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Ann. of Math. 37 (1936), 681-697

Persistent URL: http://dml.cz/dmlcz/501047

ANNALS OF MATHEMATICS
Vol. 37, No. 3, July, 1936

MULTIPLICATIONS ON A COMPLEX

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(Received February 20, 1936)

In their communications at the First International Topological Conference (Moscow, September 1935), J. W. Alexander and A. Kolmogoroff introduced the notion of a dual cycle[1] and defined a product of a dual p-cycle and a dual q-cycle, this product being a dual $(p + q)$-cycle. A different multiplication of the same sort is considered in this paper. It may be shown that the Alexander-Kolmogoroff product, augmented by the dual boundary of a suitable $(p + q - 1)$-chain, is equal to the $\binom{p+q}{p}^{\text{th}}$ multiple of the product here introduced.[2] Moreover, I consider also a product of an ordinary n-cycle and a dual p-cycle $(n \geqq p)$, this product being an ordinary $(n - p)$-cycle. There is a simple algebraic relationship between the two kinds of multiplication, which I shall explain elsewhere. As an application of the general theory, I give a new approach to the duality and intersection theory of a combinatorial manifold, given in a simplicial subdivision. The theory works exclusively in the given subdivision.

This is a preliminary paper of a purely combinatorial nature. In a later paper, I shall apply the same methods to general topological spaces, and in particular to the very general "manifolds" defined in my recent note in the *Proceedings of the National Academy of Sciences* (U. S. A.).

Many of the results of this paper were found independently by H. Whitney, but his methods of proof seem not much related to mine.

1. Let there be given a complex K. We shall designate by σ_i^p $(p = 0, 1, 2, \cdots)$ the (oriented) p-simplices of K and by $\eta_{i\,j}^p$ $(= 0, 1, -1)$ the incidence coefficient of σ_i^{p+1} and σ_j^p.

The word *group* will always designate an additively written abelian group. If $\mathfrak{A}$ is a group, then a $(p, \mathfrak{A})$-chain is a symbol of the form $a_i\sigma_i^p$, $a_i \in \mathfrak{A}$, where, as always in this paper, one has to sum over every subscript appearing twice.

The *boundary* FA^p of a $(p, \mathfrak{A})$-chain $A^p = a_i\sigma_i^p$ is zero if $p = 0$, and it is the $(p - 1, \mathfrak{A})$-chain

$$FA^p = \eta_{i\,j}^{p-1} a_i \sigma_j^{p-1}$$

[1] As a matter of fact the *Topology* of S. Lefschetz (1930), contains an essentially equivalent notion (pp. 282–286).

[2] In his paper "On the Connectivity Ring of an Abstract Space" in this number of the *Annals of Mathematics*, pp. 698–708, J. W. Alexander has modified his definition, and it is now in agreement with the one here presented.

if $p > 0$. If $FA^p = 0$, we say that A^p is an *ordinary* $(p, \mathfrak{A})$-*cycle*. The $(p, \mathfrak{A})$-chain FA^{p+1} is an ordinary $(p, \mathfrak{A})$-cycle for every $(p+1, \mathfrak{A})$-chain A^{p+1}. Two ordinary $(p, \mathfrak{A})$-cycles A_1^p and A_2^p are said to be of the same homology class, or to be homologous to each other (in symbols $A_1^p \sim A_2^p$) if there exists a $(p+1, \mathfrak{A})$-chain A_0^{p+1} such that

$$A_1^p - A_2^p = FA_0^{p+1}.$$

The *dual boundary* F^*A^p of a $(p, \mathfrak{A})$-chain $A^p = a_i \sigma_i^p$ is the $(p+1, \mathfrak{A})$-chain

$$F^*A^p = \eta_{ji}^p a_i \sigma_j^{p+1}.$$

If $F^*A^p = 0$, we say that A^p is a *dual* $(p, \mathfrak{A})$-*cycle*. The $(p+1, \mathfrak{A})$-chain F^*A^p is a dual $(p+1, \mathfrak{A})$-cycle for every $(p, \mathfrak{A})$-chain A^p. Two dual $(p, \mathfrak{A})$-cycles A_1^p and A_2^p are said to be of the same homology class, or to be homologous to each other (in symbols $A_1^p \sim A_2^p$) (1) in the case $p = 0$ only if they are identical, (2) in the case $p > 0$ if there exists a $(p-1, \mathfrak{A})$-chain A_0^{p-1} such that

$$A_1^p - A_2^p = F^*A_0^{p-1}.$$

2. Let $\mathfrak{B}$ be a given group. Let B^q be a given dual $(q, \mathfrak{B})$-cycle. By an *auxiliary construction* we mean an operation attaching to every simplex σ_i^p $(p = 0, 1, 2, \cdots)$ a $(p+q, \mathfrak{B})$-chain $B^{p+q}(\sigma_i^p)$ such that the following three conditions are satisfied. *First*, if the coefficient of a $(p+q)$-simplex τ^{p+q} in $B^{p+q}(\sigma_i^p)$ is different from zero, then σ_i^p must be a face of τ^{p+q}. *Second*, we must have

$$B^q = \sum_i B^q(\sigma_i^0). \tag{2.1}$$

Third, we must have for every simplex σ_i^p $(p = 0, 1, 2, \cdots)$

$$F^*B^{p+q}(\sigma_i^p) = \sum_j \eta_{ji}^p B^{p+q+1}(\sigma_j^{p+1}). \tag{2.2}$$

3. We shall prove that *the auxiliary construction is always possible*. Let there be given a fixed ordering of the vertices of K. Let σ^p be a given p-simplex, written as

$$\sigma^p = (v_0, v_1, \cdots, v_p)$$

corresponding to the given ordering of vertices (i.e. v_0 precedes v_1 etc.). We shall define the $(p+q, \mathfrak{B})$-chain $B^{p+q}(\sigma^p)$ as follows. The only $(p+q)$-simplices appearing in $B^{p+q}(\sigma^p)$ will have, corresponding to the given ordering of vertices, the form

$$(v_0, v_1, \cdots, v_p, \cdots, v_{p+q}), \tag{3.1}$$

i.e. the first $p+1$ vertices will be those of σ^p. The coefficient of any such simplex (3.1) in $B^{p+q}(\sigma^p)$ will be equal to the coefficient of the q-simplex $(v_p, \cdots, v_{p+q})$ in B^q.

The first two properties of the auxiliary construction being evident, we have only to prove (2.2) for

$$\sigma_i^p = \sigma^p = (v_0, v_1, \cdots, v_p).$$

The only $(p + q + 1)$-simplices τ^{p+q+1} appearing on either side of (2.2) must all have σ^p as their common face and, moreover, corresponding to the given ordering of vertices, the vertex v_p must be either the $(p + 1)^{\text{th}}$ or the $(p + 2)^{\text{th}}$ vertex of τ^{p+q+1}. We have to prove that any such τ^{p+q+1} has equal coefficients on both sides of (2.2). This being quite evident in the case where v_p is the $(p + 2)^{\text{th}}$ vertex of τ^{p+q+1}, we only have to examine the case when, in the given order of vertices, we have

$$\tau^{p+q+1} = (v_0, v_1, \cdots, v_p, \cdots, v_{p+q+1}).$$

Let b_{p+i} $(0 \leqq i \leqq q + 1)$ be the coefficient, in the $(q, \mathfrak{B})$-chain B^q, of the oriented q-simplex obtained from $(v_p, \cdots, v_{p+q+1})$ by omitting the vertex v_{p+i}. The coefficients of τ^{p+q+1} in both sides of (2.2) are respectively equal to

$$(-1)^{p+1}b_{p+i} \text{ and to } (-1)^{p+1}b_p.$$

But since B^q is a dual $(q + 1, \mathfrak{B})$-cycle, the coefficient of the $(q + 1)$-simplex $(v_p, \cdots, v_{p+q+1})$ in F^*B^q must vanish, i.e.

