Jan Mařík; Clifford E. Weil

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Jan Mařík and Clifford Weil, Michigan State University, East Lansing, MI 48824

SUMS OF POWERS OF DERIVATIVES

Let $R = (-\infty,\infty)$, $D = \{F';\ F$ is differentiable on $R\}$, and let C $[C_{ap}]$ be the system of all continuous [approximately continuous] functions on R. For each system Z of functions let $bZ = \{f \in Z;\ f$ is bounded$\}$ and $Z^+ = \{f \in Z;\ f \geq 0\}$. It is well known that the following implications hold:

(1) $(f \in D^+,\ \alpha \in C) \Rightarrow \alpha f \in D,$

(2) $(f \in bD,\ \alpha \in bC_{ap}) \Rightarrow \alpha f \in D.$

The authors are working on a paper whose title will be the same as the title of this talk. A special case of their investigation is the equation

(3) $f^2 + g^2 = h^2,\quad f,g,h \in D.$

If, e.g.,

(4) $\varphi \in D^+,\quad \alpha,\beta \in C,\quad f = \alpha\varphi,\quad g = \beta\varphi,\quad h = \sqrt{\alpha^2 + \beta^2}\ \varphi,$

then, by (1), the relation (3) is fulfilled. To get an analogy involving approximate continuity we introduce the system $S = \{f \in D;\ \alpha f \in D$ for each $\alpha \in bC_{ap}\}$. It follows from (2) that $S \supset bD$. It is not difficult to prove that we have, for instance, also $S \supset W$, where $W = \{f \in D;\ f^2 \in D\}$. (Hence S contains some functions that are not locally bounded.) Now we see that (3) holds as well, if

(5) $\varphi \in S,\quad \alpha,\beta \in bC_{ap},\quad f = \alpha\varphi,\quad g = \beta\varphi,\quad h = \sqrt{\alpha^2+\beta^2}\ \varphi.$

It is natural to ask the following question: If $f,g \in D$ are given, how do we recognize whether the relation

(6) $$\sqrt{f^2+g^2} \in D$$

holds? It is easy to find sufficient conditions for (6). For example, it is not difficult to prove that (6) holds, if $f,g \in W$. If we use the Darboux property of derivatives and (1), we obtain at once:

(7) If $f,g \in D$, $h = \sqrt{f^2+g^2} > 0$ and if $f/h, g/h \in C$, then $h \in D$.

The following analogy of (7) is almost obvious:

(8) If $f, g \in S$, $h = \sqrt{f^2+g^2} > 0$ and if $f/h, g/h \in C_{ap}$,

then $h \in D$.

(Since the functions f/h and g/h are bounded, (8) follows from the identity $h = f \cdot \frac{f}{h} + g \cdot \frac{g}{h}$.)

Now let us try to get a theorem "going in the other direction" (if (3) holds, then ...). Looking at (8) we are tempted to prove that $f/h \in C_{ap}$, if (3) holds and if $h > 0$. However, it is not difficult to construct derivatives (even bounded derivatives) f, g, h fulfilling (3) such that $h > 0$ and that neither of the functions f/h, g/h is approximately continuous. It turns out that for our purpose the requirement $h > 0$ is too weak. We need, for example,

(9) $\liminf_{y \to x} \text{ap } h(y) > 0$ for each $x \in R$.

(It is easy to see that the relations $h \in D$ and (9) imply that $h > 0$.) As a special case of the main theorem of the mentioned paper we now obtain the following:

(10) Let (3) and (9) hold. Then $f/h, g/h \in C_{ap}$.

Hence we get all (and some more) triples $f, g, h \in S$ fulfilling (3) and (9) applying the method (5); it suffices to take $\varphi = h$, $\alpha = f/h$, $\beta = g/h$.

If (3) holds and if, e.g., $g \geq 1$, then (9) is obvious and it follows from (10) that also $f/g \in C_{ap}$. If, moreover, $g = 1$, then $f \in C_{ap}$. This observation enables us to construct simple examples of pairs $f, g \in D$ for which there is no h fulfilling (3); we take an $f \in D \setminus C_{ap}$ and $g = 1$. (It is easy to see that "simple" discontinuous derivatives, like the function f defined by $f(x) = \sin 1/x$ $(x \neq 0)$ and $f(0) = 0$, are not even approximately continuous.)

As mentioned earlier, our paper deals with equations that are more general than (3); e.g., the results for the equation $f^4 + g^4 = h^4$ $(f, g, h \in D)$ are analogous. We investigate, however, also equations like

$$(11) \qquad f^4 + g^4 = h^2, \qquad f,g,h \in D.$$

In this case the results are even better; namely, the relations (11) and (9) imply that all the functions f, g, and h are approximately continuous.