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In: Miloš Ráb and Jaromír Vosmanský (eds.): Proceedings of Equadiff III, 3rd Czechoslovak Conference on Differential Equations and Their Applications. Brno, Czechoslovakia, August 28 - September 1, 1972. Univ. J. E. Purkyně - Přírodovědecká fakulta, Brno, 1973. Folia Facultatis Scientiarum Naturalium Universitatis Purkynianae Brunensis. Seria Monographia, Tomus I. pp. 31--42.

Persistent URL: http://dml.cz/dmlcz/700088

ON LIMIT-POINT AND SEPARATION CRITERIA FOR LINEAR DIFFERENTIAL EXPRESSIONS

by W. N. EVERITT and M. GIERTZ

INTRODUCTION

This paper is concerned mainly with reporting on some properties of the ordinary differential expression $M[\cdot]$ defined, in terms of two real-valued coefficients p and q, on an interval I of the real line R by

$$M[f] = -(pf')' + qf \quad \text{on} \quad I \quad \left(' = \frac{\mathrm{d}}{\mathrm{d}x}\right). \tag{0.1}$$

However some of the properties considered extend in a natural way to the partial differential expression $\mathscr{M}[\cdot]$ given by

$$\mathscr{M}[f] = -\Delta_n[f] + qf \text{ on } G, \tag{0.2}$$

where $\Delta_n[\cdot]$ is the Laplacian in R^n, G is an open connected domain of R^n and q is a real-valued coefficient defined on G.

The paper is in four sections.

The first section is concerned with properties of $M[\cdot]$ and details limit-point and separation results. The second section gives some corresponding properties of the differential expression $M^n[\cdot]$ obtained by taking formal powers of $M[\cdot]$.

In the third section we give some separation results for the partial differential expression $\mathscr{M}[\cdot]$.

Some unsolved problems are listed in the fourth section.

Finally there is a list of references.

NOTATIONS

The real and complex number fields are denoted by R and C; euclidean space of n dimensions by R^n. We use [and (to indicate that an interval of R is closed or open, respectively, at its left-hand end point. Similarly at a right-hand end point.

For any interval I of R the complex Hilbert function space on I is denoted by $L^2(I)$. The symbol $L^2_{\text{loc}}(I)$ denotes the collection of all complex-valued functions on I which are integrable-square (Lebesgue) on every compact sub-interval of I. Similarly for the function spaces $L^r(I)$ where $1 < r < \infty$.

$AC_{\text{loc}}(I)$ denotes the collection of all complex-valued functions on I which are absolutely continuous on every compact sub-interval of I.

1. THE DIFFERENTIAL EXPRESSION $M[\cdot]$

Let the coefficients p and q in the differential expression $M[\cdot]$, given by (0.1), satisfy the following basic conditions, here I is a given interval of R,

$$p, q : I \to R, \quad q \in L^2_{\text{loc}}(I) \quad \text{and} \tag{1.1}$$
$$p \in AC_{\text{loc}}(I), \quad p(x) > 0 \quad (x \in I), \quad p' \in L^2_{\text{loc}}(I).$$

It then follows, in the sense of NAIMARK [15] section 15.1, that $M[\cdot]$ is *regular* at every point of I, i.e. every initial value problem of the differential equation

$$M[y](x) = 0 \quad (x \in I) \tag{1.2}$$

may be solved at any point of I; see [15] sections 16.1 and 16.2.

Let the left-hand end point of I be a. We say that this end point is *singular* for $M[\cdot]$ if *either* a is infinite, i.e. $a = -\infty$, *or* if a is finite and the initial value problem for (1.2) cannot be solved in general at that point. A similar definition holds at the right-hand end point of I, say b.