$$(-1)^i b_{p+i} = 0 \quad \text{or} \quad (-1)^{p+i}b_{p+i} = (-1)^{p+1}b_p.$$

4. Let us suppose that the dual $(q, \mathfrak{B})$-cycle B^q is identically zero. $B^{p+q}(\sigma_i^p)$ being the elements of an auxiliary construction chosen in any manner corresponding to $B^q = 0$, we shall prove that we may attach to every p-simplex $\sigma_i^p (p = 1, 2, 3, \cdots)$ a $(p + q - 1, \mathfrak{B})$-chain $C^{p+q-1}(\sigma_i^p)$ such that the following three conditions are satisfied. *First,* if the coefficient of a $(p + q - 1)$-simplex τ^{p+q-1} in $C^{p+q-1}(\sigma_i^p)$ is different from zero, then σ_i^p must be a face of τ^{p+q-1}. *Second,* we have for every 0-simplex σ_i^0

$$B^q(\sigma_i^0) = \eta_{ji}^0 C^q(\sigma_j^1). \tag{4.1}$$

Third, we have for every p-simplex σ_i^p, where $p = 1, 2, 3, \cdots,$

$$B^{p+q}(\sigma_i^p) = \eta_{ji}^p C^{p+q}(\sigma_j^{p+1}) + F^*C^{p+q-1}(\sigma_i^p). \tag{4.2}$$

We begin by the construction of $C^q(\sigma_j^1)$. Let τ^q be any q-simplex and let $b_i(\tau^q)$ be its coefficient in $B^q(\sigma_i^0)$. If σ_i^0 is not a vertex of τ^q, we have $b_i(\tau^q) = 0$. Moreover, since $B^q = 0$, it follows from (2.1) that $\sum_i b_i(\tau^q) = 0$. Therefore, $b_i(\tau^q) \cdot \sigma_i^0$ is an ordinary $(0, \mathfrak{B})$-cycle of the q-simplex τ^q having zero as the sum of its coefficients. It is well known that such a $(0, \mathfrak{B})$-cycle is equal to the boundary of a $(1, \mathfrak{B})$-chain of the q-simplex τ^q. Therefore there exists, for every 1-simplex σ_j^1, an element $c_j(\tau^q)$ of the group $\mathfrak{B}$ such that (1) $c_j(\tau^q) = 0$ if σ_j^1 is not a face of τ^q, (2) $b_i(\tau^q) = \eta_{ji}^0 c_j(\tau^q)$ for every σ_i^0. Let us put

$$C^q(\sigma_j^1) = \sum c_j(\tau^q)\tau^q,$$

the summation running over all q-simplices τ^q. Then σ_j^1 is a face of every q-simplex appearing in $C^q(\sigma_j^1)$ and the relations (4.1) hold true.

If we put $C^{q-1}(\sigma_i^0) = 0$, the relation (4.2) corresponding to $p = 0$ reduces to (4.1). Therefore, we may suppose our construction carried through up to the relations (4.2), where p is given, and we have to construct $(p + q + 1)$-chains $C^{p+q+1}(\sigma_k^{p+2})$ satisfying the analogous relations

$$(4.3) \qquad B^{p+q+1}(\sigma_j^{p+1}) = \eta_{kj}^{p+1} C^{p+q+1}(\sigma_k^{p+2}) + F^* C^{p+q}(\sigma_j^{p+1}).$$

Since $F^* C^{p+q-1}(\sigma_i^p)$ is a dual $(p + q, \mathfrak{B})$-cycle, it follows from (4.2) that

$$F^* B^{p+q}(\sigma_i^p) = \eta_{ji}^p F^* C^{p+q}(\sigma_j^{p+1}).$$

Comparing with (2.2) we get

$$(4.4) \qquad \eta_{ji}^p B^{p+q+1}(\sigma_j^{p+1}) - F^* C^{p+q}(\sigma_j^{p+1}) = 0.$$

Now let τ^{p+q+1} be any $(p + q + 1)$-simplex and let $b_j(\tau^{p+q+1})$ be its coefficient in the $(p + q + 1, \mathfrak{B})$-chain

$$B^{p+q+1}(\sigma_j^{p+1}) - F^* C^{p+q}(\sigma_j^{p+1}).$$

If σ_j^{p+1} is not a face of τ^{p+q+1}, we have $b_j(\tau^{p+q+1}) = 0$. Moreover, it follows from (4.4) that $\eta_{ji}^p b_j(\tau^{p+q+1}) = 0$. Therefore, $b_j(\tau^{p+q+1})\sigma_j^{p+1}$ is an ordinary $(p + 1, \mathfrak{B})$-cycle of the $(p + q + 1)$-simplex τ^{p+q+1}. It is well known that such a $(p + 1, \mathfrak{B})$-cycle is equal to the boundary of a $(p + 2, \mathfrak{B})$-chain of the simplex τ^{p+q+1}. Therefore there exists, for every $(p + 2)$-simplex σ_k^{p+2}, an element $c_k(\tau^{p+q+1})$ of the group $\mathfrak{B}$ such that (1) $c_k(\tau^{p+q+1}) = 0$ if σ_k^{p+2} is not a face of τ^{p+q+1}, (2) $b_j(\tau^{p+q+1}) = \eta_{kj}^{p+1} c_k(\tau^{p+q+1})$ for every σ_j^{p+1}. Let us put

$$C^{p+q+1}(\sigma_k^{p+2}) = \sum c_k(\tau^{p+q+1}) \cdot \tau^{p+q+1}$$

the summation running over all $(p + q + 1)$-simplices τ^{p+q+1}. Then σ_k^{p+2} is a face of every $(p + q + 1)$-simplex appearing in $C^{p+q+1}(\sigma_k^{p+2})$ and the relations (4.3) hold true.

5. Let there be given three groups $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$. Let there be given a law attaching to every couple a, b, where $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$, an element $c \in \mathfrak{C}$, called the *product* of a and b and designated by ab or $a \cdot b$. Furthermore, let us suppose the validity of the distributive laws

$$(a_1 + a_2)b = a_1 b + a_2 b, \qquad a(b_1 + b_2) = ab_1 + ab_2.$$

In such circumstances, we put $\mathfrak{C} = (\mathfrak{A}, \mathfrak{B})$ and say that there is given an $(\mathfrak{A}, \mathfrak{B})$-*multiplication.*

Any $(\mathfrak{A}, \mathfrak{B})$-multiplication defines an "*inverse*" $(\mathfrak{B}, \mathfrak{A})$-multiplication (with the same group $\mathfrak{C}$), if we define the new product ba to be equal to the original product ab.

6. Let there be given an $(\mathfrak{A}, \mathfrak{B})$-multiplication. Let

$$A^p = a_i \sigma_i^p$$

be a dual $(p, \mathfrak{A})$-cycle. Let B^q be a dual $(q, \mathfrak{B})$-cycle. We shall define their product A^pB^q as a dual $[p + q, (\mathfrak{A}, \mathfrak{B})]$-cycle which, however, will be affected with a slight indetermination. To this end, we start with B^q and choose an auxiliary construction (sect. 2), which is always possible by sect. 3. Then we put

$$A^pB^q = a_i B^{p+q}(\sigma_i^p).$$

From (2.2) we have

$$F^* a_i B^{p+q}(\sigma_i^p) = a_i F^* B^{p+q}(\sigma_i^p) = \eta_{ji}^p a_i B^{p+q+1}(\sigma_j^{p+1})$$

which is equal to zero, since $\eta_{ji}^p a_i = 0$, A^p being a dual p-cycle. Therefore, $F^*(A^pB^q) = 0$, i.e., the product A^pB^q is indeed a dual $(p + q)$-cycle.

