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THE GEOMETRIC THEORY OF VOLTERRA INTEGRAL EQUATIONS—A PRELIMINARY REPORT

by GEORGE R. SELL[1])

I. INTRODUCTION

In [1] the author and R. K. Miller presented a point of view for imbedding the solutions of the Volterra integral equation

$$x(t) = f(t) + \int_0^t a(t, s)\, g(x(s), s)\, \mathrm{d}s \tag{V}$$

into a semiflow. Specifically we assume $f(t)$ is a continuous mapping of R^+ into R^n, the solution $x(t)$ is an n-dimensional vector, g a mapping of $R^n \times R^+$ into R^n and $a(t, s)$ an $n \times n$ matrix—valued function. If a and g belong to compatible spaces A and G, respectively, (see [1] for the definition) and $f \in C = C(R^+, R^n)$ then the semiflow on $C \times A \times G$ is given by

$$\pi(f, a, g, \tau) = (T_\tau f, a_\tau, g_\tau), \tag{FV}$$

where

$$g_\tau(x, s) = g(x, \tau + s)$$

$$a_\tau(t, s) = a(\tau + t, \tau + s)$$

and

$$T_\tau f(\vartheta) = f(\tau + \vartheta) + \int_0^\tau a(\tau + \vartheta, s)\, g(x(s), s)\, \mathrm{d}s, \ \vartheta \geqq 0,$$

where $x(.)$ is the given solution of (V). The mapping $T_\tau f$ actually depends on $a(t, s)$, $g(x, s)$ and $x(t)$, as well as $f(t)$, but we shall not include this in our notation.

The purpose of this report is to present some preliminary results from an investigation into the geometric theory of this semiflow. At this point, the results are somewhat fragmentary, so instead of exposing a highly developed theory, my lecture will concentrate more on topics that will lead to further research. The proofs of theorems as well as applications will appear later.

Specifically in this lecture we wish to study the fixed points of this flow, that is points (f, a, g) with the property that $\pi(f, a, g, \tau) = (f, a, g)$ for all $\tau \geqq 0$. We shall see that this gives rise to a study of the solutions of the nonlinear renewal equation, as well as the linear renewal equation.

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II. FIXED POINTS OF THE FLOW

We assume now that $f \in C = C(R^+, R^n)$ and that a and g belong to compatible spaces A and G, respectively. The first theorem characterizes the fixed points of the flow (FV).

Theorem 1. *Let (f, a, g) be a fixed point for the flow (FV). Then a and g have the following form*

$$a(t, s) = a(t - s) \tag{1}$$

$$g(x, s) = g(x), \tag{2}$$

so (V) *becomes the nonlinear renewal equation*

$$x(t) = f(t) + \int_0^t a(t - s)\, g(x(s))\, ds. \tag{N}$$

Furthermore, the corresponding solution $x(t)$ is a constant

$$x(t) \equiv x_0 = f(0), \qquad t \geqq 0,$$

and the function $f(t)$ is absolutely continuous and satisfies

$$f'(t) = -a(t)\, g(x_0), \text{ a.e.} \tag{3}$$

Conversely, if a, g and f satisfy (1), (2) *and* (3), *then the function $x(t) \equiv f(0)$ is a solution of* (V) *and (f, a, g) is a fixed point for the flow* (FV).

III. THE NONLINEAR RENEWAL EQUATION

Let us now turn our attention to the nonlinear renewal equation (N). In this case, the flow (FV) reduces to a flow on C given by

$$\pi(f, \tau) = T_\tau f, \tag{FN}$$

where a and g are now fixed in advance. Let $\mathscr{F}$ denote the fixed points of the flow (FN), the one can prove the following result.

Theorem 2. *With a and g fixed, the collection of fixed points $\mathscr{F}$ is an n-dimensional manifold in $C(R^+, R^n)$. Furthermore, if g is linear, that is $g(x) = Gx$ where G is an $n \times n$-matrix, then $\mathscr{F}$ is a linear subspace of $C(R^+, R^n)$.*

