Helena Durnová

Otakar Boruvka (1899-1995) and the Minimum Spanning Tree

In Mathematatik im Wandel. 1. vyd. Hildesheim, Berlin: Franzbecker, ISBN 3-88120-427-X
(kapitola v knize, 264-274)

Persistent URL: http://dml.cz/dmlcz/500001

Otakar Boruvka (1899-1995) and the Minimum Spanning Tree

Helena Durnová

Introduction

The minimum spanning tree is a problem which has become fairly popular among mathematicians after the Second World War. However, already before the War, before the graph-theoretical terminology that is now used in connection with this problem was developed, there existed algorithms for solving this problem. As will become clearer from the description of the individual algorithms, the algorithms of BORŮVKA (1926) and JARNÍK (1930) were not inferior to those of KRUSKAL (1956), PRIM (1957), LOBERMAN and WEINBERGER (1957), or DIJKSTRA (1959).

Otakar Boruvka: A Short Biography

Before paying attention to the minimum spanning tree algorithms and their development, I would like to introduce a twentieth-century Czech mathematician, an outstanding personality of Brno mathematical life: OTAKAR BORŮVKA.

He was born in 1899 in Uherský Ostroh, a small town in Moravia, the eastern part of the Czech Republic (then, of course, the Austro-Hungarian Empire). We shall skip his childhood with a single remark that he was a very good pupil and student. This, probably together with the fact that his father was a school teacher, predestined him to university studies. He himself recollects that he was not quite sure about what exactly he wanted to study. In the end, his choice was the university closest to his hometown, at that time the Technical University of Brno[1]. Thus, he began studying civil engineering. Then, as well as now, mathematics played a major role in the education of future engineers. BORŮVKA was captured by the lectures of

[1] Technical University of Brno was founded in 1899 in was the first Czech university in Brno. There was also a German Technical University of Brno (1849-1945).

his teacher and future colleague MATYÁŠ LERCH (1860-1922). LERCH, a former student of WEIERSTRASS, persuaded BORŮVKA to study mathematics at the newly open Masaryk University in Brno.

After graduating from the University, Boruvka became an assistant lecturer of mathematics at the university. He continued his mathematical studies also abroad, especially in Paris and Hamburg. His mathematical interests were multiple: differential geometry, algebra, and differential equations. Graph theory was not a field BORŮVKA was especially interested in: for him, the minimum spanning tree was a minor piece of work. However, even this short article shows the rigour with which he always worked.

Graph Theory: Basic Notions

Graph theory is a branch of mathematics whose history is short when compared to other disciplines and rather complicated. It sprang from many different sides: from *Analysis situs* and *Topology* or *Algebra,* but also from applied fields, especially from economics. It does have some common denominators - the basic notions - but it also has two quite distinct, yet overlapping sections: one of them is more interested in the properties of graphs, the other is centred around the usage of graphs for solving applied problems. The latter one thus overlaps with the so-called *discrete optimization.*

In the graph-theoretical part of discrete optimization, numerous problems, and of course many more algorithms for solving them, can be found. Their history has something in common: the problems pop up somewhere, and mathematicians try to solve them. Then other mathematicians either face the same problems, or come across the previous solution and find it a fertile subject for their work. The methods for solving may differ in the approach, in the level of abstraction, or in further applications. Basic definitions of notions used in graph theory are given below.

Node (vertex) can be imagined as a point in the plane. Between two nodes, there may be an *edge (branch, arc)* connecting them. The two sets - the set of nodes and the set of edges - together form a *graph.*

It is also interesting to know which nodes are - either directly or indirectly - connected to each other. If there is a possibility to pass from one node to another using edges and nodes, we say that there exists a *path* between the two edges. Further, if such paths exist between all couples of vertices, we say that the graph is *connected* - i.e. there are no isolated groups of points in the graph.

If there exists a possibility to go from one node to some **other** nodes and mutually different edges back to the original node, we call this way a *cycle* in the graph. If, on the other hand, there is no cycle in the graph, we call this graph a *tree*. And finally, *spanning tree of a graph* is such a graph that contains **all** nodes of the original graph, is connected, contains no cycles, and its set of edges is a subset of edges of the original graph.

Minimum Spanning Tree: Problem Formulation

Before the "official" terminology was introduced, the minimum spanning tree problem was disguised in various words. The task a mathematician was then solving was rather concrete.

In 1925, BORŮVKA was asked to help with the design of the electrical network for a certain area by an employee of the Western Moravian Power Plant Company. The task he was faced with could be formulated as follows: *Design the cheapest electrical network connecting all the given towns in western Moravia.* From the definitions stated above it is clear that BORŮVKA was asked to say what the minimum spanning tree for this area was. However, BORŮVKA did not know graph terminology we are familiar with today: it was ten years before KONIG wrote his classic book *Theorie der endlichen und unendlichen Graphen* (Leipzig 1936). Thus, BORŮVKA's mathematical formulation of the problem uses matrix terminology [Boruvka 1926a]. He also published a solution comprehensible to engineers in [Boruvka 1926b].