It can easily be seen that the product A^pB^q is *not* uniquely determined, depending really on the choice of the auxiliary construction. But the *homology class* of the product A^pB^q *is* determined without ambiguity, i.e. any two values $(A^pB^q)_1$ and $(A^pB^q)_2$ are connected by the homology

$$(A^pB^q)_1 \sim (A^pB^q)_2.$$

This fact is an easy consequence of the following statement: If $B^q = 0$, then $A^pB^q \sim 0$ for any choice of the auxiliary construction. We proceed to the proof of that statement. If $B^q = 0$, we saw in sect. 4 that there exist chains $C^{p+q-1}(\sigma_i^p)$ such that (4.1) and (4.2) hold true. If $p = 0$, it follows from (4.1) that

$$A^0B^q = a_i B^q(\sigma_i^0) = \eta_{ji}^0 a_i C^q(\sigma_i^0) = 0,$$

because $\eta_{ji}^0 a_i = 0$. If $p > 0$, it follows from (4.2) that

$$\begin{aligned} F^* a_i C^{p+q-1}(\sigma_i^p) &= a_i F^* C^{p+q-1}(\sigma_i^p) = a_i B^{p+q}(\sigma_i^p) - \eta_{ji}^p a_i C^{p+q}(\sigma_j^{p+1}) \\ &= a_i B^{p+q}(\sigma_i^p) = A^pB^q \end{aligned}$$

because $\eta_{ji}^p a_i = 0$. Therefore $A^pB^q = F^* a_i C^{p+q-1}(\sigma_i^p) \sim 0$.

The homology class of the product A^pB^q is uniquely determined by the homology classes of A^p and B^q. This is an easy consequence of the following statement: If either $A^p \sim 0$ or $B^q \sim 0$, then $A^pB^q \sim 0$. Let us first suppose that $A^p \sim 0$. If $p = 0$, then $A^p = 0$, which implies $A^pB^q = 0$. If $p > 0$, then there exists a $(p - 1, \mathfrak{A})$-chain $\alpha_j \sigma_j^{p-1}$ such that $A^p = a_i \sigma_i^p = F^*(\alpha_j \sigma_j^{p-1})$, i.e. $a_i = \eta_{ij}^{p-1} \alpha_j$. According to (2.2), we have

$$F^* \alpha_j B^{p+q-1}(\sigma_j^{p-1}) = \alpha_j F^* B^{p+q-1}(\sigma_j^{p-1}) = \eta_{ij}^{p-1} \alpha_j B^{p+q}(\sigma_i^p) = a_i B^{p+q}(\sigma_i^p) = A^pB^q$$

so that $A^pB^q = 0$. Now we suppose that $B^q \sim 0$. If $q = 0$, then $B^q = 0$, which we know to imply $A^pB^q \sim 0$. If $q > 0$, then there exists a $(q - 1, \mathfrak{B})$-chain H^{q-1} such that $B^q = F^*H^{q-1}$. One finds easily $(q - 1, \mathfrak{B})$-chains $H^{q-1}(\sigma_i^0)$ such that (1) σ_i^0 is a vertex of every $(q - 1)$-simplex appearing in $H^{q-1}(\sigma_i^0)$,

(2) $H^{q-1} = \sum_i H^{q-1}(\sigma_i^0)$. If we put $B^q(\sigma_i^0) = F^*H^{q-1}(\sigma_i^0)$ and $B^{p+q}(\sigma_i^p) = 0$ for $p > 0$, we evidently have an auxiliary construction in the sense of sect. 2. With this choice of auxiliary construction, we have $A^pB^q = 0$ if $p > 0$, and

$$A^0B^q = F^*a_iH^{q-1}(\sigma_i^0) \sim 0 \text{ if } p = 0.$$

7. Let there be given an ordering of the vertices of the complex K. Then we can use the particular auxiliary construction described in sect. 3, which leads to the following simple definition of the product A^pB^q. Given a $(p + q)$-simplex σ^{p+q}, we write it as

$$\sigma^{p+q} = (v_0, v_1, \cdots, v_p, \cdots, v_{p+q})$$

according to the given ordering of the vertices. Let a be the coefficient of the p-simplex $(v_0, v_1, \cdots, v_p)$ in the $(p, \mathfrak{A})$-chain A^p; let b be the coefficient of the q-simplex $(v_p, \cdots, v_{p+q})$ in the $(q, \mathfrak{B})$-chain B^q. Then ab is the coefficient of σ^{p+q} in the product A^pB^q.

This definition leads to a simple proof of the *commutative law*:

$$B^qA^p \sim (-1)^{pq}A^pB^q. \tag{7.1}$$

Here we suppose that, $\mathfrak{A}$ and $\mathfrak{B}$ being two groups, A^p is a dual $(p, \mathfrak{A})$-cycle and B^q is a dual $(q, \mathfrak{B})$-cycle. Furthermore, an $(\mathfrak{A}, \mathfrak{B})$-multiplication is given, and hence an inverse $(\mathfrak{B}, \mathfrak{A})$-multiplication also (sect. 5). The products A^pB^q and B^qA^p are formed according to the first and second of these multiplications, respectively. To prove (7.1), we fix the value of A^pB^q according to a given ordering of the vertices, and fix B^qA^p according to the *inverse* ordering of the vertices. Let a $(p + q)$-simplex

$$(v_0, v_1, \cdots, v_p, \cdots, v_{p+q})$$

be written in the original ordering of the vertices. Since

$$(v_p, \cdots, v_0) = (-1)^{\frac{1}{2}p(p+1)}(v_0, \cdots, v_p),$$

$$(v_{p+q}, \cdots, v_p) = (-1)^{\frac{1}{2}q(q+1)}(v_p, \cdots, v_{p+q}),$$

$$(v_{p+q}, \cdots, v_p, \cdots, v_0) = (-1)^{\frac{1}{2}(p+q)(p+q+1)}(v_0, \cdots, v_p, \cdots, v_{p+q}),$$

$$\tfrac{1}{2}(p + q)(p + q + 1) = \tfrac{1}{2}p(p + 1) + \tfrac{1}{2}q(q + 1) + pq,$$

it is readily seen that, with our particular choice of the auxiliary construction, we have $B^qA^p = (-1)^{pq}A^pB^q$. It seems difficult to prove the commutative law (7.1) directly from the general definition given in sect. 6.

The *distributive laws*

$$(A_1^p + A_2^p)B^q \sim A_1^pB^q + A_2^pB^q, \tag{7.2}$$

$$A^p(B_1^q + B_2^q) \sim A^pB_1^q + A^pB_2^q \tag{7.3}$$

are immediate consequences of either of the two definitions of the product.

