ABSTRACT SEMILINEAR EQUATIONS WITH SMALL PARAMETER

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Given two normed spaces U, X over the real or complex number field, let L be a linear operator from U into X, F a (nonlinear) transformation on U into X and S an open subset of U.

In this note, we search for general properties of L, F and S under which *the abstract equation*

$$Lu = \varepsilon F(u) \qquad [*]$$

is solvable in the set S for sufficiently small $\varepsilon > 0$.

To this purpose, we require first certain *general regularity properties of L and F*: the closedness of the null and range sets of L; the boundedness or compactness of the quotient inverse of L; the local Lipschitz property or continuity of F.

Further we require the *coincidence property of L and S*: the intersection of S with the null set of L is not empty.

The simplest case arises if L is a one-to-one operator and its range set is X. This case is sometimes called *noncritical*. In this case, we can transform easily [*] to the form $u = \varepsilon L^{-1}(F(u))$ which appears solvable for small $\varepsilon > 0$ in the set S under assumption of the general regularity properties and the coincidence property. In fact, the general regularity properties ensure the boundedness or compactness of L^{-1} and the local Lipschitz property or continuity of F and the coincidence property ensures that S contains zero. Hence we may apply in the first case the Banach fixed point theorem and in the second that of Schauder for sufficiently small $\varepsilon > 0$ in a suitable subset of S.

Nevertheless, our main purpose is to solve the so-called *critical* case if the operator L is not one-to-one or its range set differs from X. The most basic assumption here is the so called *bifurcation property* which describes the fundamental algebraic relation between the range sets of L and F with respect to S: for every $u \in S$, there exists a unique $\bar{u} \in S$ so that $u - \bar{u}$ lies in the null set and $F(\bar{u})$ in the range set of L.

In the noncritical case, the bifurcation property is clearly automatically fulfilled.

To transform our equation [*], we need the quotient space of U over the null space of L, which we denote by $\tilde{U}$.

Now, let $\tilde{L}$ be the operator from $\tilde{U}$ into X, naturally associated with L. This operator is called the *quotient operator* to L. It is clear that the quotient operator $\tilde{L}$ is one-to-one and that the range sets of $\tilde{L}$ and L are the same.

Further, let $\tilde{S}$ be the set of all elements of $\tilde{U}$ whose intersections with S are not empty. Clearly $\tilde{S}$ is an open subset of $\tilde{U}$.

We see easily that the bifurcation property is equivalent with the existence of a unique mapping J on $\tilde{S}$ into S such that $J(\tilde{u})$ lies in $\tilde{u}$ and $F(J(\tilde{u}))$ lies in the range set of L for $\tilde{u} \in \tilde{S}$. This mapping J will be called the *bifurcation mapping*.

Using the quotient operator and the bifurcation mapping, we can put a new problem to search for general properties of L, F and S under which the *associated equation*

$$\tilde{L}\tilde{u} = \varepsilon F(J(u)) \qquad [**]$$

is solvable in the set $\tilde{S}$ *for sufficiently small* $\varepsilon > 0$.

The decisive importance of the associated equation [**] consists in the easily provable fact that u is a solution of the equation [*] if and only if there exists a solution $\tilde{u}$ of the associated equation [**] such that $\tilde{u} = J(u)$. This fact is the main consequence of the bifurcation property and its proof is based on the above mentioned properties of the quotient operator and bifurcation mapping. Hence the solvability of the equation [*] in the set S for sufficiently small $\varepsilon > 0$ is equivalent with the solvability of the associated equation [**] in the set $\tilde{S}$ for sufficiently small $\varepsilon > 0$.

Since the quotient operator is one-to-one, we see that the associated equation [**] may be transformed to the form $\tilde{u} = \varepsilon \tilde{L}^{-1} F(J(\tilde{u}))$ and consequently, the main question is under what conditions the transformed equation may be solved. We use of course at the first place the general regularity properties ensuring the boundedness or compactness of $\tilde{L}^{-1}$ and the local Lipschitz property or continuity of F and further the coincidence property ensuring that $\tilde{S}$ contains zero. But this does not suffice. We must add certain *regularity property of bifurcation* equivalent with: the bifurcation mapping J is locally lipschitzian or continuous on $\tilde{S}$. In the first case if $\tilde{L}^{-1}$ is bounded and F and J are locally lipschitzian, we can solve the transformed equation for sufficiently small $\varepsilon > 0$ in a suitable subset of $\tilde{S}$ by means of the Banach fixed point theorem, while in the second case if $\tilde{L}^{-1}$ is compact and F and J are continuous, by means of the Schauder fixed point theorem. The first case is contained in Theorem 1 and the second in Theorem 3.