BORŮVKA's colleague, Vojtech JARNÍK, replied in a letter to BORŮVKA, stating that he found a better solution to the problem. This solution, published in 1930, does not use graph terminology either.

BORŮVKA's article had a German summary, which somehow got to Princeton in the USA. Here. J.B. KRUSKAL JR., inspired by it, came to his solution of the minimum spanning tree problem. In his recollections about the minimum spanning tree (or, as he prefers to say, the "shortest spanning subtree") he says:

> Someone (...) handed me two pages of very flimsy paper stapled together. He told me it was "floating around in the math department". (...) the pages were typewritten, carbon copy, and in German. (...) I never found out who did the typing or why. [Kruskal 1997]

The "flimsy paper" was evidently an abstract of BORŮVKA's article. KRUSKAL found his solution a bit too elaborate and suggested a more comprehensible way of dealing with the problem. KRUSKAL published his solution in 1956, and since then, few people bothered to find BORŮVKA's 1926 articles. This resulted in neglecting BORŮVKA's solution for many years. His younger Czech colleagues "promoted" his solution - however, even the versions of BORŮVKA's algorithm that appear in Czech textbooks differ. Independently of KRUSKAL, LOBERMAN and WEINBERGER arrived at the same result (algorithm) in 1957. [Loberman & Weinberger 1957]

Also in 1957, an American mathematician ROBERT C. PRIM published his solution of the minimum spanning trees. He says: "Kruskal refers to an obscure Czech paper as giving a solution inferior to his." PRIM obviously believes he found a new and better solution to the problem - however, as will be clear from the description of algorithms, BORŮVKA's and PRIM's algorithms may proceed in the same way. Finally, in 1959, DIJKSTRA published his solution. He is very brief, and so little can be said about how exactly his algorithm should work. However, his reasoning seems to work in the same way as JARNÍK's.

Solutions of the Minimum Spanning Tree Problem

To demonstrate the process of work of the algorithms, we will use the following complete graph [Fig.1]. This graph complies with Boruvka's definition of the minimum spanning tree problem: the distances between the individual vertices or nodes are mutually different. This is also a necessary and sufficient condition for the uniqueness of the minimum spanning tree of a weighted connected graph.

For the sake of clarity, the weights of the individual edges are also given in [Table 1] below. The table (matrix) is symmetric - this property of the matrix follows from the definition of the minimum spanning tree problem.

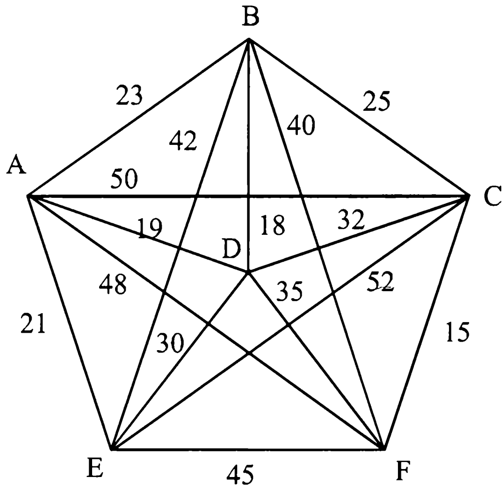

Fig. 1: Complete graph K(5)

Table 1: Weights of edges for graph from Fig. 1

	A	B	C	D	E	F
A	0	23	50	19	21	48
B	23	0	25	18	42	40
C	50	25	0	32	52	15
D	19	18	32	0	30	35
E	21	42	52	30	0	45
F	48	40	15	35	45	0

The description of the individual algorithms follows. The attempt is to compare couples of algorithms: BORŮVKA with PRIM, JARNÍK with DIJKSTRA, and KRUSKAL with LOBERMAN and WEINBERGER. BORŮVKA's algorithm is described in most detail, as it is the least known and most complex one.

Algorithms of Boruvka and Prim

Borůvka: 1926

From the two articles BORŮVKA published on the minimum spanning tree problem, one is mathematically precise, the other is written in plain language, in a more demonstrative way. Without reading the latter, it is hard to understand the former. However, reading both, it becomes obvious that the process being described is the same. The algorithm - re-written in modern terminology - runs as follows:

Input: A matrix A(n*n) of positive integers and zeroes on the main diagonal is given. (The numbers in the matrix represent the distances between the towns.)

Task: From each row or column, one number must be chosen. The sum of these (n-1) numbers should be minimum of all such sets.

Step 1 Choose a row in the matrix.[2]

Step 2 Choose the smallest positive number in this row. (This means that two vertices - the row index and the column index - have been chosen, together with the edge between them. This is T1, the first partial minimum spanning tree.)

