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Real Anal. Exchange 17 (1) (1991_92), 37-38

Persistent URL: http://dml.cz/dmlcz/502163

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THE Equation $f^2 + g^2 = h^2$, where f, g, and h Are Derivatives

It is easy to show that the sum of squares of two derivatives is not always the square of a derivative. (Take, e.g., $f(x) = \sin\frac{1}{x}$, $g(x) = \cos\frac{1}{x}(x \neq 0)$, $f(0) = g(0) = 0$.) To investigate our equation we introduce the following notation: $I = [0,1]$; D is the class of all derivatives on I; $C[C_{ap}]$ is the class of all continuous [approximately continuous] functions on I; bC_{ap} is the class of all bounded elements of C_{ap}; $M = \{f \in D;\ fg \in D \text{ for each } g \in bC_{ap}\}$. It can be proved that $M \cap C_{ap}$ is the class of all Lebesgue functions and that each bounded derivative is in M.

It is easy to see that $\sqrt{f^2+g^2} \in D$, if $f, g \in D$ and $g/f \in C$. This simple result leads to the question whether the relation

$$f^2 + g^2 = h^2, \quad f, g, h \in D \tag{1}$$

implies something about g/f, if $f \neq 0$. The following theorem points in this direction:

Theorem 1. *Let (1) hold and let*

$$\liminf \operatorname{ap} h(y) > 0 \ \ (y \to x, y \in I) \ \ \textit{for each } x \in I. \tag{2}$$

Then $f/h,\ g/h \in C_{ap}$.

(This follows from [1], Proposition 4.6 with $m = 2$ and $|(x,y)| = \sqrt{x^2+y^2}$.) If, moreover, $f \neq 0$, then, clearly, $g/f \in C_{ap}$. Now it is natural to ask whether the relations $f, g \in D$ and $g/f \in C_{ap}$ imply that $\sqrt{f^2+g^2} \in D$. The next theorem gives a negative answer to this question.

Theorem 2. *Let $f \in D \setminus M$, $f > 0$. Let $\varepsilon \in (0,1)$. Then there is a $\beta \in C_{ap}$ such that $|\beta - 1| < \varepsilon$, $g = \beta f \in D$ and $\sqrt{f^2+g^2} \notin D$.*

We get, however, an h fulfilling (1) if we impose some restrictions on f and g; at the same time the requirement $g/f \in C_{ap}$ can be weakened, as Theorems 3 and 4 show.

Theorem 3. *Let $f, g \in M$; let $\alpha, \beta \in C_{ap}$, $\alpha^2 + \beta^2 > 0$; let ψ be a function such that $f = \alpha\psi$, $g = \beta\psi$. Set $\gamma = \sqrt{a^2 + \beta^2}$, $h = \frac{\alpha}{\gamma} f + \frac{\beta}{\gamma} g$. Then (1) holds.*

(The proof is easy.)

Theorem 4. *Let $f \in M$, $g \in D$, $f^2 + g^2 > 0$; let $\alpha, \beta \in C_{ap}$ and let ψ be a function such that $f = \alpha\psi$, $g = \beta\psi$. Suppose that there is an $A \in (-\infty, 0)$ such that $g \geqq A|f|$. Then $\sqrt{f^2 + g^2} \in D$.*

Example 5.12 in [1] shows that in Theorem 1 we cannot replace the requirement (2) by $h > 0$. However, we have Theorem 5 that points in the same direction as Theorem 1:

Theorem 5. *Let $f \in M$, $f > 0$ and let (1) hold. Then $g, h \in M$.*

A characterization of M and the proofs of Theorems 2, 4, and 5 will be published later.

Reference

[1] Jan Mařík and Clifford E. Weil, *Sums of powers of derivatives*, Proc. Amer. Math. Soc. 112 (1991), 807–817