According to the original definition of HERMANN WEYL, see [19] or [16] sections 2.1 and 2.19, every singular end point of $M[\cdot]$ is classified as either *limit-point* or *limit-circle*. If the left-hand end point a of I is singular for $M[\cdot]$ and c is an arbitrary interior point of I then $M[\cdot]$ is limit-circle at a if and only if every solution y of (1.2) is in $L^2(a, c)$; otherwise $M[\cdot]$ is limit-point at a. Similar definitions hold at the right-hand end point b of I when this point is singular for $M[\cdot]$.

To define separation of $M[\cdot]$ in $L^2(I)$ we introduce the linear manifold $D_1 \equiv \equiv D_1(p, q) \subset L^2(I)$ with the definition

$$D_1 = \{f \in L^2(I) : f' \in AC_{\text{loc}}(I) \quad \text{and} \quad M[f] \in L^2(I)\}. \tag{1.3}$$

(D_1 is the maximal set within $L^2(I)$ for which $M[f]$ is defined and in $L^2(I)$.) The differential expression $M[\cdot]$, for which the coefficients p and q satisfy the basic conditions (1.1), is said to be *separated in* $L^2(I)$ if

$$qf \in L^2(I) \quad \text{for all} \quad f \in D_1 \tag{1.4}$$

or, equivalently, if

$$(pf')' \in L^2(I) \quad \text{for all} \quad f \in D_1.$$

In general, $M[\cdot]$ is not separated, see [14] VI remark 4.1 and also the examples in [4] sections 6 and 7.

We note that the conditions (1.1) on p and q, and the definition (1.3) of D_1, imply that

$$qf \in L^2_{\text{loc}}(I) \quad \text{for all} \quad f \in D_1 \tag{1.5}$$

so that whether (1.4) is satisfied or not, i.e. whether $M[\cdot]$ is separated in $L^2(I)$ or not, depends only on the properties of $M[\cdot]$ at the end points of I. It is for this reason that the L^2_{loc} conditions on p, p' and q are introduced into the basic conditions (1.1).

This also means that each end point of I may be treated separately in attempting to establish separation of $M[\cdot]$ in $L^2(I)$.

The concept of separation is connected with inequalities of the following form, here A, B, C and D are non-negative real numbers,

$$A\int_I |(pf')'|^2 + B\int_I p\,|q|\,|f'|^2 + C\int_I q^2\,|f|^2 \leqq \int_I |M[f]|^2 + D\int_I |f|^2 \quad (1.6)$$

which are to hold for all $f \in D_1$. We give examples of such inequalities below.

Separation is also connected with the properties of unbounded differential operators generated by $M[\cdot]$ in the Hilbert space $L^2(I)$; see for example [4] section 1 and [7] sections 3, 4 and 5. In particular inequalities of the form (1.6) may be used to obtain results in the theory of relatively bounded perturbations of such operators; see [7] section 9. However we shall not consider such aspects of separation in this paper.

We consider now some separation results under various conditions on the coefficients p and q.

(1) *The case when $I = [a, \infty)$ for general p and q*

Suppose in addition to p and q satisfying the basic conditions (1.1) we have also

$$q \in AC_{\mathrm{loc}}[a, \infty) \quad \text{and} \quad q(x) \geqq 0 \ (x \in [a, \infty)), \quad (1.7)$$

and for some ε satisfying $0 < \varepsilon < 1$

$$\{p(x)\}^{1/2}\,|q'(x)| \leqq (1 - \varepsilon)\,\{q(x)\}^{3/2} \quad (x \in [a, \infty)), \quad (1.8)$$

then $M[\cdot]$ is limit-point at ∞, since q is bounded below, and so the result in [3] chapter XIII, section 6.14 applies; and $M[\cdot]$ is separated in $L^2(a, \infty)$ from the result in [7], (2) of theorem 2. If in addition to (1.1, 7 and 8) the coefficient p is such that the following inequality is satisfied

$$\{\int_a^\infty p\,|f'|^2\}^2 \leqq K\int_a^\infty |f|^2 \int_a^\infty |(pf')'|^2 \quad (f \in D_1) \quad (1.9)$$