Now suppose that three groups $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_3$ are given. Let there be given an $(\mathfrak{A}_1, \mathfrak{A}_2)$-multiplication and an $(\mathfrak{A}_2, \mathfrak{A}_3)$-multiplication. Further, putting

$$(\mathfrak{A}_1, \mathfrak{A}_2) = \mathfrak{A}_{12}, \qquad (\mathfrak{A}_2, \mathfrak{A}_3) = \mathfrak{A}_{23},$$

let us suppose that there is given an $(\mathfrak{A}_{12}, \mathfrak{A}_3)$-multiplication and an $(\mathfrak{A}_1, \mathfrak{A}_{23})$-multiplication. Suppose, finally, that the associative law

$$a_1 a_2 \cdot a_3 = a_1 \cdot a_2 a_3$$

holds true for $a_1 \in \mathfrak{A}_1$, $a_2 \in \mathfrak{A}_2$, $a_3 \in \mathfrak{A}_3$. Then we have, if $A_i^{p_i}$ $(i = 1, 2, 3)$ is a dual $(p_i, \mathfrak{A}_i)$-cycle, the *associative law*

$$A_1^{p_1} A_2^{p_2} \cdot A_3^{p_3} \sim A_1^{p_1} \cdot A_2^{p_2} A_3^{p_3}. \tag{7.4}$$

The proof based on a given ordering of the vertices is quite trivial. A proof based directly on our general definition of the product is not difficult, however.

It would be interesting to prove, using only definitions based on the ordering of the vertices, that the homology class of the product $A^p B^q$ is independent of the choice of the ordering.[3]

8. Let there be given an $(\mathfrak{A}, \mathfrak{B})$-multiplication. If $A^p = a_i \sigma_i^p$ is a $(p, \mathfrak{A})$-chain and if $B^p = b_i \sigma_i^p$ is a $(p, \mathfrak{B})$-chain, let us put

$$\varphi(A^p, B^p) = a_i b_i \in (\mathfrak{A}, \mathfrak{B}).$$

If A^{p+1} is a $(p + 1, \mathfrak{A})$-chain and if B^p is a $(p, \mathfrak{B})$-chain, it is readily seen that

$$\varphi(FA^{p+1}, B^p) = \varphi(A^{p+1}, F^*B^p); \tag{8.1}$$

similarly we have

$$\varphi(A^p, FB^{p+1}) = \varphi(F^*A^p, B^{p+1}) \tag{8.2}$$

for any $(p, \mathfrak{A})$-chain A^p and any $(p + 1, \mathfrak{B})$-chain B^{p+1}.

9 Let there be given an $(\mathfrak{A}, \mathfrak{B})$-multiplication. Let A^{p+q} be an *ordinary* $(p + q, \mathfrak{A})$-cycle. Let B^q be a *dual* $(q, \mathfrak{B})$-cycle. We shall define a product $A^{p+q}B^q$ (not quite uniquely determined), which will be an *ordinary* $[p, (\mathfrak{A}, \mathfrak{B})]$-cycle. We choose an auxiliary construction $B^{p+q}(\sigma_i^p)$ associated with B^q (sect. 2), and we put

$$A^{p+q} B^q = c_i \sigma_i^p,$$

where (see sect. 8)

$$c_i = (-1)^{pq} \varphi[A^{p+q}, B^{p+q}(\sigma_i^p)].$$

That $A^{p+q}B^q$ is an ordinary $[p, (\mathfrak{A}, \mathfrak{B})]$-cycle, is trivial if $p = 0$. If $p > 0$,

[3] Such a proof has now been given by J. W. Alexander; see his paper cited above.

it follows from (2.2) and (8.1) that, for any $(p-1)$-simplex σ_j^{p-1},

$$(-1)^{pq}\eta_{i\,j}^{p-1}c_i = \eta_{i\,j}^{p-1}\varphi[A^{p+q}, B^{p+q}(\sigma_i^p)] = \varphi[A^{p+q}, \eta_{i\,j}^{p-1}B^{p+q}(\sigma_i^p)]$$
$$= \varphi[A^{p+q}, F^*B^{p+q-1}(\sigma_j^{p-1})] = \varphi[FA^{p+q}, B^{p+q-1}(\sigma_j^{p-1})] = \varphi[0, B^{p+q-1}(\sigma_j^{p-1})] = 0,$$

i.e. $F(A^{p+q}B^q) = 0$.

Suppose that $B^q = 0$. If $p = 0$, it follows from (4.1) that

$$\varphi[A^q, B^q(\sigma_i^0)] = \eta_{j\,i}^0\varphi[A^q, C^q(\sigma_j^1)],$$

so that

$$A^qB^q = F(\gamma_j\sigma_j^1), \qquad \gamma_j = \varphi[A^q, C^q(\sigma_j^1)],$$

i.e. $A^qB^q \sim 0$. If $p > 0$, it follows from (4.2) that

$$\varphi[A^{p+q}, B^{p+q}(\sigma_i^p)] = \eta_{j\,i}^p\varphi[A^{p+q}, C^{p+q-1}(\sigma_j^{p+1})] + \varphi[A^{p+q}, F^*C^{p+q-1}(\sigma_i^p)].$$

But the last summand is zero, from (8.1), since $FA^{p+q} = 0$. Therefore

$$A^{p+q}B^q = F(\gamma_j\sigma_j^{p+1}), \qquad \gamma_j = (-1)^{pq}\varphi[A^{p+q}, C^{p+q-1}(\sigma_j^{p+1})],$$

i.e. again $A^{p+q}B^q \sim 0$.

It follows readily from the preceding proof that, in any case, the homology class of the $[p, (\mathfrak{A}, \mathfrak{B})]$-cycle $A^{p+q}B^q$ is independent of the choice of the auxiliary construction. As a matter of fact, this homology class is uniquely determined by the homology classes of the ordinary $(p+q, \mathfrak{A})$-cycle A^{p+q} and the dual $(q, \mathfrak{B})$-cycle B^q. It is sufficient to prove that $A^{p+q}B^q \sim 0$, if either $A^{p+q} \sim 0$ or $B^q \sim 0$. If $A^{p+q} \sim 0$, there exists a $(p+q+1, \mathfrak{A})$-chain H^{p+q+1} such that $A^{p+q} = FH^{p+q+1}$. It follows easily from (2.2) and (8.1) that

$$A^{p+q}B^q = F(\gamma_j\sigma_j^{p+1}) \sim 0, \gamma_j = (-1)^{pq}\varphi[H^{p+q+1}, B^{p+q+1}(\sigma_j^{p+1})].$$

If $B^q \sim 0$ and $q = 0$, we have $B^q = 0$, which we know to imply $A^{p+q}B^q \sim 0$. If $B^q \sim 0$ and $q > 0$, we choose the auxiliary construction as at the end of sect. 6: $B^q(\sigma_i^0) = F^*H^{q-1}(\sigma_i^0)$ and $B^{p+q}(\sigma_i^p) = 0$ for $p > 0$. If $p > 0$, we have then $A^{p+q}B^q = 0$. If $p = 0$, we have again $A^qB^q = 0$ from (8.1), since $FA^q = 0$.

If A^p is a *dual* $(p, \mathfrak{A})$-cycle and if B^{p+q} is an *ordinary* $[(p+q), \mathfrak{B}]$-cycle, we put

$$A^pB^{p+q} = c_i\sigma_i^q,$$

where

$$c_i = \varphi[A^{p+q}(\sigma_i^p), B^{p+q}],$$

the $(p+q, \mathfrak{A})$-chains $A^{p+q}(\sigma_i^q)(q = 0, 1, 2, \cdots)$ being the elements of an auxiliary construction associated with A^p. Again, the product is an ordinary $[q, (\mathfrak{A}, \mathfrak{B})]$-cycle and only its homology class is uniquely determined, this class being indeed given by the mere knowledge of the homology classes of the factors. If A^{p+q} is an ordinary $(p+q, \mathfrak{A})$-cycle and if B^q is a dual $(q, \mathfrak{B})$-cycle, we have evidently

(9.1) $$A^{p+q}B^q \sim (-1)^{pq}B^qA^{p+q},$$

where the left-hand member is defined according to the given $(\mathfrak{A}, \mathfrak{B})$-multiplication and the right-hand member according to the inverse $(\mathfrak{B}, \mathfrak{A})$-multiplication.

