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ON CENTRAL DISPERSIONS OF THE DIFFERENTIAL EQUATION $Y'' = q(t)Y$ WITH PERIODIC COEFFICIENTS

O. Borůvka

I. Introduction.

In the following lecture I shall deal with ordinary 2^{nd} order linear differential equations of Jacobi's type

$$(q) \qquad y'' = q(t)y \ ,$$

mostly oscillatory.

Suppose that the coefficient q, the so-called carrier of the equation (q), is a continuous function in the interval $j = (-\infty, \infty)$. The equation (q) is called oscillatory if each of its integrals has an infinite number of zeros which accumulate towards both ends of j. The prototype of these oscillatory equations is the equation (-1): $y'' = -y$, $t \in j$, which we shall often deal with in what follows.

Let us consider, in particular, the equations (q) with periodic carriers q. The theory of these equations is governed by Floquet's theory. According to the latter, every equation (q) with a periodic carrier q is either disconjugate, i. e., without conjugate points, or oscillatory. To oscillatory equations (q), on the other hand, one may apply the theory of dispersions, based, as we know, on other principles than Floquet's theory. We may therefore expect that by combining the notions and the results of both theories we may arrive at a new approach to oscillatory equations (q) with periodic carriers, yielding new results. And this is exactly the leading idea of the following considerations.

II. Generalities.

Let me, first, introduce some basic notions and results from the theory of dispersions, needed in our study: 1. phases, 2. central dispersions, 3. inverse equations. ([1],[2])

All the equations (q) we shall deal with are oscillatory in the interval $j = (-\infty, \infty)$. A composite function, e. g., $a[b(t)]$ will often be written in the form $ab(t)$ or ab. Furthermore, we shall use the notation: $c_n(t) = t+n\pi$ $(n=0,\pm 1,\ldots;\ t\in j)$ and instead of c_1 we shall generally only put c.

Let (q) stand for an arbitrary (oscillatory) equation.

1. Phases. Let us note, first, that by a basis of the equation (q) or of the carrier q we mean an ordered pair, u,v, of independent integrals of (q). As to the phases, more precisely: the first phases, we distinguish the phases of a given basis of (q) and the phases of the equation (q) or of the carrier q. By a phase of the basis u,v we mean every function in the interval j, $\alpha(t)$, which is continuous in j and, for $v(t) \neq 0$ satisfies the relation: $\tan\alpha(t) = u(t):v(t)$. By a phase of the equation (q) (of the carrier q) we mean any phase of some basis of (q). Every phase α of (q) has the following properties:

1. $\alpha(t)\in C_j^3$, 2. $\alpha'(t)\neq 0$ for $t\in j$, 3. $\lim\limits_{t\to \sigma.\infty}\alpha(t)=\sigma.\operatorname{sgn}\alpha'.\infty$ $(\sigma=\pm 1)$

Every phase α of (q) uniquely determines its carrier in the sense of formula:

$$q(t) = -\{\tan\alpha, t\} \qquad (t\in j), \tag{1}$$

where the symbol $\{\ \}$ denotes Schwartz's derivative of the function $\tan\alpha$ at the point t. The carrier q with the phase α is also denoted q_α.

α is called an upper or a lower phase if the property 2. is reinforced by the inequality $\alpha'(t) > 0$ and if, moreover, there holds $\alpha(t) > t$ or $\alpha(t) < t$ $(t \in j)$, respectively. Both the upper and the lower phases are called dispersion phases.

The phase α is called elementary if

$$\alpha(t+\pi) = \alpha(t) + \pi.\operatorname{sgn}\alpha' \qquad (t \in j),$$

i. e., $\alpha c = c_{\operatorname{sgn}\alpha'}\alpha$. Every elementary phase α is of the form

$$\alpha(t) = t.\operatorname{sgn}\alpha' + p(t) \qquad (t \in j)$$

where p is a periodic function with period π : $pc = p$.

The set formed of all the elementary phases, together with the operation given by composing the functions, forms a group called the group of elementary phases, $\mathcal{G}$.