In the noncritical case, it is easy to see that the regularity property of bifurcation is always fulfilled since $\tilde{S}$ may be identified with S and J with the identity operator.

The regularity of bifurcation may be deduced from the general regularity properties and from the so-called *coercivity property*: there exists a nondecreasing positive function n on $\langle 0, \infty)$ such that $\inf_{x \text{ in the range set of } L} \| F(u') - F(u'') + x \| \geqq n(\| u' - u'')$ $u', u'' \in S$ with $u' - u''$ in the null set of L. If F is continuous, we obtain from the coercivity that the bifurcation mapping is continuous, while if F is locally lipschitzian and the function n in the coercivity property is linear, the bifurcation mapping is also lipschitzian. The coercivity property is used in Theorems 2 and 4 (cf. also Proposition 2).

Let us still remark that in the noncritical case the proofs of Theorems 1 and 3 very simplify and Theorems 2 and 4 become irrelevant.

In applications of general theory (cf. Examples 1 and 2) the most difficult concerns are on the one hand the closedness of the range set of L and on the other hand the bifurcation property which must be decided as the case may be by different technical means.

It seems that most special results may be included in the general theory presented here, but in some cases this is not clear definitively (cf. [6] and [8]).

Our approach to the problem is geometrical. As we see from preceding considerations, an important role in this approach is played by the null and range sets of the operator L. Besides the closedness of these subspaces which is indispensable as we have seen above, one supposes usually also the existence of closed complementary subspaces (which is equivalent with the existence of continuous projectors onto these subspaces). As well known, these complements exist in finite dimensional and Hilbert spaces (more generally, in all spaces isomorphic with Hilbert spaces), but they do not exist in general in practically all frequent function spaces, except of course square Lebesgue spaces. Hence the assumption of existence of closed complements is formally restrictive and may be removed by the use of quotient spaces as carried out in this note.

Geometrical approach to these problems was initiated by Cesari and then used or rediscovered by many authors (cf. [1] – [13]). The method of quotient spaces was developed in the last time independently by the author [12] and by W. S. Hall [13].

If L is an evolution operator (i.e. $Lu(t) = \sum_{i=0}^{n} A_i u^{(i)}(t)$ in an appropriate vector valued function space), also other methods may be used, in particular that of Poincaré based on the solutions of the corresponding initial value problem (cf. [14] – [16]).

* * *

For the sake of simplicity, we shall denote by $\mathsf{D}(L)$ the definition set, by $\mathsf{R}(L)$ the range set and by $\mathsf{N}(L)$ the null set of the operator L. Naturally, $\mathsf{D}(L)$, $\mathsf{N}(L)$ are linear subspaces of U and $\mathsf{R}(L)$ of X.

Theorem 1. *If*

(I) $\mathsf{N}(L)$, $\mathsf{R}(L)$ *are closed subspaces,*

(II) *for every bounded subset $K \subseteq X$, there exists a bounded complete subset $H \subseteq U$ such that for every $u \in \mathsf{D}(L)$ for which $Lu \in K$, we can find a $\bar{u} \in H$ for which $u - \bar{u} \in$ $\in \mathsf{N}(L)$,*

(III) *for every $q \in U$, there exist a neighborhood M and a constant m so that $\| F(u_1) -$ $- F(u_2) \| \leqq m \| u_1 - u_2 \|$ for every $u_1, u_2 \in M$,*

then for every open subset $S \subseteq U$ satisfying

(α) $S \cap \mathsf{N}(L) = \emptyset$,

(β) *for every* $u \in S$, *there exists a* $\bar{u} \in S$ *so that* $u - \bar{u} \in \mathsf{N}(L)$ *and* $F(\bar{u}) \in \mathsf{R}(L)$,