10. Let there be given an ordering of the vertices of the complex K. The particular auxiliary construction described in sect. 3 leads to following simple definition of the product A^pB^{p+q} of a dual $(p, \mathfrak{A})$-cycle A^p and an ordinary $(p + q, \mathfrak{B})$-cycle B^{p+q}. Given a q-simplex σ^q, we write it as

$$\sigma^q = (v_0, v_1, \cdots, v_q)$$

according to the given ordering of the vertices, and consider all the $(p + q)$-simplices

$$\sigma_k^{p+q} = (v_0, v_1, \cdots, v_q, \cdots, v_{p+q})$$

having σ^q as their common face and such that, in the given ordering, v_q precedes any vertex of σ_k^{p+q} which is not a vertex of σ^q. For every such σ_k^{p+q} put

$$\sigma_k^p = (v_q, \cdots, v_{p+q}).$$

Let a_k be the coefficient of σ_k^p in A^p; let b_k be the coefficient of σ_k^{p+q} in B^{p+q}. Then the coefficient of σ^q in A^pB^{p+q} is equal to

$$\sum_k a_k b_k.$$

Now let us consider the product $A^{p+q}B^q$ of an ordinary $(p + q, \mathfrak{A})$-cycle A^{p+q} and a dual $(q, \mathfrak{B})$-cycle B^q. This time we use the auxiliary construction based on the *inverse* ordering of the vertices, but we describe the result in terms of the original ordering. Given a p-simplex σ^p, we write it as

$$\sigma^p = (v_q, \cdots, v_{p+q})$$

according to the given ordering of the vertices, and consider all the $(p + q)$-simplices

$$\sigma_k^{p+q} = (v_0, v_1, \cdots, v_q, \cdots, v_{p+q})$$

having σ^p as their common face and such that, in the given ordering, v_q follows any vertex of σ_k^{p+q} which is not a vertex of σ^p. For every such σ_k^{p+q}, put

$$\sigma_k^q = (v_0, \cdots, v_q).$$

Let a_k be the coefficient of σ_k^{p+q} in A^{p+q}; let b_k be the coefficient of σ_k^q in B^q. Then the coefficient of σ^p in $A^{p+q}B^q$ is equal to

$$\sum_k a_k b_k.$$

These definitions, in connection with that given at the beginning of sect. 7

(for the product of two dual cycles) lead to a simple proof of the *associative laws*:

$$A_1^{p_1+p_2+p_3} B_2^{p_2} \cdot B_3^{p_3} \sim A_1^{p_1+p_2+p_3} \cdot B_2^{p_2} B_3^{p_3}, \tag{10.1}$$

$$B_1^{p_1} A_2^{p_1+p_2+p_3} \cdot B_3^{p_3} \sim B_1^{p_1} \cdot A_2^{p_1+p_2+p_3} B_3^{p_3}, \tag{10.2}$$

$$B_1^{p_1} B_2^{p_2} \cdot A_3^{p_1+p_2+p_3} \sim B_1^{p_1} \cdot B_2^{p_2} A_3^{p_1+p_2+p_3}. \tag{10.3}$$

Here we suppose given three groups $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$, an $(\mathfrak{A}_1, \mathfrak{A}_2)$-multiplication, an $(\mathfrak{A}_2, \mathfrak{A}_3)$-multiplication, an $(\mathfrak{A}_{12}, \mathfrak{A}_3)$-multiplication with $\mathfrak{A}_{12} = (\mathfrak{A}_1, \mathfrak{A}_2)$ and an $(\mathfrak{A}_1, \mathfrak{A}_{23})$-multiplication with $\mathfrak{A}_{23} = (\mathfrak{A}_2, \mathfrak{A}_3)$. It is supposed that $a_1 a_2 \cdot a_3 = a_1 \cdot a_2 a_3$ for $a_i \in \mathfrak{A}_i$ $(i = 1, 2, 3)$. $A_i^{p_1+p_2+p_3}$ $(i = 1, 2, 3)$ is an ordinary $(p_1 + p_2 + p_3, \mathfrak{A}_i)$-cycle and $B_i^{p_i}$ $(i = 1, 2, 3)$ is a dual $(p_i, \mathfrak{A}_i)$-cycle. Of course, any of the three formulas (10.1), (10.2) and (10.3) implies the others using (7.1) and (9.1). We omit writing explicitly the trivial *distributive laws*.

11. In the remaining part of this paper the coefficients of all chains are taken from the additive group of all integer numbers. Moreover, we suppose that $K = M_n$ is an orientable simple n-circuit, i.e. that the following four conditions are satisfied. *First*, each simplex of M_n is either an n-simplex or a face of an n-simplex. *Second*, each $(n - 1)$-simplex of M_n is a common face of precisely two n-simplices of M_n. *Third*, any two n-simplices of M_n may be connected by a sequence of n-simplices of M_n such that any two consecutive n-simplices of the sequence have a common $(n - 1)$-face. *Fourth*, the n-simplices σ_i^n of M_n can be given such orientations that their sum $\Gamma^n = \sum_i \sigma_i^n$ is an ordinary n-cycle. (We always suppose the orientation of the n-simplices chosen in this manner.)

If σ_i^p is any p-simplex of M_n, we denote by Lk. $[\sigma_i^p]$ its *link*, i.e. the subcomplex of M_n composed of all the simplices τ of M_n having no common vertex with σ_i^p but having the property that there exists a simplex of M_n having both τ and σ_i^p among its faces.

If $0 \leqq p \leqq n$, we say that M_n is *p-regular* if the following two conditions are satisfied. *First* (requiring nothing if $p = n$ or $p = n - 1$), the link Lk. $[\sigma_i^p]$ on any p-simplex of M_n is an orientable simple $(n - p - 1)$-circuit. Second (requiring nothing if $p = 0$), for each k such that $0 \leqq k \leqq p - 1$, any dual $(n - p - 1)$-cycle of any Lk. $[\sigma_i^k]$ is homologous to zero in Lk. $[\sigma_i^k]$. It is easily seen that the orientable combinatorial n-manifolds are identical with orientable simple n-circuits, which are p-regular for any $0 \leqq p \leqq n$.

12. For $0 \leqq p \leqq n$, we denote by $\mathfrak{B}_p$ the group of all the homology classes of ordinary p-cycles of M_n and by $\bar{\mathfrak{B}}_p$ the group of all the homology classes of dual p-cycles of M_n.

Given any dual $(n - p)$-cycle B^{n-p} of M_n $(0 \leqq p \leqq n)$, we put

$$\psi_p(B^{n-p}) = \Gamma^n \cdot B^{n-p},$$

where $\Gamma^n = \sum_i \sigma_i^n$. Evidently, ψ_p is a homomorphic mapping of the group $\bar{\mathfrak{B}}_{n-p}$ on a subgroup $\psi_p(\bar{\mathfrak{B}}_{n-p})$ of the group $\mathfrak{B}_p$.

13. *If M_n is p-regular, then the mapping ψ_p is* $1 - 1$, so that the group $\bar{\mathfrak{B}}_{n-p}$ is isomorphic with a subgroup [i.e. $\psi_p(\bar{\mathfrak{B}}_{n-p})$] of the group $\mathfrak{B}_p$.