An important part is played by phases of the equation (-1): $y'' = -y$. Each phase of the latter, $\varepsilon(t)$, may be expressed in the form

$$(2) \qquad \varepsilon(t) = {}_n\mathrm{Arc}\ \tan\left[C \cdot \frac{\sin\ (t+a)}{\sin\ (t+b)}\right],$$

where n is an integer and $C;a,b$ constants such that $a \in (0,\pi]$, $b \in [0,\pi)$, $C\sin(b-a) \neq 0$. The expression on the right-hand side in (2) denotes the continuous function in the interval j uniquely determined by the conditions:

$$\varepsilon(-a-\overline{n-1}\,\pi)=0,\ \tan\ \varepsilon(t)=C.\sin(t+a):\sin(t+b) \text{ for } \sin(t+b)\neq 0.$$

ε is an upper phase if $C.\sin(b-a) > 0$ and, moreover, either $n = 1$ and between the constants $C;a,b$ there are further relations, or $n \geqslant 2$; ε is a lower phase if, again, $C.\sin(b-a) > 0$ and, moreover, either $n = 0$ and between the constants $C;a,b$ there are further relations, or $n \leq -1$.

The set of all phases of the equation (-1), together with the operation given by the composing of functions, forms again a group called the fundamental group $\mathfrak{F}$. The latter is a subgroup in $\mathfrak{G}$, $\mathfrak{F} \subset \mathfrak{G}$, hence all the phases of the equation (-1) are elementary.

An important result: Let α be an arbitrary phase of the equation (q). Then all the phases of the equation (q) are exactly the composite functions $\varepsilon\alpha(t)$, $\varepsilon \in \mathfrak{F}$.

2. Central dispersions. To every number n $(=0, \pm 1, \ldots)$ there corresponds the central dispersion (of the first kind) with the index n of the equation (q) or of the carrier q, denoted φ_n. It is a function in the interval j and its value $\varphi_n(t)$ $(t \in j)$ is determined as follows: If $n \neq 0$, then $\varphi_n(t)$ is the $|n|^{th}$ conjugate number (of the first kind) with the number t, greater or smaller than t according as $n > 0$ or $n < 0$. In case of $n = 0$, we put $\varphi_0(t) = t$. The dispersion φ_1, in particular, is called fundamental and is often denoted φ.

From the number of important properties of the dispersions φ_n I shall only introduce these:

Every dispersion φ_n has the above properties 1.-3. of phases while the property 2. is replaced by the inequality $\alpha'(t) > 0$; moreover, there holds $\varphi_{n-1}(t) < \varphi_n(t)$ $(t \in j)$. We see that φ_n is an upper or a lower phase of a convenient equation according as $n \geqslant 1$ or $n \leqslant -1$.

Every phase α of the equation (q) is connected with the dispersion φ_n by Abelian relation

(3) $$\alpha\varphi_n(t) = \alpha(t) + n.\pi . \operatorname{sgn} \alpha' \qquad (t \in j),$$

more briefly: $\alpha\varphi_n = c_{n.\operatorname{sgn}\alpha'}\alpha$.

Hence follows the expression of the dispersion φ_n by α :

$$\varphi_n(t) = \alpha^{-1} c_{n.\operatorname{sgn}\alpha'}\alpha(t) \qquad (t \in j) \tag{4}$$

Every integral y of (q) as well as its derivative y′ is, by the dispersion φ_n, transformed into itself in the sense of formula:

$$\begin{aligned} y\varphi_n(t) &= (-1)^n \, [\varphi_n'(t)]^{\frac{1}{2}} \, . \, y(t), \\ y'\varphi_n(t) &= (-1)^n \, [\varphi_n'(t)]^{-\frac{1}{2}} \, . \, y'(t), \quad \text{if } y(t)=0. \end{aligned} \tag{5}$$