(γ) *for every* $q \in S$, *there exist a neighborhood* B *and a constant* b *so that for every* $u' \in B$, $u'' \in B$, $\bar{u}' \in S$, $\bar{u}'' \in S$ *fulfilling* $u' - \bar{u}' \in \mathsf{N}(L)$, $u'' - \bar{u}'' \in \mathsf{N}(L)$, $F(\bar{u}') \in \mathsf{R}(L)$, $F(\bar{u}'') \in \mathsf{R}(L)$, *there is* $\| \bar{u}' - \bar{u}'' \| \leqq b \| u' - u'' \|$,

there exist a constant $\vartheta > 0$ *and a function* u *on* $\langle 0, \vartheta \rangle$ *into* U *such that*

(a) $\mathsf{u}(\varepsilon) \in S$ *for every* $0 \leqq \varepsilon \leqq \vartheta$,

(b) $\mathsf{u}(\varepsilon) \in \mathsf{D}(L)$ *and* $L\mathsf{u}(\varepsilon) = \varepsilon F(\mathsf{u}(\varepsilon))$ *for every* $0 < \varepsilon \leqq \vartheta$,

(c) $\mathsf{u}(\varepsilon) \to \mathsf{u}(0)$ $(\varepsilon \to 0_+)$,

(d) *there exists a constant* c *such that* $\| \mathsf{u}(\varepsilon_1) - \mathsf{u}(\varepsilon_2) \| \leqq c | \varepsilon_1 - \varepsilon_2 |$ *for every* $0 < \varepsilon_1, \varepsilon_2 \leqq \vartheta$,

(e) *for every constant* $\bar{\vartheta} > 0$ *and every function* $\bar{\mathsf{u}}$ *on* $\langle 0, \bar{\vartheta} \rangle$ *into* U, *satisfying* (a) – (c), *there exists a constant* $0 < \gamma \leqq \vartheta$, $0 < \gamma \leqq \bar{\vartheta}$ *so that* $\bar{\mathsf{u}}(\varepsilon) = \mathsf{u}(\varepsilon)$ *for* $0 \leqq \varepsilon \leqq \gamma$.

The proof may be constructed easily from the second part of the proof of Theorem 1 [12] with some little changements. A theorem of this type was proved also by W. S. Hall [13] by another method. But our theorem is rather general without unnecessary restrictions (in particular $X = U^*$).

Proposition 1. *If the spaces* U, X *are complete (i.e. Banach spaces), then the hypotheses (I), (II) of Theorem 1 hold if and only if* L *is a closed operator with closed range set.*

For the proof see [12], Proposition 1.

Theorem 2. *The preceeding Theorem 1 remains valid if we replace* (γ) *by*

(γ') *there exists a constant* $n > 0$ *so that for every* $u', u'' \in S$, $u' - u'' \in \mathsf{N}(L)$ *and for every* $x \in \mathsf{R}(L)$, *there is* $\| F(u') - F(u'') - x \| \geqq n \| u' - u'' \|$.

For the proof, it suffices to verify that (γ) of Theorem 1 holds. This was made essentially by the author in the first part of the proof of Theorem 1 [12].

Theorem 3. *If*

(I) *as in Theorem 1*,

(II) *for every bounded subset* $K \subseteq X$, *there exists a compact subset* $H \subseteq U$ *such that for every* $u \in \mathsf{D}(L)$, $Lu \in K$, *we can find* $\bar{u} \in H$, $u - \bar{u} \in \mathsf{N}(L)$,

(III) *for every* $q \in U$ *and* $\mu > 0$, *there exists a neighbourhood* M *so that for every* $u \in M$, *there is* $\| F(u) - F(q) \| \leqq \mu$,

then for every open subset $S \subseteq U$ *satisfying*

(α), (β) *as in Theorem 1*,

(γ) *for every* $q \in S$ *and* $\beta > 0$, *there exists a neighbourhood* B *so that for every* $u \in B$, $\bar{u} \in S$, $\bar{q} \in S$ *fulfilling* $u - \bar{u} \in \mathsf{N}(L)$, $q - \bar{q} \in \mathsf{N}(L)$, $F(\bar{u}) \in \mathsf{R}(L)$, $F(\bar{q}) \in \mathsf{R}(L)$ *there is* $\| \bar{u} - \bar{q} \| \leqq \beta$,

there exist a constant $\vartheta > 0$ *and a function* u *on* $\langle 0, \vartheta \rangle$ *into* U *such that*

(a) – (c) *as in Theorem 1*,

(d) *the set* $\{\mathsf{u}(\varepsilon), 0 < \varepsilon \leqq \vartheta\}$ *is relatively compact in* U,

(e) *for every constant* $\bar{\vartheta} > 0$ *and function* $\bar{u}$ *on* $\langle 0, \bar{\vartheta}\rangle$ *into* U *satisfying* (a) – (c), *there is* $u(\varepsilon) - \bar{u}(\varepsilon) \to 0(\varepsilon \to 0_+)$.

