution belong to the class of functions $\mathfrak{M}$ given by

$$\mathfrak{M} = \{\, u(x, t) \mid \| u(., t) \|_\Omega < M\, \mathrm{e}^{\delta t}, \quad \forall \delta > 0\}. \tag{3.4}$$

Then the solution can be sought in the form

$$w_n = a_\alpha(t)\, \varphi_\alpha \quad (\alpha = 1, 2, \dots, n), \tag{3.5}$$

where $\varphi_\alpha$ are the first $n$ terms of a sequence of coordinate functions complete in $W_2^2(\Omega)$.

Inserting (3.5) into (3.1) we arrive at

$$2V(w) = \int_0^t \Big[ \sum_{\nu=0}^{P} a_\alpha^{(\nu)}(\tau)\, a_\beta(t - \tau) \big(K^{(\nu)}\varphi_\alpha, \varphi_\beta\big) - \\ -2\big(Lq(\tau), a_\beta(t - \tau)\, \varphi_\beta\big)\Big]\, \mathrm{d}\tau, \tag{3.6}$$

where we have denoted

$$K = \sum_{\nu=0}^{P} K^{(\nu)} \frac{\partial^\nu}{\partial t^\nu}. \tag{3.7}$$

Then the variation $\delta_1 w_n(t - \tau)$ is given by $\delta_1 a_\alpha(t - \tau)$ and from $\delta_1 V = 0$ one obtains

$$\delta_1 V = \int_0^t \Big[ \sum_{\nu=0}^{P} a_\alpha^{(\nu)}(\tau) \big(K^{(\nu)}\varphi_\alpha, \varphi_\beta\big) - \\ -\big(Lq(\tau), \varphi_\beta\big)\Big]\, \delta_1 a_\beta(t - \tau)\, \mathrm{d}\tau = 0, \tag{3.8}$$

Hence

$$\sum_{\nu=0}^{P} a_\alpha^{(\nu)}(\tau) \big(K^{(\nu)}\varphi_\alpha, \varphi_3\big) - \big(Lq(\tau), \varphi_\beta\big) = 0. \tag{3.9}$$

Thus we have arrived at the system of ordinary differential equations for unknown functions $a_\alpha(t)$ with the initial conditions

$$a_\alpha^{(\nu)}(0) = 0 \quad (\nu = 0, 1, \dots p - 1). \tag{3.10}$$

This is a Cauchy problem and has a unique solution. Inserting this solution in (3.5) we obtain the $n$-th approximation of the considered problem.

Applying Laplace transform to the system (3.9) we arrive at the algebraic system (2.10) for the determining the Laplace transform $\tilde{a}_\alpha(p)$ of $a_\alpha(t)$. Thus the problem of convergence of the $n$-th approximation transforms into the well analysed problem of convergence of the Ritz method for the Laplace transform of the problem [5].

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