If $q(t) < 0$, $t \in j$, then there holds, for $n \geqslant 1$,

$$\varphi_n'(t) = \frac{q(t_1)}{q(t_3)} \cdot \frac{q(t_5)}{q(t_7)} \cdot \ldots \cdot \frac{q(t_{4n-3})}{q(t_{4n-1})} \tag{6}$$

with convenient numbers $t_{2\nu-1}$ ($\nu=1,\ldots,2n$). The latter separate the zeros $(t=)a_0 < a_1 < \ldots < a_n (= \varphi_n(t))$ of every integral y of (q), which vanishes at the number t, and the zeros $b_1 < \ldots < b_n$ of its derivative y′, lying in the interval $(t, \varphi_n(t))$:

$$(t=)a_0 < t_1 < b_1 < t_3 < a_1 < t_5 < \ldots < t_{4n-3} < b_n < t_{4n-1} < a_n (= \varphi_n(t)).$$

By means of the central dispersions φ_n we define the distance functions of the equation (q) or of the carrier q by the relation

$$d_n(t) = \varphi_n(t) - t \qquad (n=0,\pm 1,\ldots;\ t \in j)$$

with the evident meaning: $d_n(t)$ is the distance of the number $\varphi_n(t)$ from t. Clearly, $d_n(t) \gtreqless 0$ according as $n \gtreqless 0$. d_1 (>0) is the basic distance function; it is often denoted by d.

3. Inverse equations. Let me now introduce a new notion, namely that of the inverse equations with regard to (q).

A differential equation $(\bar{q})$ is called inverse of the equation (q) if it has a phase $\bar{\alpha}$ which is the inverse function of some phase α of (q): $\bar{\alpha} = \alpha^{-1}$.

From this definition there follows a number of properties of inverse equations; I shall introduce only those we shall need in what follows.

Symmetry: If $(\bar{q})$ is inverse of (q), then (q) is inverse of $(\bar{q})$.

The carrier of the equation inverse of (q), with the phase $\bar{\alpha}$, is given by the formula

$$q_{\bar{\alpha}}(t) = -1 - [1+q\bar{\alpha}(t)]\,(\bar{\alpha}(t))'^{2}$$

The set of all (oscillatory) equations (q) is decomposed into nonempty disjoint subsets, called blocks, so that the latter form a decomposition U of the set in question. To each block $u \in U$ there exists exactly one inverse block, $u^{-1} \in U$, with the characteristic property: Any two equations $(q) \in u$, $(\bar{q}) \in u^{-1}$ are inverse of each other.

The proof of this theorem lies deep in the algebraic theory of oscillatory equations (q) and cannot be introduced here for want of space.

III. Study of the problem.

Well, applying the above notions, I may now proceed to the main subject of my lecture dealing, according to the title, with the properties of the central dispersions of the oscillatory equations (q) with periodic carriers q: $qc = q$, $t \in j$. The set of these equations will be denoted A_p.

Let, first, $(q) \in A_p$. Then there holds: if y is an integral of the equation (q), then the function yc is an integral of (q) as well. Hence we easily deduce that to every phase α of (q) there exists a phase ε of the equation (-1), $\varepsilon \in \mathfrak{G}$, such that $\alpha c = \varepsilon\alpha$. Consequently α is a solution of the equation

$$(7) \qquad \alpha(t+\pi) = \varepsilon\alpha(t) \qquad (t \in j)$$

or

$$(\bar{\varphi}_{\operatorname{sgn}\alpha'} \equiv)\ \alpha c \alpha^{-1}(t) = \varepsilon(t).$$

The function $\bar{\varphi}_{\operatorname{sgn}\alpha'}$ is, by (4), the fundamental dispersion of $(q_{\alpha^{-1}})$ or the function inverse of this dispersion, according as $\alpha' > 0$ or $\alpha' < 0$.

Consequently:

$\varepsilon(t) \in \mathfrak{G}$ is a dispersion phase, upper or lower, according as $\alpha' > 0$ or $\alpha' < 0$.

The fundamental dispersion of the inverse equation $(q_{\alpha^{-1}})$ and, therefore, every central dispersion of the latter, lies in the fundamental group $\mathfrak{G}$.

These results suggest the question whether the above properties of phases and central dispersions of the equations of the class A_p or of the corresponding inverse equations are characteristic of the equations of A_p. The answer is affirmative in the sense of the following theorems:

The equation (q) belongs to the class A_p if and only if each of its phases α satisfies the equation (7) with a dispersion phase $\varepsilon(t) \in \mathfrak{G}$.

