Miloš Kössler
On the zeros of analytic functions

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It was agreed that Resolutions 1–8 be submitted for confirmation to a Special General Meeting to be held on Thursday, February 10th, 1921.

ABSTRACTS.

On the Zeros of Analytic Functions

Dr. MILOŠ KÖSSLER.

I start with the equation

$$\phi(x) - uf(x) = 0, \tag{1}$$

where $\phi(x)$ and $f(x)$ are analytic functions.

If $a_1, a_2, a_3, \ldots$, the roots of $\phi(x) = 0$ are supposed known, I form the power series

$$x_k = \sum_{n=0}^{\infty} a_n^{(k)} u^n, \tag{2}$$

where

$$a_0^{(k)} = a_k, \quad a_m^{(k)} = \frac{1}{m!}\frac{d^{m-1}}{dx^{m-1}}\left[\left(\frac{x-a_k}{\phi(x)}\right)^m f^m(x)\right]_{x=a_k} \quad (k = 1, 2, 3, \ldots). \tag{3}$$

These power series, which represent the roots of (1), are convergent inside a definite circle $|u| = R$. I transform them into the polynomial developments of Mittag-Leffler,

$$(4) \qquad x_k = \sum_{m=0}^{\infty} P_m^{(k)}(u),$$

which are convergent in the whole star, and it is now possible to calculate the roots of (1) for every value of u.

In the case of multiple roots of $\phi(x) = 0$, it is necessary to make a slight modification of the series (2).

This method is very general and powerful; the three following results are obtained as special cases:—

(I) The roots of the general algebraic equation

$$x^n - (a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n) = 0,$$

are expressible in the form

$$x_k = \sum_{m=1}^{\infty} \frac{e^{2km\pi i/n}}{m!} \frac{d^{m-1}}{dx^{m-1}} \left[(a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n)^{m/n}\right]_{x=0} \qquad (k = 0, 1, 2, \ldots, n-1),$$

if the coefficients $a_1, a_2, \ldots, a_n$ satisfy certain definite conditions; and the roots are expressible in the form

$$x_k = \sum_{m=1}^{\infty} P_m^{(k)}(e^{2k\pi i/n}),$$

when the coefficients have arbitrary values.

(II) *All* the zeros of such functions as

$$R(x, e^x), \quad R(x, \sin x), \quad R(x, e^{h(x)}), \quad R[\wp(x), e^x],$$

where $R(u, v)$ denotes a rational function of u and v, $h(x)$ is a polynomial in x and $\wp(x)$ is the Weierstrassian elliptic function, can be developed in expansions of the type (4).

(III) All the zeros of a given integral function $F(x)$ can be developed in this manner by using the equation

$$\sin x - u[F(x) + \sin x] = 0,$$

and calculating the zeros when $u = 1$.

As an example consider the zeros of

$$F(x) \equiv \sin x - ie^x.$$

For small values of $|u|$ we solve the equation

$$\sin x - ue^x = 0,$$

by an ascending series

$$x_k = \sum_{m=0}^{\infty} a_m^{(k)} u^m \quad (k = 0, 1, 2, 3, \ldots),$$

where $$a_0^{(k)} = \pm k\pi, \quad a_m^k = \frac{1}{m!} \frac{d^{m-1}}{dx^{m-1}} \left[\left(\frac{x \mp k\pi}{\sin x} \right)^m e^{mx} \right]_{x=\pm k\pi}.$$

The zeros of $F(x)$ are then given by Borel's formula

$$x_k = \int_0^\infty e^{-t} F_k(it)\, dt,$$

by putting $$F_k(u) = \sum_{m=0}^{\infty} \frac{a_m^{(k)} u^m}{m!}.$$

On Dr. Sheppard's Method of Reduction of Error by Linear Compounding

Prof. A. S. EDDINGTON.

Dr. W. F. Sheppard's theory (*Phil. Trans.*, Vol. 221, A, pp. 199–237) is here treated according to the methods and notation of the tensor calculus. In this way great compactness is attained, and the symmetry of the formulæ becomes apparent. A geometrical interpretation is given of the significance of the processes employed. This method of treating the problem is likely to appeal chiefly to those who already have some familiarity with the theory of tensors; but since it provides an illustration of the elementary notions of tensors, it may also be of use as a first introduction to that subject.

On the Linear Differential Equation of the Second Order

Prof. H. J. PRIESTLEY.

The following results, arrived at in a paper to be communicated to the forthcoming meeting of the Australasian Association for the Advancement of Science, may be of interest to the members of the London Mathematical Society.