Joachim A. Nitsche

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In: Miloš Ráb and Jaromír Vosmanský (eds.): Proceedings of Equadiff III, 3rd Czechoslovak Conference on Differential Equations and Their Applications. Brno, Czechoslovakia, August 28 - September 1, 1972. Univ. J. E. Purkyně - Přírodovědecká fakulta, Brno, 1973. Folia Facultatis Scientiarum Naturalium Universitatis Purkynianae Brunensis. Seria Monographia, Tomus I. pp. 235--239.

Persistent URL: http://dml.cz/dmlcz/700056

# INTERIOR ERROR ESTIMATES OF PROJECTION METHODS

by JOACHIM A. NITSCHE

**1.** Let $\Omega$ be a bounded domain in $R^N$ and $\Omega_1$, $\Omega'$ etc. subdomains of $\Omega$. By $W_2^n(\Omega')$ the Sobolev-space of functions with generalized $L_2$-integrable $n$-th derivatives is meant with the norm $\| \,.\, \|_{n.\Omega'}$.

As typical example we consider the boundary value problem

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{1}$$

and restrict ourselves with respect to numerical methods to that of Ritz: For any subspace $S_h \subseteq \mathring{W}_2^1(\Omega)$ the approximation $u_h = R_h u \in S_h$ is defined by

$$D(u - u_h, \chi) = 0 \text{ for } \chi \in S_h\,. \tag{2}$$

The convergence $u_h \to u$ will heavily depend on the approximability properties the used subspaces will have. Roughly speaking spline subspaces $\{S_h \,|\, 0 < h < 1\}$ fulfill conditions of the following type:

Let $\Omega_i$ with $i = 1, 2, \ldots, I$ be a finite set of subdomains of $\Omega$ and $\Omega_i'$ be such that $\Omega_i' \subset\subset \Omega_i$. If a function $u$ with $u \in L_2(\Omega)$ restricted to $\Omega_i$ is in $W_2^{r_i}(\Omega_i)$ then there is for any $h \in (0, 1)$ a $\chi_h \in S_h$ such that

$$\| u - \chi_h \|_{k.\Omega_i'} \leqq ch^{r_i - k}\{\| u \|_{0.\Omega} + \| u \|_{r_i.\Omega_i}\} \tag{3}$$

with $c$ independent of $u$ and $h$.

While the Ritz approximations are defined globally they may under certain conditions have this interior and local convergence property as is shown in the next sections. The presented result is a simplified version of a joint work with A. SCHATZ, which will be published later. For the corresponding problem in case of $L_2$-projections see Appl. Anal. **2** (1972), pp. 161–168.

**2.** We will consider approximating subspaces according to the following conditions: Let $n$ be a fixed integer. For $h \in (0, 1)$ there are given

i. a finite dimensional subspace $S_h \in \mathring{W}_2^1(\Omega) \cap W_2^n(\Omega)$
ii. a finite set $\mathfrak{T}_h$ of subdomains of $\Omega$

such that the four properties are satisfied.

**Proposition 1.** *For any $\Omega' \subseteq \Omega$ there is a $T \in \mathfrak{T}_h$ with*

$$\text{dist}\,(\Omega', T) \leqq ch,$$

*$c$ being independent of $\Omega'$ and $h$.*

**Proposition 2.** *For any $T \in \mathfrak{T}_h$ and any $\chi \in S_h$ inverse relations*

$$\| \chi \|_{k+1.T} \leqq ch^{-1} \| \chi \|_{k.T} \qquad (k = 0, 1, \ldots, n-1)$$

*hold true with c independent of $\chi$, T and h.*

**Proposition 3.** *For $u \in \mathring{W}_2^1(\Omega) \cap W_2^m(\Omega)$ with $m \leqq n + 1$ and $\Omega' = \operatorname{supp}(u)$ there is a $\chi \in S_h$ with*

$$\operatorname{dist}(\Omega', \operatorname{supp}(\chi)) \leqq ch$$

*such that*

$$\| u - \chi \|_{k.\Omega} \leqq ch^{m-k} \| u \|_{m.\Omega} \qquad (k = 0, 1, \ldots, m-1)$$