The equation (q) belongs to the class A_p if and only if all the central dispersions of each inverse equation of (q) lie in the fundamental group $\mathcal{G}$.

Furthermore, one may, in this connection, prove the theorems:

The fundamental dispersions of the equations inverse of the equations of the class A_p are exactly all the upper phases from the fundamental group $\mathcal{G}$.

All the equations (q) of the same block simultaneously either belong or do not belong to the class A_p.

Remember that the class A_p consists of oscillatory equations (q) with periodic carriers: $q(t+\pi) = q(t)$, $t \in (-\infty,\infty)$.

2. Into our considerations there have entered, as an important element, the central dispersions of the equations inverse of those from A_p.

In this connection there arises the question concerning the properties of the central dispersions of the equations of the class A_p themselves. The central dispersions of the equations $(q) \in A_p$ have, in fact, the following remarkable property:

All central dispersions of every equation $(q) \in A_p$ are elementary.

In other words:

All distance functions of every equation $(q) \in A_p$ are periodic.

Indeed, let $(q) \in A_p$ and φ be the fundamental dispersion of (q).

We easily ascertain that the proof need not be given but for the dispersion φ.

Well, let α (e. g. $\alpha' > 0$) be a phase of (q). From (7) and (3) there follows

$$\alpha c \varphi = \varepsilon \alpha \varphi = \varepsilon c \alpha = c \varepsilon \alpha = c \alpha c,$$

consequently, $c\varphi = (\alpha^{-1} c\, \alpha) c = \varphi c$ so that $\varphi c = c\varphi$ and the proof is accomplished.

3. In this place I have the opportunity to mention the class of the equations (q) characterized by the fact that their fundamental dispersions are elementary. Let the class of these equations be denoted by A, so that (q) $\in$ A $\Longleftrightarrow \varphi c = c\varphi$. We have just seen that $A_p \subset A$.

The equations (q) $\in$ A may be characterized by the following geometric property: Let C be an arbitrary integral curve of the equation (q) $\in$ A, whose parametric expression is given by some of the bases of (q) $\in$ A. Let O stand for the origin of the coordinates and $\overrightarrow{OP(t)}$ for the radius vector from the point O to the point P(t) $\in$ C determined by the value of the parameter (time) t. Then the oriented areas traced out by the radius vector $\overrightarrow{OP(t)}$ and the opposite radius vector $\overrightarrow{OP(\varphi(t))}$ in the time from t to $t+\pi$ are the same.

The equations (q) with elementary fundamental dispersions occur even in other connections, e. g., if there is a question of determining pairs of equations (q) with interchangeable fundamental dispersions.

The theory of the equations (q) $\in$ A is extensive, so I cannot - for want of time - deal with it in detail but shall confine myself to a few remarks.

The theory of the equations $(q) \in A$ is fundamentally analogous to the theory of the equations $(q) \in A_p$, the part of the group $\mathfrak{G}$ in the latter being taken over by the group of the elementary phases, $\mathfrak{F}$.

In particular, there holds:

The equation (q) belongs to the class A if and only if each of its phases α satisfies the equation $\alpha(t+\pi) = h\alpha(t)$ $(t \in j)$ with a dispersion phase $h(t) \in \mathfrak{F}$.

The class A is closed with regard to the operation of forming inverse equations.

The equation (q) belongs to the class A if and only if the carriers $q(t)$, $q(t+\pi)$ have the same fundamental dispersion.

If $(q) \in A$, then there exist, in every interval $[t, \varphi(t))$, at least four mutually different numbers t_1, t_2, t_3, t_4 such that $q(t_i+\pi) = q(t_i)$ $(i = 1,2,3,4)$.

Note that, by the first theorem, the class A is wider than A_p: $A_p \subset A \neq A_p$.