The proof may be constructed from the second part of the proof of Theorem 2 [12]. A theorem of this type was proved also by W. S. Hall [13] by another method. But the present theorem is rather general without unnecessary restrictions (e.g. $X = U^*$). Moreover, our proof is simpler and more direct without use of Michael selection principle based on the axiom of choice.

Theorem 4. *The preceding Theorem* 3 *remains valid if we replace* (γ) *by*

(γ') *for every* $\varkappa > 0$ *there exists a* $\nu > 0$ *so that for every* $u', u'' \in S$ *fulfilling* $u' - u'' \in \mathsf{N}(L)$ *and* $\| F(u') - F(u'') - x \| \leqq \nu$ *for some* $x \in \mathsf{R}(L)$, *there is* $\| u' - u'' \| \leqq \varkappa$.

For the proof it suffices to verify that (γ) of Theorem 3 holds. This was made by the author in the first part of the proof of Theorem 2 [12].

Proposition 2. *The condition* (γ') *of Theorem 4 is equivalent to*

(γ'') *there exists a nondecreasing function* n *on* $\langle 0, \infty)$, $n(\alpha) > 0$ *for* $\alpha > 0$, *so that for every* $u', u'' \in S$ *fulfilling* $u' - u'' \in \mathsf{N}(L)$, *there is for every* $x \in \mathsf{R}(L)$ $\| F(u') - F(u'') - x \| \geqq n(\| u' - u'' \|)$.

The proof of this equivalence is easy, the condition (γ'') is technically advantageous.

Remark 1. The condition (α) in Theorems 1 and 3 is obviously the coincidence property, while (β) is essentially the bifurcation property without unicity. The condition (γ) gives the needed regularity property of bifurcation and simultaneously the unicity of bifurtion.

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In the following Examples 1 and 2, we shall apply the general theory to *the problem of periodic generalized solutions for semilinear string equation with small parameter* $u_{tt} - u_{\xi\xi} = \varepsilon f(t, \xi, u(t, \xi))$.

Example 1. *Let* f *be a real function on* $(-\infty, \infty) \times \langle 0, \pi\rangle \times (-\infty, \infty)$ *If*
(X) $f(t + 2\pi, \xi, r) = f(t, \xi, r)$ *for every* t, ξ, r *and* f *is continuous in all variables, then for every* $-\infty < s < \infty$ *and* $0 < \omega \leqq \infty$ *satisfying*

(χ) *there exists a convex nondecreasing function* c *on* $\langle 0, \infty)$, $c(0) = 0$, $c(\alpha) > 0$ *for* $0 \leqq \alpha < \infty$ *so that* $|f(t, \xi, s)| < \sup_{0 \leqq \eta < 2\omega} c(\eta)$ *for* $-\infty < t < \infty$, $0 \leqq \xi \leqq \pi$, and $|f(t, \xi, r_1) - f(t, \xi, r_2)| \geqq c(|r_1 - r_2|)$ *for* $-\infty < t < \infty$, $0 \leqq \xi \leqq \pi$, $|r_1 - s| < \omega$, $|r_2 - s| < \omega$,

there exist a constant $\vartheta > 0$ *and a function* $\boldsymbol{u}$ *on* $(-\infty, \infty) \to \langle 0, \pi\rangle \to \langle 0, \vartheta\rangle$ *continuous on* $(-\infty, \infty) \times \langle 0, \pi\rangle$ *for each* $0 \leqq \varepsilon \leqq \vartheta$, *such that*