Usually, the first step in the analysis of a nonlinear problem is to try to "linearize" the problem. In our case, if one assumes that g is a C^1-function, then it is possible to compute the Fréchet derivative of $T_\tau f$. Let us denote the derivative by $L_\tau(f)$. Then if $h \in C$ and $\hat{h} = L_\tau(f)\, h$ one has

$$\hat{h}(\vartheta) = h(\tau + \vartheta) + \int_0^\tau a(\tau + \vartheta - s)\, g_x(x(s))\, y(s)\, ds \tag{4}$$

where $x(t)$ is the solution of

$$x(t) = f(t) + \int_0^t a(t - s)\, g(x(s))\, ds,$$

and $y(t)$ is the solution of

$$y(t) = h(t) + \int_0^t a(t - s)\, g_x(x(s))\, y(s)\, ds. \tag{5}$$

In the special case where $x(t) \equiv x_0$, Equation (5) becomes

$$y(t) = h(t) + \int_0^t a(t - s)\, g_x(x_0)\, y(s)\, ds,$$

which is, of course, the linear renewal equation. Furthermore, the formula for $\hat{h}$ given by Equation (4) can also be written in terms of the translation operator T by $\hat{h} = T_\tau h$, where now T_τ depends on the terms given in Equation (5).

The problem of linearization of Volterra integral equations has been studied by several authors, see for example H. Antosiewicz [2], C. Corduneanu [3], [4] and R. K. Miller [5] and [6].

The basic idea here is to use the theory of admissibility to show that if the linearized equation is "admissible" then the nonlinear equation has some appropriate geometric property, such as stability or boundedness. This theory should be pursued further in the context described above, but rather than do that here let us turn our attention to a study of the linear renewal equation.

IV. THE LINEAR RENEWAL EQUATION

We now study the linear equation

$$x(t) = f(t) + \int_0^t a(t - s)\, x(s)\, ds. \tag{L}$$

For this equation the associated flow

$$T_\tau f(\vartheta) = f(\tau + \vartheta) + \int_0^\tau a(\tau + \vartheta - s)\, x(s)\, ds \tag{FL}$$

is linear in f and continuous in f and τ. Furthermore the solution $x(t)$ of τ(L) can be written in the form

$$x(t) = f(t) - \int_0^t r(t - s) f(s)\, ds \tag{6}$$

where the resolvent kernel $r(t)$ is a solution of

$$r(t) = -a(t) + \int_0^t a(t - s)\, r(s)\, ds, \tag{R}$$

or equivalently,

$$r(t) = \sum_{n=1}^{\infty} r_n(t)$$

where $r_1(t) = a(t)$ and

$$r_{n+1}(t) = \int_0^t a(t - s)\, r_n(s)\, \mathrm{d}s,$$

$$n = 1, 2, \ldots$$

Since we can represent solutions of (L) by Equation (6) we see that the asymptotic behavior of solutions of (L) is determined by the asymptotic behavior of the solutions of the resolvent equation (R).

In this sense, the study of the asymptotic behavior of (L) reduces to a study of the asymptotic behavior of (R). This observation was used by R. K. MILLER [7] when he studied scalar-valued equations ($n = 1$) where the kernel a was nonpositive. We shall discuss this observation further in a subsequent paper. Let us now examine the question of eigenvectors for the flow (FL).

V. EIGENVECTORS FOR THE FLOW (FL)

A function $f \in C$ is said to be an eigenvector for the flow $T_\tau f$ if there is a continuous function $\lambda(\tau)$ such that

$$T_\tau f = \lambda(\tau) f. \tag{7}$$

If $\lambda(\tau) \equiv 1$, then we see that the eigenvectors associated with $\lambda(\tau)$ are precisely the fixed points of T_τ. Equation (7) can be rewritten as

$$\lambda(\tau) f(\vartheta) = f(\tau + \vartheta) + \int_0^\tau a(\tau + \vartheta - s)\, x(s)\, \mathrm{d}s. \tag{8}$$

If we set $\vartheta = 0$ in Equation (8), then we see that

$$x(\tau) = \lambda(\tau) f(0),$$

that is $\lambda(\tau) f(0)$ is a solution of (L).

Let us now look for eigenvectors with $\lambda(\tau)$ assuming a special form, say $\lambda(\tau) = e^{\nu\tau}$.

One can then prove the following interesting result, which includes the linear version of Theorem 1 as a special case.

Theorem 3. *A function $e^{\nu t} x_0$ is a solution of* (L) *if and only if the function $f(t)$ has the form*

$$f(t) = e^{\nu t}[x_0 - \int_0^t e^{-\nu s} a(s)\, x_0\, \mathrm{d}s]. \tag{9}$$

Furthermore, in this case f is an eigenvector satisfying

$$T_\tau f = e^{\nu\tau} f. \tag{10}$$

Conversely, if f is an eigenvector satisfying Equation (10), *then f also satisfies Equation* (9) *and* $e^{\nu t}f(0)$ *is a solution of* (L).

In addition for each ν, *the collection of vectors f satisfying Equation* (10) *forms an n-dimensional linear subspace of* $C(R^+, R^n)$.

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