4. Now we arrive at the last point of my lecture, where we return to the equations (q) with periodic carriers: $(q) \in A_p$. It will be a question of expressing real periodicity factors (characteristic roots) of the equations $(q) \in A_p$ by means of the values of the derivatives of central dispersions and, furthermore, of estimating the absolute values of the periodicity factors by means of the extreme values of the function $|q|$. ([3])

Well, let $(q) \in A_p$ be an arbitrary equation whose periodicity factors s_σ $(\sigma = \pm 1)$ are real. Denote

$$A = s_1 + s_{-1} ,$$

so that the characteristic equation corresponding to (q) is, by Floquet's theory, $s^2 - A.s + 1 = 0$ and, according to the supposition, we have $|A| \geqslant 2$.

Let s be one of the roots s_σ . Then there exists a nontrivial integral y_0 of the equation (q), with the property

(8) $$y_0 c = s.y_0 .$$

The equation (q) being oscillatory, the function y_0 has at least one zero x: $y_0(x) = 0$. From (8) then follows $y_0(x+\pi) = 0$. We see that the point $x+\pi$ is right conjugate with the point x. So we have, for a natural n:

(9) $$\varphi_n(x) = x + \pi .$$

Let u,v denote the basis of the equation (q), determined by the initial values:

(10) $$u(x)=1, \quad u'(x)=0; \qquad v(x)=0, \quad v'(x)=1.$$

Then $A = u(x+\pi)+v'(x+\pi)$ whence, by (9),(5),(10), there follows

$$A = (-1)^n \left([\varphi_n'(x)]^{\frac{1}{2}} + [\varphi_n'(x)]^{-\frac{1}{2}} \right).$$

We see that the periodicity factors of (q) are:

(11) $$s_\sigma = (-1)^n [\varphi_n'(x)]^{\frac{\sigma}{2}} \qquad (\sigma = \pm 1).$$

Now suppose that the carrier q of the equation (q) is always different from zero: $q(t) < 0$ $(t \in j)$.

Then, by (11) and (6), we have:

(12) $$s_\sigma = (-1)^n \left[\frac{q(x_1)}{q(x_3)} \cdot \frac{q(x_5)}{q(x_7)} \cdot \ldots \cdot \frac{q(x_{4n-3})}{q(x_{4n-1})} \right]^{\frac{\sigma}{2}} \quad (\sigma = \pm 1)$$

with convenient numbers $x_{2\nu-1}$ $(\nu=1,\dots,2n)$ separating the zeros $(x=)a_0 < a_1 < \dots < a_n(=x+\pi)$ of the integral y_0 and the zeros $b_1 < \dots < b_n$ of its derivative y_0', lying in the interval $(x,x+\pi)$:

$$(x=)a_0 < x_1 < b_1 < x_3 < a_1 < x_5 < \dots < x_{4n-3} < b_n < x_{4n-1} < a_n(=x+\pi)$$

Denote

$$m = \min_{t\in j} |q(t)|, \qquad M = \max_{t\in j} |q(t)|$$

so that

$$0 < m \leq M.$$

n stands for the number of zeros of the function y_0 in the interval $[x,x+\pi)$ and therefore, by a classical theorem, satisfies the inequalities

$$\sqrt{m} \leq n \leq \sqrt{M}. \tag{13}$$

From (6) we have

$$\left(\frac{m}{M}\right)^n \leq \varphi_n'(x) \leq \left(\frac{M}{m}\right)^n$$

which, together with (11) and (13), yields

$$\left(\frac{m}{M}\right)^{\frac{1}{2}\sqrt{M}} \leq |s_\sigma| \leq \left(\frac{M}{m}\right)^{\frac{1}{2}\sqrt{M}}. \tag{14}$$

Thus we have arrived at the following result:

If the periodicity factors s_σ of the equation $(q) \in A_p$ are real $(\sigma = \pm 1)$, then their values are given by the values of the function φ_n' in a number $x \in j$, in the sense of formula (11). If, furthermore, the carrier q is always different from zero, then the numbers s_σ may be expressed by values of the carrier q in the sense of formula (12). In that case there hold, for absolute values of the numbers s_σ, the inequalities (14).

Note that the inequalities (14) may be employed to obtain information as to the absolute values of the periodicity factors for Hill's equation in case of instability.

