GENERIC ONE-PARAMETER FLOWS ON THE TORUS AND BIFURCATIONS OF PERIODIC ORBITS

by PAVOL BRUNOVSKÝ

§ 1

We consider differential equations of the form

$$\begin{aligned} \dot{x} &= \varphi(\mu, x, y) \\ \dot{y} &= \psi(\mu, x, y) \end{aligned} \qquad (E_\mu)$$

where $x, y \in R^1$, the scalar parameter $\mu \in P(= R^1$ or $S^1)$, φ, ψ are $C^r(2 \leqq r \leqq \infty)$ and periodic with period 1 in x and y, and $\varphi(\mu, x, y) > 0$ for all μ, x, y. Because of the periodicity assumption (E_μ) can be identified with a vector field on the torus $S^1 \times S^1$.

The basic theory of differential equations on the torus has become a standard part of the books on differential equations (cf., e.g. [1]). Recently, the problems of topological classification and structural stability as well as dependence of the flow of a differential equation on the torus on parameters has been brought up by several authors (cf. [2–6]).

To study the qualitative behavior of the flow of (E_μ) one associates with (E_μ) a C^r diffeomorphism $f_\mu : R^1 \to R^1$ obtained as follows: $f_\mu(y)$ is equal to the y-coordinate of the first intersection with the line $x = 1$ of the trajectory of (E_μ) through $(0, y)$. Obviously,

$$f_\mu(y + 1) = f_\mu(y) + 1 \text{ for all } y \qquad (1)$$

and f_μ mod 1 is a C^r diffeomorphism of S^1.

From the basic theory one knows of the existence of an important topological invariant of f_μ, the rotation number $\varrho(f_\mu) = \lim_{n\to\infty} n^{-1} f_\mu^n(y)$, which is independent of y. Furthermore, one knows that if $\varrho(f_\mu)$ is rational (equal, say, to kl^{-1}, k, l, relatively prime), then f_μ has at least one l-periodic point (i.e. a point y such that $f_\mu^l(y) - y$ is integer and $f_\mu^j(y) - y$ is not integer for $0 < j < l$), i.e. (E_μ) has at least one periodic trajectory; if $\varrho(f_\mu)$ is irrational, then by a continuous transformation of coordinates, f_μ can be brought to the translation $y + \varrho(f_\mu)$.

We are interested in the typical behavior of $\varrho(f_\mu)$ if μ varies. Our motivation for this problem stems from the problem of generic bifurcation of periodic trajectories of differential equations: for other motivations, cf., e.g. [7]. This problem is also closely related to the problems studied in [2–6].

To make precise what we mean by typical (or, *generic*) behavior, we denote by $\mathscr{F}$ the space of all C^r maps $f : P \times R^1 \to R^1$ such that $f_\mu : R^1 \to R^1$ defined as $f_\mu(y) =$

$= f(\mu, y)$ is a diffeomorphism satisfying (1), endowed with the C^r Whitney topology. A property of maps from $\mathscr{F}$ is called generic if it is valid for every f from some residual subset of $\mathscr{F}$. One can make the corresponding definition for the space of differential equations (E_μ) and show the equivalence of genericity in the space $\mathscr{F}$ and the space of differential equations (E_μ).

Denote $\sigma_f(\mu) = \varrho(f_\mu)$. By a *μ-orbit* of a point y_0 we understand the set of points $f_\mu^j(y_0)$, j integer.

From [2, 3] we know that σ_f is continuous, but not Lipschitz continuous in general. From the results of [8] it follows that, generically, the loci $Z_l(f)$ of l-periodic points of f_μ, $\mu \in P$ are 1-dimensional submanifolds of $P \times R^1$, intersecting every set $\Sigma_\mu = \{\mu\} \times R^1$ in isolated points. Furthermore, if $Z_l(f)$ does not intersect Σ_μ transversally at y, then $(f_\mu^l)''(y) \neq 0$, $\dfrac{\partial f_\mu^l}{\partial \mu}(y) \neq 0$.

From these facts we obtain the following.