for some K satisfying $0 < K < \infty$, then for any $\delta \in (0, 1)$ we have an inequality of the form (1.6)

$$\delta\int_a^\infty |(pf')'|^2 + (1 + \varepsilon)\int_a^\infty pq\,|f'|^2 + \varepsilon\int_a^\infty |f|^2 \leqq$$

$$\leqq \int_a^\infty |M[f]|^2 + (1 + \sqrt{K})^2\{q(a)\}^2(1 - \delta)^{-1}\int_a^\infty |f|^2 \quad (f \in D_1);$$

for a proof of this result see [7] theorem 3.

A condition of a seemingly different nature is given by DUNFORD and SCHWARTZ in [3] chapter XIII, 9 B5, where it is stated that separation occurs if p is positive, q is real and

$$\limsup_{\infty}\,\{|(pq')'|\,q^2\} < 1. \quad (1.10)$$

However there appears to be some misprint here, since (1.10) admits the case $p(x) = 1, q(x) = -x$ $(x \in [0, \infty))$ for which separation does not occur; see [4] section 7.

(2) *The case when* $I = [a, \infty)$ *and* $p = 1$

Consider the case when $I = [a, \infty)$ and

$$p(x) = 1 \quad (x \in [a, \infty)); \tag{1.11}$$

in this case a number of conditions on q are known which result in separation. Before stating them it is convenient to make the following definition of a function $P(\cdot)$ which maps $[0, \infty)$ into sets of the coefficient q, as follows:

$$q \in P(\gamma), \text{ where } \gamma \geqq 0, \text{ if } q \in L^{2\gamma}_{\mathrm{loc}}[a, \infty) \cap L^2_{\mathrm{loc}}[a, \infty)$$
$$\text{and } |q|^{\gamma} f \in L^2(a, \infty) \text{ for all } f \in D_1 \equiv D_1(q).$$

We note that $P(\gamma) \supseteq P(\beta)$ if $0 \leqq \gamma < \beta$ and, from (1.4), that $M[\cdot]$ is separated in $L^2(a, \infty)$ if and only if $q \in P(1)$. We may note also that if $q \in P(1/2)$ then $M[\cdot]$ is limit-point at ∞; this follows from an argument similar to that given in [4] section 1 or [12] section 3, lemma 2.

Assuming (1.1) to hold, i.e. $q \in L^2_{\mathrm{loc}}[a, \infty)$, each of the following additional conditions on q implies that $M[\cdot]$ is separated in $L^2(a, \infty)$ or, equivalently, that $q \in P(1)$.

The method of proof in these cases below are all entirely different in nature. It should be noted that in these conditions separation still occurs if q is replaced by $q + k$, for any real number k, and the conditions are then satisfied on an interval of the form $[X, \infty)$ where $X \geqq a$ is taken to be sufficiently large.

Since $q \in P(1)$ implies that $q \in P(1/2)$ each condition also ensures that $M[\cdot]$ is limit-point at ∞.

i) The first condition, given in [4], is

$$q' \in AC_{\mathrm{loc}}[a, \infty), \quad q(x) > 0 \quad (x \in [a, \infty))$$
$$\text{and } \int_a^{\infty} \{q^{-1/4}(q^{-1/4})''\} < \infty \tag{1.12}$$

(ii) The condition given in (1.8) above may in the case $p = 1$ be improved to

$$q \in AC_{\mathrm{loc}}[a, \infty), \quad q(x) \geqq 0 \quad (x \in [a, \infty))$$
$$\text{and for some } c \in (0, 2) \tag{1.13}$$
$$|q'(x)| \leqq c\{q(x)\}^{3/2} \qquad (x \in [a, \infty)).$$

(iii) F. V. ATKINSON gives, in [1], a condition which is less demanding on the regularity of q. In the circumstances when q is differentiable this condition takes the form

$$q \in AC_{\text{loc}}[a, \infty), \; q(x) > 0 \quad (x \in [a, \infty))$$
$$\text{and} \quad -(1/3)\,[-C + \sqrt{16 + C^2}] < q'(x)\,\{q(x)\}^{-3/2} < C \qquad (1.14)$$
$$\text{for some } \; C \in (0, 4/\sqrt{3}).$$

An example in [8] shows that the upper bound of $4/\sqrt{3}$ is best possible.