*and c being independent of u and h.*

**Proposition 4.** *Let $\omega \in C^\infty(\Omega)$ be given with $\Omega_2 = \operatorname{supp}(\omega)$ and let $\Omega_1$ contain $\Omega_2$ properly. Any function $\omega\chi$ with $\chi \in S_h$ can be approximated by an element $\varphi \in S_h$ with* supp $(\varphi) \subseteq \Omega_1$ *according to*

$$\| \omega\chi - \varphi \|_{k.\Omega} \leqq ch^{n+1-k} \| \chi \|_{n.\Omega_1} \qquad (k = 0, 1, \ldots, n)$$

*with c depending possibly on $\omega$ resp. $\Omega_1$, $\Omega_2$ but independent of $\chi$ and h.*
These assumptions are typical for splines. For $N = 2$, i.e. in two dimensions, let $\Gamma$ be a regular triangulation of $R^2$ with a maximum edge length $h$. Then $\mathfrak{T}_h$ consists of all unions of triangles contained in $\Omega$. $S_h$ may be the space of all continuous functions which are piecewise linear in the triangles of $\Gamma$. As is easily checked the propositions are true with $n = 1$.

An immediate consequence of the propositions is

**Lemma 1.** *Let $\Omega_1$, $\Omega_2$ be subdomains of $\Omega$ with $\Omega_2 \subset\subset \Omega_1$. Any function $u \in W_2^m(\Omega_1)$ with $m \leqq n + 1$ may be approximated by elements $\chi_h \in S_h$ according to*

$$\| u - \chi_h \|_{k.\Omega_2} \leqq ch^{m-k} \| u \|_{m.\Omega_1} \qquad (k = 0, 1, \ldots, m-1)$$

*with c independent of u and h.*

Though the formulated conditions will give the wanted interior error estimates for the Ritz approximations in case of a sufficiently smooth boundary $\partial\Omega$ there may be a certain 'pollution' effect caused by corners or other irregularities of $\partial\Omega$. We will formulate the conditions on $\partial\Omega$ implicitely by

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i. *For $f \in L_2(\Omega)$ the solution u of* (1) *is in $W_2^2(\Omega)$ and*

$$\| u \|_{2.\Omega} \leqq c \| f \|_{0.\Omega}.$$

ii. *If $\Omega_2 = \operatorname{supp}(f) \subset\subset \Omega$ and $\Omega_1$ contains $\Omega_2$ properly then u restricted to $\Omega - \Omega_1$ is in $W_2^\lambda(\Omega - \Omega_1)$ with $\lambda$ independent of f and $\Omega_1$, $\Omega_2$ and*

$$\| u \|_{\lambda.\Omega-\Omega_1} \leqq c \| f \|_{0.\Omega_2}.$$

**3.** Now we turn over to the question of local interior error estimates for the Ritz approximations. We assume the propositions of #2 to be valid. The main step will be

**Lemma 2.** *Let $\Omega_1$, $\Omega_2$ be domains with $\Omega_2 \subset\subset \Omega_1 \subseteq \Omega$. Let further $u$ be in $\mathring{W}_2^1(\Omega)$ and restricted to $\Omega_1$ in $W_2^p(\Omega_1)$ with $p \leqq n + 1$. The error $e = e_h = u - R_h u$ then fulfills the recurrence relation*

$$\| e \|_{1.\Omega_2} \leqq ch \| e \|_{1.\Omega_1} + ch^{p-1} \| u \|_{p.\Omega_1} + ch^{\lambda-1} \| e \|_{1.\Omega}.$$

As a direct consequence by iteration with respect to domains we have then

**Theorem.** *Under the conditions of Lemma 2 the error estimate*

$$\| e \|_{1.\Omega_2} \leqq ch^{p-1} \| u \|_{p.\Omega_1} + ch^{\lambda-1} \| e \|_{1.\Omega} \tag{5}$$

*is valid with $c$ depending only on $\Omega_1$, $\Omega_2$.*

This theorem shows that the error of the Ritz method locally depends on the one hand on the local regularity of the solution of the boundary value problem and on the other hand on an additional overall error term caused by possible irregularities of the boundary. For $\lambda$ sufficiently large, i.e. for smooth boundaries, the second term will be of order $h^{p-1}$ at least for $u \in \mathring{W}_2^1(\Omega)$ already since $\| e \|_{1.\Omega}$ is bounded by $c \| u \|_{1.\Omega}$. Otherwise if $u$ has in all of $\Omega$ a certain smoothness then $\| e \|_{1.\Omega}$ will be small and so the second term in (5) may still be dominated by the first one.