Proposition 1. *There is a residual subset $\mathscr{F}_1$ of $\mathscr{F}$ such that for every $f \in \mathscr{F}_1$ the following is valid*:

For ϱ rational $\sigma_f^{-1}(\varrho)$ is a disjoint union of closed intervals with non-empty interiors. If $\varrho = kl^{-1}$ (k, l relatively prime), then for any right (*left*) *endpoint μ_0 of such interval, there is exactly one μ_0-orbit of l-periodic points, for any point y_0 of which the quantities $\dfrac{\partial}{\partial \mu} f_{\mu_0}^l(y_0)$ and $(f_{\mu_0}^l)''(y_0)$ are different from* 0 *and have the same* (*the opposite*) *sign.*

Remark 1. *If we make a restriction $l \leqq L < \infty$ in proposition* 1, *"residual" in its statement can be replaced by "open dense".*

Proposition 2. *Let the assumptions of proposition* 1 *be satisfied and let μ_0 be a right endpoint of an interval of $\sigma_f^{-1}(\varrho)$ satisfying* (2). *Then, σ_f is increasing or decreasing in some neighbourhood of μ_0 according to the sign of $\left.\dfrac{\partial f_\mu^l}{\partial \mu}(y_0)\right|_{\mu=\mu_0}$ but not constant in any neighbourhood of y_0.*

A similar statement is true for the left endpoints of the intervals of $\sigma_f^{-1}(\varrho)$.

This proposition is a corollary of the following lemma formulated only for the case

$$\left.\frac{\partial (f_\mu^l)}{\partial \mu}(y_0)\right|_{\mu=\mu_0} > 0. \tag{3}$$

Lemma 1. *Let the assumptions of proposition* 2 *and* (3) *be satisfied. Then, there are $\eta > 0$ and n_0 such that for all $n > n_0$, $\mu_0 \leqq \mu_1 < \mu_2 < \mu_0 + \eta$, $y \in R^1$,*

$$f_{\mu_1}^{nl}(y) < f_{\mu_2}^{nl}(y).$$

The proof of this lemma is based upon the observation that the closer μ is to μ_0, the more points of the μ-orbit of any point are contained in any choosen neighbourhood of the μ_0-orbit of y_0.

From Proposition 2 and Lemma 1, using Remark 1 we obtain two theorems on the global behavior of the rotation number. Theorem 1 shows that although σ_f is not Lipschitz continuous in general, generically its behavior is still quite regular. Theorem 2 shows a certain stability of σ_f.

Theorem 1. *There is an open dense subset $\mathscr{F}_0$ of $\mathscr{F}$ such that for every $f \in \mathscr{F}_0$, every compact subset of P can be written as a finite union of closed intervals with nonitersecting interiors, on each of which σ_f is monotone. The maximal and minimal values of σ_f on each of these intervals (which are the values of σ_f at the endpoints of the intervals) are rational.*

Corollary. *Generically, σ_f is of bounded variation.*

Theorem 2. *Let $f \in \mathscr{F}_1$. Then, for every $g \in \mathscr{F}_1$ sufficiently close to f, σ_g is topologically equivalent to σ_f in the following sense: There is a homeomorphism $h : P \to P$ such that $\sigma_g \circ h = \sigma_f$. Moreover, h tends to identity as $g \to f$.*

The next proposition can be considered as a certain counterpart to proposition 2 for $\sigma_f(\mu_0)$ being irrational.

Proposition 3. *Let $f_{\mu_0}(y) = y + \alpha$, where α is irrational. Let $\beta_0 = \int_0^1 \beta(x)\,\mathrm{d}x \neq 0$, where $\beta(x) = \dfrac{\partial f}{\partial \mu}(\mu_0, x)$. Then, σ_f is increasing or decreasing in some neighbourhood of μ_0 according to the sign of β_0.*

This proposition follows from the following lemma:

Lemma 2. *Let the assumptions of proposition 3 be satisfied and let $\beta_0 > 0$. Then, there exist $\eta > 0$, k such that for all $\mu_1 < \mu_2$, $|\mu_i - \mu_0| < \eta$, $i = 1, 2$ and all y, $n > 0$,*

$$f^{nk}_{\mu_1}(y) < f^{nk}_{\mu_2}(y).$$

For the proof we note that

$$f^n_\mu(y) = y + n\alpha + (\mu - \mu_0) \sum_{j=0}^{n-1} \beta(y + j\alpha) + o(\mu - \mu_0)$$

and that by the ergodic theorem $\lim \sum_{j=0}^{n-1} \beta(y + j\alpha) \to \beta_0$.