(iv) The condition

$$q \in L^r(a, \infty) \quad \text{for some} \quad r \in [2, \infty) \qquad (1.15)$$

is discussed in detail in [12].

We make the following remarks on the conditions (i) to (iv) above. Firstly (i) to (iii) essentially restrict the oscilatory behaviour of q; however it may be verified that such conditions are satisfied by all reasonable monotonic increasing functions, e.g. x^τ for all $\tau \geqq 0$, $\log x$, e^x, $\exp\{e^x\}$, $\log\log x$, and also by moderately oscillating functions. Secondly (iv) above does not restrict the oscillatory nature of q, which may be unbounded above and below, but does require q to be globally small at ∞.

As to the $P(\cdot)$ classes to which q may belong we record here the following results (again note the basic condition (1.1) on q, i.e. $q \in L^2_{\text{loc}}[a, \infty)$):

(i) if q is bounded below on $[a, \infty)$ then $q \in P\left(\frac{1}{2}\right)$; see [4] section 1.

(ii) if (1.12) holds and if, in addition, $q(x) \geqq (\log x)^\beta$ (for all sufficiently large x) where $\beta > 0$, and for K satisfying $0 < K < \infty$,

$$q^{-3/4}\,|\,(q^{-1/4})''\,| < K \quad \text{on} \quad [a, \infty), \qquad (1.16)$$

then $q \in P(1)$ but $q \notin P(1 + \delta)$ for any $\delta > 0$; see [4] section 5; note that (1.16) is satisfied by all the examples listed in the previous paragraph.

(iii) given any $\beta > 0$ it is possible to construct a coefficient q_β which is bounded below on $[a, \infty)$ $\left(\text{i.e. } q \in P\left(\frac{1}{2}\right) \text{ from (i) above}\right)$ such that $q_\beta \in P\left(\frac{1}{2} + \beta\right)$ but $q_\beta \notin P\left(\frac{1}{2} + \beta + \delta\right)$ for any $\delta > 0$; see [4] sections 1 and 6.

(iv) from results in [12] it is shown that if (1.15) is satisfied for some index $r \in [1, \infty)$ then $q \in P\left(\frac{1}{2} r\right)$ and that this result is best possible; we note that with the index r in this range we have $q \in P\left(\frac{1}{2}\right)$ and so $M[\cdot]$ is limit-point at ∞.

(v) if

$$q(x) = -x^\tau \qquad (x \in [a, \infty), \, a \geqq 0)$$

where $0 < \tau \leqq 2$ then $q \in P(0)$ but $q \notin P(\delta)$ for any $\delta > 0$; see [4] section 7; in this case $M[\cdot]$ is limit-point at ∞.

Finally in this case when $I = [a, \infty)$ and $p = 1$ we mention that there are many interesting inequalities of the form (1.6); some of these may be found in [7] but a more detailed account is given [9].

(3) *The case when* $I = (-\infty, \infty)$ *and* $p = 1$

Again (1.1) is to be satisfied but now on $(-\infty, \infty)$.

We give here some of the number of inequalities of the form (1.6) taken from the detailed account in [9].