In order to prove the lemma we choose subdomains $\Omega_1'$, $\Omega_1''$ and $\Omega_2'$ according to

$$\Omega_2 \subset\subset \Omega_2' \subset\subset \Omega_1' \subset\subset \Omega_1'' \subset\subset \Omega_1 .$$

The constants $c$ in the subsequent inequalities may depend on the choice of these domains. In order to get an estimate for the error $e = e_h = u - u_h = u - R_h u$ in $\Omega_2$ we introduce a cut-off function $\omega$, i.e. a function $\omega \in C^\infty$ with $0 \leqq \omega \leqq 1$ and

$$\omega(x) = \begin{cases} 1 & \text{for} \quad x \in \Omega_2, \\ 0 & \text{for} \quad x \in \Omega - \Omega_2'. \end{cases}$$

We will use the abbreviation $\tilde{u} = \omega u$ etc., obviously we have $\tilde{e} = e$ in $\Omega_2$. We may write

$$\tilde{e} = \{\tilde{u} - R_h\tilde{u}\} - \{\tilde{u}_h - R_h\tilde{u}_h\} + R_h\tilde{e}. \tag{6}$$

All functions are in $\mathring{W}_2^1(\Omega)$, so for them the norm $\| . \|_{1.\Omega}$ is equivalent to $D(.)^{1/2}$ with $D(v) = D(v, v)$. Since $R_h$ is the minimal projection with respect to $D(.)$ we have

$$\| \tilde{u} - R_h\tilde{u} \|_{1.\Omega} \leqq cD(\tilde{u} - R_h\tilde{u})^{1/2} \leqq c \inf_{\chi \in S_h} D(\tilde{u} - \chi)^{1/2} \leqq$$
$$\leqq c \inf_{\chi \in S_h} \| \tilde{u} - \chi \|_{1.\Omega}.$$

Because of $\tilde{u} \in \mathring{W}_2^1(\Omega) \cap W_2^p(\Omega)$ and $\| \tilde{u} \|_{p.\Omega} \leqq c \| u \|_{p.\Omega_1}$ proposition 3 gives

$$\| \tilde{u} - R_h\tilde{u} \|_{1.\Omega} \leqq ch^{p-1} \| u \|_{p.\Omega_1}. \tag{7}$$

Next we estimate the second term in (6). Proposition 4 guarantees in a similar way

$$\| \tilde{u}_h - R_h \tilde{u}_h \|_{1.\Omega} \leqq ch^n \| u_h \|_{n.\Omega_1'}.$$

For $h$ sufficiently small we can because of proposition 1 find a $T \in \mathfrak{T}_h$ with $\Omega_1' \subseteq \subseteq T \subseteq \Omega_1''$. Then we get

$$h^n \| u_h \|_{n.\Omega_1'} \leqq h^n \| u_h \|_{n.T} \leqq ch^{p-1} \| u_h \|_{p-1.T}.$$

Now let $U_h \in S_h$ be an approximation according to lemma 1.
Then we have

$$\begin{aligned} h^{p-1} \| u_h \|_{p-1.T} &\leqq h^{p-1} \| u_h - U_h \|_{p-1.T} + h^{p-1} \| U_h \|_{p-1.T} \leqq \\ &\leqq ch \| u_h - U_h \|_{1.T} + h^{p-1} \{ \| u \|_{p-1.\Omega_1''} + \| u - U_h \|_{p-1.\Omega_1''} \}. \end{aligned}$$

The second term in the last inequality is bounded by $ch^{p-1} \| u \|_{p.\Omega_1}$. The first can further be estimated in the way

$$\begin{aligned} h \| u_h - U_h \|_{1.T} &\leqq h \| u - u_h \|_{1.T} + h \| u - U_h \|_{1.T} \leqq \\ &\leqq h \| e \|_{1.\Omega_1} + ch^p \| u \|_{p.\Omega_1} \end{aligned}$$

and so we have

$$\| \tilde{u}_h - R_h \tilde{u}_h \|_{1.\Omega} \leqq ch \| e \|_{1.\Omega_1} + ch^{p-1} \| u \|_{p.\Omega_1}. \tag{8}$$