It is sufficient to prove that $\Gamma^n B^{n-p} \sim 0$ implies $B^{n-p} \sim 0$.

Let $B^{n-p+k}(\sigma_i^k)$ be the elements of a given auxiliary construction associated with the dual $(n - p)$-cycle B^{n-p}. Since $\Gamma^n \cdot B^{n-p} \sim 0$, there exists a $(p + 1)$-chain $c_j \sigma_j^{p+1}$ such that $\Gamma^n \cdot B^{n-p} = (-1)^{p(n-p)} F(c_j \sigma_j^{p+1})$, i.e.

$$\varphi[\Gamma^n, B^n(\sigma_i^p)] = \eta_{j\,i}^{p} c_j .$$

For any σ_j^{p+1}, let us choose an n-simplex τ^n such that σ_j^{p+1} is a face of τ^n, and put $H^n(\sigma_j^{p+1}) = c_j \tau^n$. Since $\Gamma^n = \sum_i \sigma_i^n$, we have $\varphi[\Gamma_j^n H^n(\sigma_j^{p+1})] = c_j$ and, therefore,

$$\varphi[\Gamma^n, B_0^n(\sigma_i^p)] = 0 , \tag{13.1}$$

where

$$B_0^n(\sigma_i^p) = B^n(\sigma_i^p) - \eta_{j\,i}^{p} H^n(\sigma_j^{p+1}) . \tag{13.2}$$

Evidently σ_i^p is a face of each n-simplex appearing in the n-chain $B_0^n(\sigma_i^p)$. Therefore there exists in the link Lk. $[\sigma_i^p]$ an $(n - p - 1)$-chain $C^{n-p-1}(\sigma_i^p)$ such that the n-chain $B_0^n(\sigma_i^p)$ can be obtained from the $(n - p - 1)$-chain C^{n-p-1} by replacing each $(n - p - 1)$-simplex

$$(v_{p+1}, \cdots, v_n)$$

by the n-simplex

$$(v_0, \cdots, v_p, v_{p+1}, \cdots, v_n)$$

where

$$(v_0, \cdots, v_p) = \sigma_i^p . \tag{13.3}$$

Since the complex Lk. $[\sigma_i^p]$ contains no $(n - p)$-simplex, the $(n - p - 1)$-chain $C^{n-p-1}(\sigma_i^p)$ of the complex Lk. $[\sigma_i^p]$ must be a dual $(n - p - 1)$-cycle. Moreover, the equation (13.1) signifies that the sum of the coefficients of $C^{n-p-1}(\sigma_i^p)$ is equal to zero. Since M_n is p-regular, Lk. $[\sigma_i^p]$ is an orientable simple $(n - p - 1)$-circuit, which implies readily the existence of an $(n - p - 2)$-chain $D^{n-p-2}(\sigma_i^p)$ in the complex Lk. $[\sigma_i^p]$ such that

$$F^* D^{n-p-2}(\sigma_i^p) = (-1)^{p+1} C^{n-p-1}(\sigma_i^p) . \tag{13.4}$$

Let $H^{n-1}(\sigma_i^p)$ signify the $(n - 1)$-chain which arises from the $(n - p - 2)$-chain $D^{n-p-2}(\sigma_i^p)$ by replacing each $(n - p - 2)$-simplex

$$(v_{p+1}, \cdots, v_{n-1})$$

by the $(n-1)$-simplex

$$(v_0, \cdots, v_p, v_{p+1}, \cdots, v_{n-1}),$$

supposing the validity of (13.3). Then (13.4) implies that

$$F^*H^{n-1}(\sigma_i^p) = B_0^n(\sigma_i^p). \tag{13.5}$$

Moreover, σ_i^p is a face of every $(n-1)$-simplex appearing in the $(n-1)$-chain $H^{n-1}(\sigma_i^p)$.

Now, let us put

$$B_{p-1}^n(\sigma_i^p) = 0,$$
$$B_{p-1}^{n-1}(\sigma_j^{p-1}) = B^{n-1}(\sigma_j^{p-1}) - \eta_{i\,j}^{p\,p-1} H^{n-1}(\sigma_i^p)$$

and

$$B_{p-1}^{n-p+k}(\sigma_i^k) = B^{n-p+k}(\sigma_i^k) \quad \text{for} \quad p-1 \neq k \neq p.$$

From (13.2) and (13.5) it is easily seen that the chains $B_{p-1}^{n-p+k}(\sigma_i^k)$ form an auxiliary construction associated with B^{n-p}.

Now let us suppose that (as we have found to be possible in the case $r = p-1$) we have found chains $B_r^{n-p+k}(\sigma_i^k)(1 \leqq r \leqq p-1)$ forming an auxiliary construction associated with B^{n-p} and such that $B_r^{n-p+r+1}(\sigma_i^{r+1}) = 0$. By the definition of an auxiliary construction, we have

$$F^*B_r^{n-p+r}(\sigma_i^r) = 0 \tag{13.6}$$

for each σ_i^r. Since σ_i^r is a face of each $(n-p+r)$-simplex appearing in $B_r^{n-p+r}(\sigma_i^r)$, there exists in the link Lk. $[\sigma_i^r]$ an $(n-p-1)$-chain $C^{n-p-1}(\sigma_i^r)$ such that the $(n-p+r)$-chain $B_r^{n-p+r}(\sigma_i^r)$ can be obtained from the $(n-p-1)$-chain $C^{n-p-1}(\sigma_i^r)$ by replacing each $(n-p-1)$-simplex

$$(v_{r+1}, \cdots, v_{n-p+r})$$

by the $(n-p+r)$-simplex

$$(v_0, \cdots, v_r, v_{r+1}, \cdots, v_{n-p+r}),$$

where

$$(v_0, \cdots, v_r) = \sigma_i^r. \tag{13.7}$$

Now the equation (13.6) signifies that $C^{n-p-1}(\sigma_i^r)$ is a dual $(n-p-1)$-cycle of the complex Lk. $[\sigma_i^r]$. Since M_n is p-regular, it follows that there exists an $(n-p-2)$-chain $D^{n-p-2}(\sigma_i^r)$ of the complex Lk. $[\sigma_i^r]$ such that

$$F^*D^{n-p-2}(\sigma_i^r) = (-1)^{r+1}C^{n-p-1}(\sigma_i^r). \tag{13.8}$$

Let $H^{n-p+r-1}(\sigma_i^r)$ denote the $(n-p+r-1)$-chain which arises from the $(n-p-2)$-chain $D^{n-p-2}(\sigma_i^r)$ by replacing each $(n-p-2)$-simplex

$$(v_{r+1}, \cdots, v_{n-p+r-1})$$

by the $(n-p+r-1)$-simplex

$$(v_0, \cdots, v_r, v_{r+1}, \cdots, v_{n-p+r-1}) .$$

supposing the validity of (13.7). Then (13.8) implies that

$$F^* H^{n-p+r-1}(\sigma_i^r) = B_r^{n-p+r}(\sigma_i^r) . \tag{13.9}$$

Now, let us put

$$\begin{gathered} B_{r-1}^{n-p+r}(\sigma_i^r) = 0 , \\ B_{r-1}^{n-p+r-1}(\sigma_j^{r-1}) = B_r^{n-p+r-1}(\sigma_j^{r-1}) - \eta_{i\,j}^{r\,r-1} H^{n-p+r-1}(\sigma_i^r) \end{gathered} \tag{13.10}$$

and

$$B_{r-1}^{n-p+k}(\sigma_i^k) = B_r^{n-p+k}(\sigma_i^k) \quad \text{for} \quad r-1 \neq k \neq r .$$

It follows readily from (13.9) that the chains $B_{r-1}^{n-p+k}(\sigma_i^k)$ form an auxiliary construction associated with B^{n-p} and such that (13.10) holds true.