Suppose that

$$q \in AC_{\text{loc}}(-\infty, \infty) \quad \text{and} \quad q(x) \geqq 0 \qquad (x \in (-\infty, \infty)) \tag{1.17}$$

and also satisfies

$$| q'(x) | \leqq c\{q(x)\}^{3/2} \qquad (x \in (-\infty, \infty)) \tag{1.18}$$

where the number $c \in (0, 2)$; then we have the following inequalities (where the constants appearing on the left-hand side are best possible, from [9] section 2)

$$\begin{gathered} \left(1 - \frac{1}{4}c^2\right) \| qf \| \leqq \| M[f] \| \qquad (f \in D_1(q)), \\ (2 - c) \| q^{1/2}f' \| \leqq \| M[f] \| \qquad (f \in D_1(q)) \\ \min\{1, 4c^{-2} - 1\} \| f'' \| \leqq \| M[f] \| \qquad (f \in D_1(q)). \end{gathered} \tag{1.19}$$

Also if in (1.18) we have $c \in (0, \sqrt{2})$ and if $\delta \in \left[\frac{1}{2}c, c^{-1}\right]$ then

$$(1 - \delta c) \| qf \|^2 + (2 - c\delta^{-1}) \| q^{1/2}f' \|^2 + \| f'' \|^2 \leqq \| M[f] \|^2 \quad (f \in D_1(q)). \tag{1.20}$$

It is clear that in all such cases $M[\cdot]$ is separated in $L^2(-\infty, \infty)$; we note also that (1.17) implies $M[\cdot]$ is limit-point at both $-\infty$ and ∞.

(4) *The case when I is bounded and* $p = 1$

Separation and limit-point results are also available in the case of bounded intervals and such problems are considered in detail in [9] and [10]. Many inequalities of the type (1.19 and 20) are given in [9] thus yielding separation results. In [10] results are proved which lead to establishing $q \in P\left(\frac{1}{2}\right)$ in the case of finite intervals with singular behaviour in q at the end points.

2. THE CASE OF POWERS OF $M[\cdot]$

Separation and generalised limit-point results for the formal powers $M^n[\cdot]$, where n is a positive integer, of the differential expression

$$M[f] = -f'' + qf \quad \text{on} \quad [a, \infty) \tag{2.1}$$

are given in [5] and [6].

We write, inductively, $M^n[\cdot] = M[M^{n-1}[\cdot]]$ and then put

$$M^n[f] = \sum_{k=0}^{2n} a_{nk} f^{(k)} \quad \text{on} \quad [a, \infty) \qquad (n = 2, 3, \ldots), \tag{2.2}$$

where $f^{(k)}$ denotes the k-derivative of f and the coefficients a_{nk} depend on the powers and derivatives of q. We shall assume here, given the integer n, that

$$q^{(2n-3)} \in AC_{\text{loc}}[a, \infty) \tag{2.3}$$

in which case all the coefficients a_{nk} are defined almost everywhere on $[a, \infty)$. As an example we have

$$M^2[f] = f^{(4)} - 2qf^{(2)} - 2q'f' + (q^2 - q^{(2)})f. \tag{2.4}$$

Roughly speaking separation in $L^2(a, \infty)$ for $M^n[\cdot]$ is the property that f and $M^n[f]$ in $L^2(a, \infty)$ should imply that every term on the right-hand side of (2.2) should separately also be in $L^2(a, \infty)$.

When (2.3) is satisfied the power $M^n[\cdot]$ exists as a differential expression and is formally symmetric on $[a, \infty)$. As an example we may rewrite (2.4) above in the symmetric form

$$M^2[f] = f^{(4)} - (2qf')' + (q^2 - q^{(2)})f \quad \text{on} \quad [a, \infty). \tag{2.5}$$

Thus in the general case $M^n[\cdot]$ may be used to generate self-adjoint, unbounded differential operators in $L^2(a, \infty)$. Separation and the generalisation of the limit-point/limit-circle classification for $M^n[\cdot]$ are connected with the properties of these operators and their relatively bounded perturbations but we shall not be concerned with such matters in this paper; however some details may be found in [5].

We state two separation results here. The first is the partial separation result as given in [5] and the second is the complete separation result from [6]. As in the case when $n = 1$ some control on the coefficient q is necessary in order to obtain these separation results.