(a) $|\boldsymbol{u}(t, \xi, \varepsilon) - s| < \omega$,

(b) $\boldsymbol{u}(t + 2\pi, \xi, \varepsilon) = \boldsymbol{u}(t, \xi, \varepsilon), \boldsymbol{u}(t, 0, \varepsilon) = \boldsymbol{u}(t, \pi, \varepsilon) = 0$

and for every infinitely differentiable 2π-periodic function φ on $(-\infty, \infty)$, the function $\int_0^{2\pi} \varphi(\tau)\, \boldsymbol{u}(\tau, \xi, \varepsilon)\, d\tau$ is twice continuously differentiable on $\langle 0, \pi\rangle$ and

$$\int_0^{2\pi} \varphi''(\tau)\, \boldsymbol{u}(\tau, \xi, \varepsilon)\, d\tau - \frac{d^2}{d\xi^2} \int_0^{2\pi} \varphi(\tau)\, \boldsymbol{u}(\tau, \xi, \varepsilon)\, d\tau = \int_0^{2\pi} \varphi(\tau)\, \boldsymbol{f}(\tau, \xi, \boldsymbol{u}(\tau, \xi, \varepsilon))\, d\tau,$$

(c) $\boldsymbol{u}(t, \xi, \varepsilon) \to \boldsymbol{u}(t, \xi, 0)$ $(\varepsilon \to 0_+)$ *uniformly on* $(-\infty, \infty) \times \langle 0, \pi\rangle$,

(d) *the function $\boldsymbol{u}$ is bounded and the functions $\boldsymbol{u}(., ., \varepsilon)$, $0 \leqq \varepsilon \leqq \vartheta$, are equicontinuous on $(-\infty, \infty) \times \langle 0, \pi\rangle$,*

(e) *for every constant $\bar{\vartheta} > 0$ and function $\bar{\boldsymbol{u}}$ on $(-\infty, \infty) \times \langle 0, \pi\rangle \to \langle 0, \vartheta\rangle$ continuous on $(-\infty, \infty) \times \langle 0, \pi\rangle$ for each $0 \leqq \varepsilon \leqq \bar{\vartheta}$, satisfying* (a) – (c), *there is* $\boldsymbol{u}(t, \xi, \varepsilon) - \bar{\boldsymbol{u}}(t, \xi, \varepsilon) \to 0(\varepsilon \to 0_+)$ *uniformly on* $(-\infty, \infty) \times \langle 0, \pi\rangle$.

The proof is based on Theorem 4 using Proposition 2. We choose $U = X =$ the Banach space of all continuous, in t 2π-periodic functions on $(-\infty, \infty) \times \langle 0, \pi\rangle$, L is the extended (or generalized) string operator $\Box\, u = u_{tt} - u_{\xi\xi}$ and F is defined by $\boldsymbol{f}(t, \xi, u(t, \xi))$ in above spaces. The set S is an open subset of U determined by constants s, ω. The verification of the hypothesis (γ'') of Proposition 2 is based on some maximum principle and was given by the author (unpublished). To verify (β) of Theorem 4, it is possible to approximate $\boldsymbol{f}$ by a smoother function and then use a raisoning known from [5], [6].

Remark 2. The simplest case in Example 1 arises if $s = 0$, $\omega = \infty$, $c(\alpha) = \alpha^\varrho$ for some $\varrho \in \{1, 2, \ldots\}$. This case was studied by Torelli in [7] under additional restriction on the growth of $\boldsymbol{f}$ since he worked in appropriate Lebesgue spaces, not in continuous functions. Also the proof differs from ours and is based on monotone mappings theory. See also W. S. Hall [8] and H. Petzeltová [17].

Example 2. *Let $\boldsymbol{f}$ be as in Example 1. If*

(X) *as in Example 1 and $\boldsymbol{f}$ is locally lipschitzian in r uniformly with respect to $-\infty < t < \infty$, $0 \leqq \xi \leqq \pi$,*

then for every $-\infty < s < \infty$ and $0 < \omega \leqq \infty$ satisfying

(χ) *as in Example 1 with c linear,*

then there exist a constant $\vartheta > 0$ and a function $\boldsymbol{u}$ on $(-\infty, \infty) \times \langle 0, \pi\rangle \to \langle 0, \vartheta\rangle$ such that