IV. Final remark.

Let me now finish my lecture with a short look at the algebraic theory of (oscillatory) equations (q) and stress the connection of the classes A_p and A with other elements of the theory in question.

The basic notion of the algebraic theory of the equation (q) is the group of phases.

The group of phases, $\mathfrak{G}$, is the set of all phases of all the equations (q) with the group-operation given by the composing of functions. The unit of $\mathfrak{G}$, $\underline{1}$, is the function t ($\in$ j): $\underline{1}$ = t.

The increasing phases form, in $\mathfrak{G}$, an invariant subgroup $\mathfrak{G}_o$, with the index 2; the decreasing phases form a coset of the factor group $\mathfrak{G}/\mathfrak{G}_o$, denoted G_1.

The upper phases α are characterized by the inequality $\alpha(t) > t$ ($\in$ j); they form a subset in $\mathfrak{G}_o$, called the upper complex, K_1. The lower phases α are characterized by the inequality $\alpha(t) < t$ ($\in$ j); they too form a subset in $\mathfrak{G}_o$, called the lower complex, K_{-1}. The complexes K_σ ($\sigma = \pm 1$) are disjoint and consist of functions mutually inverse. K_1 is composed of the fundamental dispersions of the equations (q); K_{-1} consists of functions inverse of these dispersions. For $\xi \in \mathfrak{G}$ there applies: $\xi^{-1} K_\sigma \xi = K_{\sigma . \operatorname{sgn} \xi'}$.

The set of all phases of the equation (-1) forms a subgroup in $\mathfrak{G}$, namely the above mentioned fundamental group $\mathfrak{F}$. The latter generates, on $\mathfrak{G}$, the right decomposition $\mathfrak{G}/_r\mathfrak{F}$ and the left decomposition $\mathfrak{G}/_\ell\mathfrak{F}$. Each element of the former has the form $\mathfrak{F}\alpha$ ($\alpha \in \mathfrak{G}$); it is the set of all phases of the equation (q) with the carrier given by the formula (1). Each element of the latter is of the form $\alpha\mathfrak{F}$ $(\alpha \in \mathfrak{G})$; it is the set of the functions inverse of the phases of $(q_{\alpha^{-1}})$. Every element $\bar{u} \in \bar{U}$ of the least common covering $\bar{U} = [\mathfrak{G}/_r\mathfrak{F}, \mathfrak{G}/_l\mathfrak{F}]$ is the union of the phases of some equations (q) [4]. The latter form the block corresponding to the element $\bar{u}$. To $\bar{u}$ there exists exactly one inverse element $\bar{u}^{-1} \in \bar{U}$ composed of the functions inverse of the phases lying in $\bar{u}$. Every equation (q) from the block corresponding to $\bar{u}$ is inverse of every equation from the block corresponding to $\bar{u}^{-1}$and vice versa.

The center of the group $\mathfrak{G}_0 \cap \mathfrak{F}$ is the infinite cyclic group $\mathfrak{Z} = \{c_n(t)\}$ $(c_n(t) = t+n\pi$; $n=0,\pm1,\ldots)$. The group $\alpha^{-1}\mathfrak{Z}\alpha$ $(\alpha \in \mathfrak{G})$ consists of the central dispersions of the equation (q_α). The normalizer of $\mathfrak{Z}$ in $\mathfrak{G}$ is the group of elementary phases, $\mathfrak{E}$. There holds: $\mathfrak{E} \supset \mathfrak{F}$.

The class A consists of all the equations (q) characterized by the fact that the inner automorphisms of the group $\mathfrak{G}$, formed by their phases α, the so-called phase-automorphisms of the equations (q), transform the center $\mathfrak{Z}$ into its normalizer: $\alpha^{-1}\mathfrak{Z}\alpha \subset \mathfrak{E}$. The same class A consists of the equations (q) inverse of the equations of class A.

The class A_p is a part of A: $A_p \subset A$. The equations of the class A_p are characterized by the fact that the phase-automorphisms of the inverse equations transform the center $\mathfrak{Z}$ into the fundamental group $\mathfrak{F}$.

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