(1) *Partial separation for* $M^n[\cdot]$

From [5] we have the following result:

$$\begin{aligned} \text{if } &\text{(i) } q^{(2n-3)} \in AC_{\text{loc}} \quad [a, \infty), \\ &\text{(ii) } q(x) \geqq d > 0 \quad (x \in [a, \infty)), \\ &\text{(iii) } q^{-1/4}(q^{-1/4})'' \in L(a, \infty), \end{aligned} \tag{2.6}$$

then for all $f \in L^2(a, \infty)$ *for which* $f^{(2n-1)} \in AC_{\text{loc}}[a, \infty)$ *and* $M^n[f] \in L^2(a, \infty)$ *we have*

$$M^r[f] \in L^2(a, \infty) \qquad (r = 1, 2, \ldots, n-1). \tag{2.7}$$

Thus all the in-between powers $M^r[f]$ are in $L^2(a, \infty)$ when both f and $M^n[f]$ are in $L^2(a, \infty)$.

It is somewhat remarkable that this result requires only the control condition (iii) on q, q' and q'' and no restriction on the behaviour of the higher derivatives of q.

(2) *Complete separation for* $M^n[\cdot]$

From [6] we have the following result:
if q satisfies (i), (ii) and (iii) of (2.6) *and, in addition, (iv) there exist positive numbers C_k such that*

$$|q^{(k)}| \leqq C_k q^{1+k\frac{1}{2}} \text{ on } [a, \infty) \qquad (k = 2, 3, \ldots, 2n-3) \tag{2.8}$$

then for all $f \in L^2(a, \infty)$ for which $f^{(2n-1)} \in AC_{\text{loc}}[a, \infty)$ and $M^n[f] \in L^2(a, \infty)$ we have

$$\text{(a)} \quad a_{nk} f^{(k)} \in L^2(a, \infty) \qquad (k = 0, 1, 2, \ldots, 2n)$$

(*where the coefficients a_{nk} are given in* (2.2))

(b) *whenever* $m_1, m_2, \ldots, m_k$ and $l_1, l_2, \ldots, l_k$

are any non-negative integers satisfying

$$\sum_{r=1}^{k} (m_r + l_r) \leqq 2n, \qquad \textit{then}$$

$$\left\{\frac{d}{dx}\right\}^{m_1} \{\sqrt{q}\}^{l_1} \left\{\frac{d}{dx}\right\}^{m_2} \cdots \left\{\frac{d}{dx}\right\}^{m_k} \{\sqrt{q}\}^{l_k} f \in L^2(a, x). \tag{2.9}$$

The result (a) shows that separation in $L^2(a, \infty)$ is complete; no separate term of $M^n[f]$ escapes membership of $L^2(a, \infty)$. Result (b) is an even more detailed separation result and contains (a) as a special case.

The proofs of these results may be found in [5] and [6]. We remark here only that it was found essential to prove partial separation as a stage in the proof of complete separation.

If q is infinitely differentiable and satisfies conditions very much like (2.6) and (2.8), then any polynomial in d/dx and $\sqrt{q}$ maps $D_1(q)$ into $L^2(a, \infty)$. This means that we have a limiting complete separation result in the sense that (2.9) holds true without any restriction on the exponents m_i and l_i. We refer to [13] § 4 for further details.

3. THE PARTIAL DIFFERENTIAL EXPRESSION $\mathscr{M}[\cdot]$

Separation results and inequalities of the type discussed in section 1 hold true for partial differential expressions of the form $\mathscr{M}[\cdot]$, as given by (0.2), in higher dimensions. In fact, they extend in a natural way to functions which posses derivatives in a generalised sense as detailed below; see also the book by KATO, [14] chapter V, section 5.2, and the book by YOSHIDA [18] chapter I, section 8.