(a) – (c) *as in Example 1,*

(d) *$\boldsymbol{u}$ is lipschitzian in $0 < \varepsilon \leqq \vartheta$ uniformly with respect to $-\infty < t < \infty$, $0 \leqq \xi \leqq \pi$,*

(e) *for every constant $\bar{\vartheta} > 0$ and function $\bar{\boldsymbol{u}}$ on $(-\infty, \infty) \to \langle 0, \pi\rangle \to \langle 0, \bar{\vartheta}\rangle$ continuous on $(-\infty, \infty) \times \langle 0, \pi\rangle$ for each $0 \leqq \varepsilon \leqq \bar{\vartheta}$, satisfying* (a) – (c), *there exists a constant $0 < \gamma \leqq \vartheta$, $0 < \gamma \leqq \bar{\vartheta}$ such that $\bar{\boldsymbol{u}}(t, \xi, \varepsilon) = \boldsymbol{u}(t, \xi, \varepsilon)$ for $-\infty < t < \infty$, $0 \leqq \xi \leqq \pi$, $0 \leqq \varepsilon \leqq \gamma$.*

The proof is based on Theorem 2, the necessary spaces and operators are defined as in Example 1.

Remark 3. Analogous results as in Examples 1 and 2 may be obtained also if f depends also on $\boldsymbol{u}_t$ and $\boldsymbol{u}_\xi$. See [12].

Remark 4. By similar methods, one can also examine periodic solutions of ordinary (pendulum) semilinear equations, even in classical sense. For beam equations, the technique of Lebesgue or Orlicz spaces must be used. Cf. [9] and [17].

Remark 5. The function $f(t, \xi, r) = f(t, \xi) + ar \mid r \mid^{p-1} + b \sin (r \mid r \mid^{q-1})$ $(ab > 0$, $p, q \in \{1, 2, \ldots\}$, f is a continuous function on $(-\infty, \infty) \to \langle 0, \pi \rangle$, 2π-periodic in $t)$ is admissible in Example 1 and with $p = q = 1$ also in Example 2 if the magnitude of $\sup\limits_{t,\xi} |f(t, \xi)|$ is conveniently limited.

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Now we shall study the smoothness of solutions in a very general setting.

To this purpose, let us first define an auxiliary notion. A linear operator G from a Banach space E into itself will be called *exponential* if there exists a function $\mathcal{T}$ on $\langle 0, \infty)$ into bounded linear operators on E into itself such that (1) $\mathcal{T}(\delta_1 + \delta_2) = \mathcal{T}(\delta_1)\, \mathcal{T}(\delta_2)$ for $\delta_1 \delta_2 \in (0, \infty)$, (2) $\mathcal{T}(\delta)\, x \to x (\delta \to 0_+)$ for every $x \in E$, (3) $x \in \mathsf{D}(G)$ if and only if $\lim\limits_{\delta \to 0_+} \delta^{-1} (\mathcal{T}(\delta)\, x - x)$ exists, (4) for $x \in \mathsf{D}(G)$, $Gx = \lim\limits_{\delta \to 0^+} \delta^{-1} (\mathcal{T}(\delta)\, x - x)$. The function $\mathcal{T}$ is called the semigroup generated by G and is uniquely determined.

Let us now have U, X, L, F as above and, moreover, two fixed operators A from U into U and B from X into X.

Theorem 5. *If*

(I), (II) *as in Theorem* 1,

(III) *for every* $u \in \mathsf{D}(A)$ *such that* $Au \in \mathsf{D}(L)$, *there is* $u \in \mathsf{D}(L)$, $Lu \in \mathsf{D}(B)$ *and* $BLu = LAu$,

(IV) F *is continuously differentiable in Fréchet sense on* U,

(V) *for every* $u \in \mathsf{D}(A)$ *there is* $F(u) \in \mathsf{D}(B)$ *and the transformation* $u \to BF(u) - F'(u)\, Au$ *is uniformly continuous on relatively compact subsets of* $u \in \mathsf{D}(A)$,

(VI) *the operators* A, B *are exponential*,

then for every constant $\vartheta > 0$ *and function* $\boldsymbol{u}$ *on* $\langle 0, \vartheta \rangle$ *into* U *satisfying*

(α) $\mathsf{u}(\varepsilon) \in \mathsf{D}(L)$ *and* $L\mathsf{u}(\varepsilon) = \varepsilon F(\mathsf{u}(\varepsilon))$ *for every* $0 < \varepsilon \leqq \vartheta$,