Applying the preceding argument successively for $r = p-1, p-2, \cdots, 2, 1$, we obtain an auxiliary construction $B_0^{n-p+k}(\sigma_i^k)$ associated with B^{n-p} and such that $B_0^{n-p+1}(\sigma_i^1) = 0$. Applying the same argument again in the case $r = 0$, we have (13.9), written now as

$$F^* H^{n-p-1}(\sigma_i^0) = B_0^{n-p}(\sigma_i^0) .$$

But since $B_0^{n-p}(\sigma_i^0)$ are elements of an auxiliary construction associated with B^{n-p}, we have $B^{n-p} = \sum_i B_0^{n-p}(\sigma_i^0) = F^* \sum_i H^{n-p-1}(\sigma_i^0)$, whence $B^{n-p} \sim 0$.

14. *If M_n is $(p-1)$-regular,*[4] *then the group $\psi_p(\bar{\mathfrak{B}}_{n-p})$ is the whole group $\mathfrak{B}_p$,* so that the group $\mathfrak{B}_p$ is a homomorphic image of the group $\bar{\mathfrak{B}}_{n-p}$. Comparing this with the result of the preceding section we see that, *if M_n is both $(p-1)$-regular and p-regular, the groups $\mathfrak{B}_p$ and $\bar{\mathfrak{B}}_{n-p}$ are isomorphic.*

Let $C^p = c_i \sigma_i^p$ be an ordinary p-cycle of M_n, so that $\eta_{i\,j}^{p\,p-1} c_i = 0$. We shall find a dual $(n-p)$-cycle B^{n-p} and an auxiliary construction $B^{n-p+k}(\sigma_i^k)$ associated with it such that $\Gamma^n \cdot B^{n-p} = C^p$, i.e.

$$\varphi[\Gamma^n, B^n(\sigma_i^p)] = c_i . \tag{14.1}$$

The construction of n-chains $B^n(\sigma_i^p)$ satisfying (14.1) is quite evident; it is sufficient to choose for each σ_i^p an n-simplex τ^n having σ_i^p among its faces and to put $B^n(\sigma_i^p) = c_i \tau^n$. Since $\eta_{i\,j}^{p\,p-1} c_i = 0$, we have for each σ_j^{p-1}

$$\varphi[\Gamma^n, \eta_{i\,j}^{p\,p-1} B^n(\sigma_i^p)] = 0 . \tag{14.2}$$

Since σ_j^{p-1} is a face of every n-simplex appearing in $\eta_{i\,j}^{p\,p-1} B^n(\sigma_i^p)$ and since the $(p-1)$-regularity of M_n implies that the link Lk. $[\sigma_j^{p-1}]$ is an orientable simple $(n-p)$-circuit, we can start with (14.2) and repeat the same argument which, in the preceding section and starting with (13.1), led us to (13.5). We thus

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[4] Any M_n is supposed to be (-1)-regular.

obtain, for every σ_j^{p-1}, an $(n-1)$-chain $B^{n-1}(\sigma_j^{p-1})$ such that σ_j^{p-1} is a face of each simplex appearing in $B^{n-1}(\sigma_j^{p-1})$ and such that

$$F^*B^{n-1}(\sigma_j^{p-1}) = \eta_{i\,j}^{p-1}B^n(\sigma_i^p) .$$

More generally, let us suppose that, for a given $r(1 \leqq r \leqq p-1)$, we have succeeded in attaching to every σ_i^k $(r \leqq k \leqq p)$ an $(n-p+k)$-chain $B^{n-p+k}(\sigma_i^k)$ having the two following properties. *First*, σ_i^k is a face of each $(n-p+k)$-simplex appearing in $B^{n-p+k}(\sigma_i^k)$. *Second*, we have for $r \leqq k \leqq p-1$

$$F^*B^{n-p+k}(\sigma_i^k) = \eta_{j\,i}^k B^{n-p+k+1}(\sigma_j^{k+1}) . \tag{14.3}$$

It follows that

$$F^*\eta_{i\,j}^{r-1}B^{n-p+r}(\sigma_i^r) = 0 . \tag{14.4}$$

Since σ_j^{r-1} is a face of every $(n-p+r)$-simplex appearing in $\eta_{i\,j}^{r-1}B^{n-p+r}(\sigma_i^r)$ and since the $(p-1)$-regularity of M_n implies that every dual $(n-p-1)$-cycle of the complex Lk. $[\sigma_j^{r-1}]$ is homologous to zero in Lk. $[\sigma_j^{r-1}]$, we can start with (14.4) and repeat the same argument which, in the preceding section and starting with (13.6), led us to (13.9). We obtain thus, for every σ_j^{r-1}, an $(n-p+r-1)$-chain $B^{n-p+r-1}(\sigma_j^{r-1})$ such that σ_j^{r-1} is a face of each simplex appearing in $B^{n-p+r-1}(\sigma_j^{r-1})$ and such that (14.3) holds true for $k = r-1$.

Starting with the chains $B^n(\sigma_i^p)$ and $B^{n-1}(\sigma_i^{p-1})$ already found, and applying the preceding argument successively for $r = p-1, p-2, \cdots, 2, 1$, we find chains $B^{n-p+k}(\sigma_i^k)$ $(0 \leqq k \leqq p)$ such that σ_i^k is a face of each simplex appearing in $B^{n-p+k}(\sigma_i^k)$ and such that (14.3) holds true for $0 \leqq k \leqq p-1$. In particular, for $k = 0$, (14.3) says that

$$F^*B^{n-p}(\sigma_i^0) = \eta_{j\,i}^0 B^{n-p+1}(\sigma_j^1) .$$

Since $\sum_i \eta_{j\,i}^0 = 0$ for every σ_j^1, we have $F^* \sum_i B^{n-p}(\sigma_i^0) = 0$, i.e.

$$B^{n-p} = \sum_i B^{n-p}(\sigma_i^0)$$

is a dual $(n-p)$-cycle. Of course our chains $B^{n-p+k}(\sigma_i^k)$ form an auxiliary construction associated with B^{n-p} and we have $\Gamma^n \cdot B^{n-p} = C^p$.

15. Let $0 \leqq p \leqq n$, $0 \leqq q \leqq n$. Suppose that M_n is r-regular both for $r = p$ and for $r = q$. Let C^p be an ordinary p-cycle belonging to the family $\psi_p(\mathfrak{B}_{n-p})$; let D^q be an ordinary q-cycle belonging to the family $\psi_q(\mathfrak{B}_{n-q})$; if M_n is r-regular also for $r = p-1$ and $r = q-1$, we know (sect. 14) that the cycles C^p and D^q are unrestricted.

We shall define the *intersection* of C^p and D^q and we shall designate it by $C^p \times D^q$. In the case $p+q < n$ we simply put

$$C^p \times D^q = 0 .$$

In the case $p+q \geqq n$, we shall define $C^p \times D^q$ as an ordinary $(p+q-n)$-cycle, but only its homology class will be uniquely determined.

Since C^p belongs to $\psi_p(\bar{\mathfrak{Z}}_{n-p})$, there exists a dual $(n - p)$-cycle A^{n-p} such that

$$\Gamma^n A^{n-p} \sim C^p . \tag{15.1}$$

Since D^q belongs to $\psi_q(\bar{\mathfrak{Z}}_{n-q})$, there exists a dual $(n - q)$-cycle B^{n-q} such that

$$\Gamma^n B^{n-q} \sim D^q . \tag{15.2}$$

We know (see sect. 13) that the homology classes of A^{n-p} and B^{n-q} are uniquely defined.