Given an open, connected domain G in n-dimensional euclidean space R^n, with points $x = (x_1, x_2, \ldots, x_n)$, we define the generalised differential operators D_k $(k = 1, 2, \ldots, n)$ as follows:

the domain of D_k consists of all $f \in L_{\text{loc}}(G)$ for which there exists a $g \in L_{\text{loc}}(G)$ such that

$$\int_G \left\{ f \frac{\partial h}{\partial x_k} \right\} = - \int_G \{gh\}, \qquad (h \in C_0^\infty(G)); \tag{3.1}$$

here $C_0^\infty(G)$ is the set of all complex-valued functions with compact support in G which are infinitely differentiable, in all variables, in the classical sense. For all such f we then put $D_k f = g$. If f is locally absolutely continuous in x_k then we have $D_k f = \dfrac{\partial f}{\partial x_k}$.

The differential expressions $\mathcal{M}[\cdot]$ we are concerned with here are of the form

$$\mathcal{M}[f] = -\Delta_n[f] + qf, \tag{3.2}$$

where now $\Delta_n[\cdot] = \sum_{k=1}^{n} D_k^2$ is the generalised Laplacian and the real-valued coefficient $q \in L_{\text{loc}}^2(G)$ in analogy with (1.1).

As in the one-dimensional case we say that $\mathcal{M}[\cdot]$ is *separated in* $L^2(G)$ if

$$qf \in L^2(G) \quad \text{for all} \quad f \in D_1(q), \tag{3.3}$$

where the set $D_1(q)$ is defined by

$$D_1(q) = \{f \in L^2(G) : \Delta_n[f] \quad \text{exists as above and} \quad \mathcal{M}[f] \in L^2(G)\} \tag{3.4}$$

Assume now that $G = R^n$ and that $q : R^n \to (0, \infty)$ is such that $D_k q$ $(k = 1, 2, \dots, n)$ exists and, for some real numbers c_k with $\sum_{k=1}^{n} c_k^2 = c^2 < 4$, satisfies the conditions

$$| D_k q(x) | \leqq c_k \{q(x)\}^{3/2} \qquad (x \in R^n;\ k = 1, 2, \dots, n) \tag{3.5}$$

Then inequalities of the form

$$a_0 \| qf \|^2 + a_{n+1} \| \Delta_n[f] \|^2 + \sum_{k=1}^{n} a_k \| q^{1/2} D_k f \|^2 \leqq \| \mathcal{M}[f] \|^2 \qquad (f \in D_1(q)) \tag{3.6}$$

hold true with $a_0, a_1, \dots, a_{n+1}$ non-negative and not all zero. Explicit values for the numbers a_k in terms of $c = (c_1, c_2, \dots, c_n)$ and some parameters are given in [11]; in particular the inequalities in (1.19) generalise to

$$\begin{gathered} (1 - c^2/4) \| qf \| \leqq \| \mathcal{M}[f] \| \qquad (f \in D_1(q)) \\ (2 - c_k) \| q^{1/2} D_k f \| \leqq \| \mathcal{M}[f] \| \qquad (f \in D_1(q);\ k = 1, 2, \dots, n) \\ \min\{1, 4c^{-2} - 1\} \| \Delta_n[f] \| \leqq \| \mathcal{M}[f] \| \qquad (f \in D_1(q)). \end{gathered} \tag{3.7}$$

As another example we have the following generalisation of (1.20): for any $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ with $\delta_k \geqq \frac{1}{2} c_k$ and $\delta c = \sum_{k=1}^{n} \delta_k c_k \leqq 1$,

$$(1 - \delta c) \| qf \|^2 + \sum_{k=1}^{n} (2 - c_k/\delta_k) \| q^{1/2} D_k f \|^2 + \| \Delta_n f \|^2 \leqq \| \mathscr{M}[f] \|^2 \quad (f \in D_1(q)). \tag{3.8}$$

These results clearly imply that $\mathscr{M}[\cdot]$ is separated in $L^2(R^n)$; in fact $D_k^2 f$ and $q^{1/2} D_k f$ $(k = 1, 2, \ldots, n)$ all belong to $L^2(R^n)$ for all $f \in D_1(q)$.