(β) $\mathsf{u}(\varepsilon) \to u(0)\ (\varepsilon \to 0_+)$,

(γ) *the set* $\{\mathsf{u}(\varepsilon) : 0 < \varepsilon \leqq \beta\}$ *is relatively compact in* U,

(δ) *for every* $0 \leqq \varepsilon \leqq \beta$ *and* $h \in \mathsf{N}(L)$ *such that* $F(\mathsf{u}(\varepsilon))\, h = 0$, *there is* $h = 0$,

(ε) $\{F'(u(\varepsilon))\,h + y : h \in N(L),\ \ y \in R(L)\} = X$ *for every* $0 \leqq \varepsilon \leqq \vartheta$,
there exists a constant $0 < \bar{\vartheta} \leqq \vartheta$ *such that*

(a) $u(\varepsilon) \in D(A)$ *and* $L(u(\varepsilon)) \in D(B)$ *for every* $0 \leqq \varepsilon \leqq \bar{\vartheta}$,

(b) $Au(\varepsilon) \to Au(0)$ and $BLu(\varepsilon) \to BLu(0)$ $(\varepsilon \to 0_+)$,

(c) *the sets* $\{Au(\varepsilon) : 0 < \varepsilon \leqq \vartheta\}$ *and* $\{BLu(\varepsilon) : 0 < \varepsilon \leqq \bar{\vartheta}\}$ *are relatively compact in* U *and* X *resp.*

The proof was given by the author (unpublished) and is based on the method of finite differences constructed by means of semigroups generated by the operators A, B.

Remark 6. If the function u in Theorem 5 is continuous, then we obtain easily also the continuity of Au and BLu.

* * *

Example 3. *Let f be as in Example 2 and $n \in \{0, 1, \ldots\}$. If*

(X) *as in Example 2,*

(XX) *$f(.,\xi,.)$ is n-times differentiable on $(-\infty, \infty) \times (-\infty, \infty)$ for every $0 \leqq \xi \leqq \pi$ and all these derivatives are continuous in all variables,*

then for every $-\infty < s < \infty$ and $0 < \omega \leqq \infty$ satisfying

(χ) *as in Example 2,*

there exist a constant $\vartheta > 0$ and a function $\boldsymbol{u}$ on $(-\infty, \infty) \times \langle 0, \pi\rangle \times \langle 0, \vartheta\rangle$, continuous in $-\infty < t < \infty$, $0 \leqq \xi \leqq \pi$ for every $0 \leqq \varepsilon \leqq \vartheta$ such that

(a) – (e) *as in Example 2,*

(f) *$\boldsymbol{u}$ is n-times continuously differentiable on $(-\infty, \infty) \times \langle 0, \pi\rangle$ for every $0 \leqq \varepsilon \leqq \vartheta$,*

(g) *all above derivatives are continuous in all variables.*

The proof is based on Example 2 and Theorem 5 using Remark 5. Let first $n = 1$. We choose U, X, L, F as in Example 1 and A, B as first derivatives in t which generate translation groups in U, X. Now it is possible to verify the hypotheses (I) – (VI). The existence of ϑ and $\boldsymbol{u}$ satisfying (a) – (e) follows from Example 2 from where we obtain immediately (α) – (γ). The remaining conditions (δ) – (ε) follow from (χ) by the same method as in [5]. Using Theorem 5 we obtain the existence of continuous derivative in t. This result may be extended by induction to order n. The differentiability in both variables t, ξ follows then as a special property of solutions of (generalized) linear string equation.

Remark 7. It is clear that for $n \geqq 2$ the function $\boldsymbol{u}$ in Example 3 fulfils the classical string equation

$$\boldsymbol{u}_{tt}(t, \xi, \varepsilon) - \boldsymbol{u}_{\xi\xi}(t, \xi, \varepsilon) = \varepsilon \boldsymbol{f}(t, \xi, \boldsymbol{u}(t, \xi, \varepsilon)).$$

Remark 8. All preceding theorems and propositions may be extended without serious difficulties to the case where F depends also on a general parameter.

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is incomplete and subjectively selected. Hence it is necessary at least to keep in review also the references in cited works.

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