This being done, we put

$$C^p \times D^q \sim \Gamma^n \cdot A^{n-p} B^{n-q} . \tag{15.3}$$

It follows from (10.1) and (15.1) that

$$C^p \times D^q \sim C^p B^{n-q} . \tag{15.4}$$

The *distributive laws*

$$\begin{aligned} (C_1^p + C_2^p) \times D^q &\sim (C_1^p \times D^q) + (C_2^p \times D^q) , \\ C^p \times (D_1^q + D_2^q) &\sim (C^p \times D_1^q) + (C^p \times D_2^q) \end{aligned} \tag{15.5}$$

are evident. The *commutative law*

$$D^q \times C^p \sim (-1)^{(n-p)\,(n-q)} C^p \times D^q \tag{15.6}$$

follows from (7.1) and (15.3). If M_n is also s-regular and if E^s is an ordinary s-cycle belonging to the family $\psi_s(\bar{\mathfrak{Z}}_{n-s})$, we see from (7.4), (10.1) and (15.3) the validity of the *associative law*

$$(C^p \times D^q) \times E^s \sim C^p \times (D^q \times E^s) . \tag{15.7}$$

16. Let M_n be an orientable combinatorial n-manifold and let M_n' be its barycentrical subdivision. It is well known that M_n' is also an orientable combinatorial n-manifold. We shall show that, on the manifold M_n', our definition of intersection of ordinary cycles is equivalent to the classical definition.

Let $\sigma_i^p (0 \leqq p \leqq n)$ denote the simplices of M_n. We choose the orientation of the n-simplices σ_i^n in such manner that $\gamma^n = \sum_i \sigma_i^n$ is an ordinary n-cycle on M_n; we choose arbitrarily the orientation of the p-simplices $\sigma_i^p (1 \leqq p \leqq n - 1)$ and, as usual, we denote by $\eta_{i\,j}^p$ the incidence coefficient of σ_i^{p+1} and $\sigma_j^p (0 \leqq p \leqq n - 1)$.

Now let us recall the definition of the complex M_n'. The vertices of M_n' are identical with the simplices $\sigma_i^p (0 \leqq p \leqq n)$ of M_n. The vertices $\sigma_{i_0}^{p_0}$, $\sigma_{i_1}^{p_1}, \cdots , \sigma_{i_r}^{p_r}$ of M_n', where $p_0 \leqq p_1 \leqq \cdots \leqq p_r$, form an r-simplex of M_n' if and only if (1) $p_0 < p_1 < \cdots < p_r$, (2) $\sigma_{i_s}^{p_s}$ is a face of $\sigma_{i_{s+1}}^{p_{s+1}}$ for $0 \leqq s \leqq r - 1$.

Put

$$\Gamma^n = \sum \eta^0_{i_1 i_0} \eta^1_{i_2 i_1} \cdots \eta^{n-1}_{i_n i_{n-1}} (\sigma^0_{i_0}, \sigma^1_{i_1}, \cdots, \sigma^n_{i_n})$$

the summation running over all the n-simplices of M'_n. It is well known that Γ^n is an ordinary n-cycle of M'_n (usually called the barycentrical subdivision of γ^n).

The classical intersection of two ordinary cycles on M'_n is obtained by choosing each factor in a particular way in its homology class, which we must describe in detail.

Let $H^p = a_i \sigma^p_i$ be an ordinary p-cycle of M_n. Put

$$C^p = \sum \eta^0_{i_1 i_0} \eta^1_{i_2 i_1} \cdots \eta^{p-1}_{i_p i_{p-1}} a_{i_p} (\sigma^0_{i_0}, \sigma^1_{i_1}, \cdots, \sigma^p_{i_p}),$$

the summation running over all the p-simplices of M'_n having the indicated form $(\sigma^0_{i_0}, \sigma^1_{i_1}, \cdots, \sigma^p_{i_p})$. Let $K^{n-q} = b_i \sigma^{n-q}_i$ be a dual $(n-q)$-cycle of M_n. Put

$$D^q = \sum \eta^{n-q}_{i_{n-q+1} i_{n-q}} \cdots \eta^{n-1}_{i_n i_{n-1}} b_{i_{n-q}} (\sigma^{n-q}_{i_{n-q}}, \sigma^{n-q+1}_{i_{n-q+1}}, \cdots, \sigma^n_{i_n}),$$

the summation running over all the q-simplices of M'_n having the indicated form $(\sigma^{n-q}_{i_{n-q}}, \sigma^{n-q+1}_{i_{n-q+1}}, \cdots, \sigma^n_{i_n})$.

In the classical theory of combinatorial manifolds it is shown that C^p is an ordinary p-cycle on M'_n, that D^q is an ordinary q-cycle on M'_n, and that we may choose the ordinary p-cycle H^p on M_n and the dual $(n-q)$-cycle K^{n-q} on M_n in such a manner that C^p and D^q are homologous to arbitrarily given ordinary p-cycle and q-cycle on M'_n. The classical intersection of C^p and D^q is zero if $p + q < n$; in the case $p + q \geqq n$, it is equal to

$$C^p \times D^q = \sum \eta^{n-q}_{i_{n-q+1} i_{n-q}} \cdots \eta^{p-1}_{i_p i_{p-1}} a_{i_p} b_{i_{n-q}} (\sigma^{n-q}_{i_{n-q}}, \cdots, \sigma^p_{i_p}), \tag{16.1}$$

the summation running over all the $(p + q - n)$-simplices of M'_n having the indicated form $(\sigma^{n-q}_{i_{n-q}}, \cdots, \sigma^p_{i_p})$.

The case $p + q < n$ being trivial, we have to show that, if $p + q \geqq n$, (16.1) holds true according to our definition of intersection.

We now choose an ordering ω of the vertices of M_n and define an $(n-q)$-chain B^{n-q} on M'_n as follows. Let

$$\tau^{n-q} = (\sigma^{h_0}_{i_0}, \sigma^{h_1}_{i_1}, \cdots, \sigma^{h_{n-q}}_{i_{n-q}})$$

be an $(n-q)$-simplex of M'_n. Let v_λ be the first vertex of the h_λ-simplex $\sigma^{h_\lambda}_{i_\lambda}$.

$(0 \leqq \lambda \leqq n - q)$, relatively to the ordering ω. If the v_λ's $(0 \leqq \lambda \leqq n - q)$ are not all different from each other, then the coefficient of τ^{n-q} in B^{n-q} will be zero. In the other case,

$$(v_0, v_1, \cdots, v_{n-q}) \tag{16.2}$$

is an $(n-q)$-simplex of M_n and the coefficient of τ^{n-q} in B^{n-q} will be equal to the coefficient of (16.2) in K^{n-q}. It is not difficult to verify that B^{n-q} is a dual cycle on M'_n.

Now we order the set of all the vertices of M'_n in such a manner that σ_i^h precedes σ_j^k, whenever $h < k$; this can be done in many ways. We form the product $\Gamma^n B^{n-q}$ in the manner explained in sect. 10, using our ordering of the vertices of M'_n. We easily verify that

$$\Gamma^n B^{n-q} = D^q,$$

so that

$$C^p \times D^q \sim C^p B^{n-q}$$

from (15.2) and (15.3). Now if we form the product $C^p B^{n-q}$ again in the manner explained in sect. 10, using the same ordering of the vertices of M'_n, we easily verify that (16.1) holds true.

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