The proofs of results of which (3.7) and (3.8) are special cases will be found in [11]; they depend in part on a result given in the book [17] by TITCHMARSH.

Similar results hold for other open subsets G of R^n; details will be found in [11].

As in the case of ordinary differential expressions, separation for partial differential expressions is connected with the properties of the unbounded differential operators generated by $\mathscr{M}[\cdot]$ in $L^2(G)$. We do not consider this aspect here but some details may be found in [11].

4. SOME UNSOLVED PROBLEMS

We list here some unsolved problems in the limit-point classification and separation areas for the ordinary differential expression $M[\cdot]$; also for the partial differential expression $\mathscr{M}[\cdot]$.

(A) *The ordinary case*

(1) When $M[f] = -(pf')' + qf$ on I, and it is known that $M[\cdot]$ is separated in $L^2(I)$, is it the case that there is always an inequality of the form (1.6) valid on $D_1(p, q)$?

In particular when $I = [a, \infty)$, $p = 1$ and $q \in L^r(a, \infty)$ for some $r \in [2, \infty)$ (so that from (2) (iv) of section 1 above, $M[\cdot]$ is separated in $L^2(a, \infty)$) is there an inequality of the form

$$A \| qf \|^2 + B \| |q|^{1/2} f' \|^2 + C \| f'' \|^2 \leqq \| M[f] \|^2 + D \| f \|^2 \quad (f \in D_1(q))? \tag{4.1}$$

The proof of separation in this case, given in [12], throws no light on this problem.

(2) When $M[f] = -f'' + qf$ on I we may define the non-negative number $V(q)$ as follows (recall the definition of $P(.)$ from (2) of section 1 above)

$$V(q) = \sup \{\gamma : q \in P(\gamma)\}$$

i.e. $q \in P(V(q) - \varepsilon)$ for all $\varepsilon > 0$; but does $q \in P(V(q))$? All the available examples give an affirmative answer to this question.

(3) There are many interesting problems associated with the inequalities of the form (1.6) given in [7], [8], [9], and [10]. If $M[f] = -f'' + qf$ on $(-\infty, \infty)$ and

$q \geqq 0$ satisfies the condition, see (1.18), $|q'| \leqq cq^{3/2}$ on $(-\infty, \infty)$ do we obtain an inequality of the form (4.1) *when* $c \in [2, 4/\sqrt{3}]$? If there is no inequality do we still obtain separation in $L^2(-\infty, \infty)$? It is known from the example given in [8], which is technically complicated, that the answer to these questions is in the negative when $c \geqq 4/\sqrt{3}$.

(4) When $M[f] = -f'' + qf$ on $[a, \infty)$ there is detailed information about the range of the $P(.)$ mapping; see the examples in [4]. Is it possible to construct similar examples when I is bounded and q has a singularity at one or both end points?

(5) When $M[f] = -f'' + qf$ on $[a, \infty)$ and $M[.]$ is *limit-circle* at ∞ is it the case that $q \in P(0)$ but $q \notin P(\delta)$ for any $\delta > 0$? In this case it is known that $q \in P\left(\frac{1}{2}\right)$.

(6) When $M[f] = -f'' + qf$ on $[a, \infty)$ and $q \in L^r(a, \infty)$ with $r \in (0,1)$ what may be said of the limit-point/limit-circle classification of $M[.]$? Is $M[.]$ separated in $L^2(a, \infty)$?

(B) *The partial case*

We mention only one problem; is the differential expression $\mathscr{M}[f] = -\Delta_n[f] + qf$ on R^n separated in $L^2(R^n)$ when $q \in L^r(R^n)$ and the index $r \in [2, \infty)$?

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