It remains to bound the last term $R_h \tilde{e}$ of (6). Since this is an element of $S_h$ we have

$$\begin{aligned} \| R_h \tilde{e} \|_{1.\Omega} &\leqq cD(R_h \tilde{e})^{1/2} \leqq \\ &\leqq c \sup \{ D(R_h \tilde{e}, \chi) \mid \chi \in S_h \wedge \| \chi \|_{1.\Omega} \leqq 1 \}. \end{aligned}$$

Now—see (2)—for $\chi \in S_h$

$$D(R_h \tilde{e}, \chi) = D(\tilde{e}, \chi).$$

The factor $\omega$ of $\tilde{e} = \omega e$ in the Dirichlet integral can be shifted over:

$$D(\tilde{e}, \chi) = D(e, \tilde{\chi}) + D(e, v) \tag{9}$$

with $v$ being the solution of

$$\begin{aligned} -\Delta v &= 2\nabla(\omega, \chi) + \chi \Delta \omega \quad \text{in } \Omega \\ v &= 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega. \end{aligned}$$

We discuss first the term $D(e, \tilde{\chi})$. By the definition of $e = u - R_h u$ we have

$$| D(e, \tilde{\chi}) | = \inf_{\varphi \in S_h} | D(e, \tilde{\chi} - \varphi) |.$$

Since $\varphi$ may be choosen for $h$ sufficiently small according to supp $(\varphi) \subseteq \Omega_1'$ we get

$$\begin{aligned} | D(e, \tilde{\chi}) | &\leqq \| \omega\chi - \varphi \|_{1.\Omega'} \| e \|_{1.\Omega_1} \leqq \\ &\leqq ch^n \| \chi \|_{n.T} \| e \|_{1.\Omega_1} \leqq \\ &\leqq ch \| \chi \|_{1.T} \| e \|_{1.\Omega_1} \leqq ch \| \chi \|_{1.\Omega} \| e \|_{1.\Omega_1}. \end{aligned}$$

In order to estimate the second term in (9) we use a partition of unity with respect to $\Omega_1''$ and $\Omega - \Omega_1'$, i.e. $C^\infty$-functions $\sigma$, $\tau$ with $\sigma + \tau = 1$ in $\Omega$ and

$$\operatorname{supp}(\sigma) \subseteq \Omega_1'', \quad \operatorname{supp}(\tau) \subseteq \Omega - \Omega_1'$$

and write $v_1 = \sigma v$, $v_2 = \tau v$. We have

$$| D(e, v) | \leqq \inf_{\varphi_1 \in S_h} | D(e, v_1 - \varphi_1) | + \inf_{\varphi_2 \in S_h} | D(e, v_2 - \varphi_2) | .$$

Here the regularity assumption comes in. We have the a priori estimates

$$\| v_1 \|_{2.\Omega} \leqq c \| v \|_{2.\Omega_1'} \leqq c \| \chi \|_{1.\Omega},$$
$$\| v_2 \|_{\lambda.\Omega} \leqq c \| v \|_{2.\Omega_1'} \leqq c \| \chi \|_{1.\Omega}$$

and therefore similar to above for $\chi$ with $\| \chi \|_{1.\Omega} \leqq 1$

$$| D(e, v)| \leqq ch \| e \|_{1.\Omega_1} + ch^{\lambda-1} \| e \|_{1.\Omega}.$$

This gives finally the same estimate for $R_h\tilde{e}$:

$$\| R_h\tilde{e} \|_{1.\Omega} \leqq ch \| e \|_{1.\Omega_1} + ch^{\lambda-1} \| e \|_{1.\Omega}. \tag{10}$$

The estimates (7), (8) and (10) lead to the stated lemma 2.

*Author's address:*

*Joachim A. Nitsche*
*Department of Mathematics*
*University of Freiburg*
*Hebelstrasse 40, 78 Freiburg i. Br.,*
*Federal Republic of Germany*