Remark 2. Although it is known, that any C^2 automorphism of S^1 with an irrational rotation number α can be brought to the translation $y + \alpha$ by a continuous change of coordinates, proposition 3 does not appeal to every such map since it is not known in general whether the change of coordinates can be made differentiable. However, in [2] it is proved that the analytic maps for which the change of coordinates can be made analytic are dense in the set of C^r maps in the C^0 topology.

From lemmas 1, 2 we can obtain also some information about the manifold of periodic points $Z_l(f)$ for $f \in \mathscr{F}_1$:

Proposition 4. *Let the assumptions of lemma 1 or lemma 2 be satisfied and let $f \in \mathscr{F}_1$. Then, there is a neighbourhood U of μ_0 (right in case of $\sigma_f(\mu_0)$ rational) such that for every $\varrho = kl^{-1}$ rational and every interval J of $\sigma_f^{-1}(\varrho)$ contained in U, $Z_l(f) \cap \cap [J \times R^1]$ is connected.*

From proposition 4 we obtain

Theorem 3. *Let $f \in \mathscr{F}_1$. Then for every $g \in \mathscr{F}_1$ sufficiently close to f the sets $Z_l(f)$ and $Z_l(g)$ are isomorphic. Moreover, their isomorphism tends to identity as $g \to f$.*

§ 2

In this section we relate the problems studied in §1 to the problem of generic bifurcations of periodic trajectories of differential equations.

Denote by $\mathscr{X}$ the space of one-parameter C^r differential equations (or, vector fields, $2 < r \leqq \infty$) on an n-dimensional manifold M, i.e. the space of maps $X : P \times \times M \to TM$ such that for every $\mu \in P$, $X_\mu : M \to TM$ defined by $X_\mu(x) = X(\mu, x)$ is a vector field (or, in the classical notation, a differential equation $\dot{x} = X(\mu, x)$), with the Whitney C^r topology (again, P is either R^1 or S^1).

Let Γ be a periodic trajectory with period T of X_{μ_0} (if we consider it as a subset of $P \times M$, we shall label it by μ_0 and write (μ_0, Γ)). It is important to know what happens to this trajectory and the flow in its neighbourhood if the parameter is slightly changed. If the local topological structure of the flow remains preserved (i.e., X_μ is locally topologically equivalent to X_{μ_0} at Γ[1])), (μ_0, Γ) is called an ordinary periodic trajectory, if not, (μ_0, Γ) is called a bifurcation periodic trajectory of X.

It follows from the structural stability theorems that (μ_0, Γ) can be a bifurcation trajectory only if at least one of its non-trivial multipliers lies on the unit circle. In [9] it is proven that generically (genericity is defined in a similar way as for flows on the torus in § 1) this is possible in three ways only:

1. one non-trivial multiplier is 1,
2. one non-trivial multiplier is -1,
3. a pair of complex conjugate multipliers not being roots of unity lie on the unit circle, and in each case none of the other non-trivial multipliers is on the unit circle.

It follows from [8, 10, 11] that generically, the bifurcations in these cases look as follows:

In case 1, for μ on one side of μ_0 there are two periodic orbits in the neighbourhood of Γ the periods of which tend to T as $\mu \to \mu_0$ and which collapse into (μ_0, Γ) and disappear for μ on the other side of μ_0.

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[1]) this means that there is a homeomorphism h of some neighborhood U of Γ onto f(U), again a neighbourhood of Γ, mapping trajectories of X_{μ_0} in U onto trajectories of X_μ in h(U).

In case 2 and 3, there are periodic trajectories (μ, Γ_μ) of X for any μ close to μ_0 in the neighbourhood of (μ_0, Γ) the periods of which tend to T as $\mu \to \mu_0$ and which together with (μ_0, Γ) form a 2-dimensional C^r manifold. Further, in case 2 there is another periodic trajectory for the values of μ on one side of μ_0 which collapses into (μ_0, Γ) and disappears for μ on the other side of μ_0, the period of which approaches $2T$ as $\mu \to \mu_0$: in case 3, for the values of μ on one side of μ_0 there is an invariant torus of X_μ which collapses into (μ_0, Γ) and disappears on the other side of μ_0.

In the cases 1 and 2 the entire topological picture of the bifurcation is clear and there are no more bifurcation trajectories in some neighbourhood of (μ_0, Γ) in $P \times M$ except of (μ_0, Γ) itself. In fact, we are able to give a topological classification of the possible bifurcations of type 1 and 2 in a certain precise sense (this will be the subject of our forthcoming paper). However, in case 3 the topology of the bifurcations depends essentially on the topology (and, thus, also, on the behavior of the rotation number) of the flow on the invariant torus that splits from (μ_0, Γ).

The theorems of § 1 have only an indirect value for the understanding of this problem. In fact, only propositions 3 and 4 can be applied directly. Namely, in proper coordinates the automorphism of S^1 associated with the flow on the invariant torus for $\mu > \mu_0$ can be represented in the form

$$f_\mu(y) = y + \alpha + (\mu - \mu_0)\,\beta(y) + o(\mu - \mu_0)$$

Since it can be proven that generically $\int_0^1 \beta(y)\,\mathrm{d}y \neq 0$ it follows from proposition 3 that, generically, in some neighbourhood of μ_0 the rotation number of the flow is either increasing or decreasing and not constant. Consequently, generically, in any neighbourhood of (μ_0, Γ) in $P \times M$ there are infinitely many bifurcation periodic trajectories. As a consequence of proposition 4 we obtain that generically, for every l there exists a neighbourhood U_l of μ_0 such that the intersection of the manifold Z_l of l-periodic points of the associated diffeomorphism of S^1 with $U_l \times S^1$ is connected.

REFERENCES

[1] P. Hartman: *Ordinary differential equations*, New York—London—Sydney, Wiley 1964.

[2] V. I. Arnold: *Malye znamenateli I. Ob otobraženijakh okružnosti na sebja*, Izvestija AN SSSR 25 (1961), 21–86.

[3] V. A. Pliss: *Nelokaľnyje problemy teorii kolebanij*, Moskva-Leningrad, Nauka 1964.

[4] A. Aeppli. L. Markus: *Integral equivalence of vector fields on manifolds and bifurcation of differential systems*, American Journal of Mathematics 85 (1963), 633–654.

[5] J. Moser: *A rapidly connvegent iteration methods and nonlinear differential equations II*, Annali della Scuola Normale Superiore di Pisa, Scienc e Fisiche Matematiche 20 (1966), 499–535.

[6] L. E. Reiziň: *Topologičeskaja klassifikacija dinamičeskikh sistem bez toček pokoja na tore*, Latvijskij matematičeskij ježegodnik 5 (1969), 113–121.

[7] J. SOTOMAYOR: *Generic one-parameter families of vector fields on two dimensional manifolds*, (to appear).
[8] P. BRUNOVSKÝ: *On one parameter families of diffeomorphisms*, Commentationes mathematicae Universiatis Carolinae 11 (1970), 559–582, 12 (1971), 765–784.
[9] M. MEDVEĎ: *Thesis* (to appear)
[10] JU. I. NEJMARK: *Metod točečnykh otobraženij v teorii nelinejnykh kolebanij*, Trudy meždunarodnogo simpoziuma po nelinejnym kolebanijam, tom 2, Kiev, Izdateľstvo Akademii nauk USSR, 1963.
[11] R. SACKER: *On Invariant surfaces and bifurcation of periodic solutions of ordinary differential equations*, New York University Report, IMM–NYU 333, October 1964.

Author's address:
Pavol Brunovský
Mathematical Institute, Slovak Academy of Sciences
Štefánikova 41, Bratislava
Czechoslovakia