Step 3 The algorithm now divides into two branches: Is there a number in the new row (determined by the column index from *Step 2.*) that is smaller than the number included in the minimum spanning tree in the previous step?

Yes: The smallest of such numbers becomes a new member of the minimum spanning tree. Perform *Step 2* and *Step 3* with the newly added number.

No: Choose a different row and perform *Step 2* and *Step 3.*

The algorithm proceeds until there are no isolated rows. This means that the maximum number of repetitions is *(n-1)*. Then, we get several partial spanning trees which need to be connected:

---

[2] It is obviously irrelevant whether a row or a column is chosen - in the original article, Boruvka starts with a row.

Step 4 Compute a new distance matrix for partial spanning trees.

Step 5 Go to *Step 1* and perform the algorithm for the new matrix.

Output: A sequence of vertices determining the minimum spanning tree.

For our specific example of a graph, the algorithm runs as follows:

Step 1 (arbitrary choice): Choose row - for example *F*.

Step 2 Smallest number in this row is *15*, thus edge *CF* is chosen.

Step 3 There is no edge smaller then *15* going from *C*. We have to choose a different row - for example, *A* - and perform *Steps 2* and *3*.

Step 2 Edge *AD* with length *19* is chosen.

Step 3 There is an edge of length smaller than *19* going from *A*: Yes - *B*.

Step 2 We choose edge *BD* of length *18*.

Step 3 There is no edge smaller than *18* going from *B*. We have to choose the remaining row: *E* (this is the row which has not been chosen yet).

Step 2 The smallest edge in this row is *AD* with length *21*.

We cannot go on like this, since there are no isolated rows. We have to compute a new matrix - its vertices will be the partial minimum spanning trees, its edges the rows between them. The components are *H={EABD}* and *G={CF}*. The new distance matrix is given in Table 2.

Table 2: New matrix for BORŮVKA algorithm

	H	G
H	0	25
G	25	0

The edge of length *25* connecting the two components is edge *BC*, which is also the last edge of the minimum spanning tree. The resulting minimum spanning tree is thus given by this sequence of vertices: *EADBCF*.

Prim: 1957

PRIM claims that minimum spanning tree may be obtained using one of these two principles at most *(n-1)* times:

P1: Any isolated terminal can be connected to the nearest neighbour.

P2: Any isolated fragment can be connected to the nearest neighbour.

For PRIM, fragment is the same as for BORŮVKA a subtree. In a certain translation, BORŮVKA in his technical paper says *exactly* the same as PRIM. In BORŮVKA's mathematical paper, we need to substitute "a row from which no number has been chosen" with "an isolated terminal", for further steps (newly computed matrices in the algorithm of BORŮVKA) "a row from which no number has been chosen" with "isolated fragment". However, PRIM's demonstration makes his algorithm more similar to those of JARNÍK and DIJKSTRA.

Both BORŮVKA and PRIM got a response to their algorithm soon: from JARNÍK in 1930 and from DIJKSTRA in 1959, respectively. It might be just a coincidence that both responses are very similar.

From the historical point of view, it is interesting to mention that BORŮVKA talked about this problem in a seminar during his stay in Paris in 1926. The topic was chosen by Professor COOLIDGE, who chaired the seminar, out of three offered by BORŮVKA. However, it was not until 1956 that this paper became more widely known outside Czechoslovakia. In Czechoslovakia, the reaction came from BORŮVKA's colleague, JARNÍK.

Algorithms of Jarník and Dijkstra

Jarník: 1930

JARNÍK connects point in the plane, instead of working with numbers in matrices. He starts at any point in the plane and ends when all the points have been connected into an uninterrupted whole. The first two steps are identical with BORŮVKA's algorithm, the third is different.

Step 1 Choose a point. (This is the same as choosing a row for BORŮVKA.)

Step 2 Choose the shortest edge going out of that point. (i.e. smallest number in the row, for BORŮVKA.) Thus, a *fragment* is formed.

Step 3 Find the point closest to the fragment. Perform *Step 3* until all the points are joined.

If we start from the same point as for BORŮVKA's algorithm, JARNÍK's algorithm gives this sequence of fragments: *F* - *FC* - *FCB* - *FCBD* - *FCBDA* and finally *FCBDAE.*

Dijkstra: 1959

DIJKSTRA's algorithm differs only in the terminology used: where JARNÍK says "point", DIJKSTRA says "node", and for "line", DIJKSTRA uses "branch". DIJKSTRA's algorithm is very similar to JARNÍK and PRIM, but he quotes neither. He only quotes [Kruskal 1956] and [Loberman & Weinberger 1957].

Algorithms of Kruskal and Loberman and Weinberger

These two algorithms can be called identical, but they were discovered independently, as follows from the footnote of the LOBERMAN and WEINBERGER article:

> This reference [to Kruskal 1956] was discovered by the present authors after their procedures had been formulated It is seen that the procedures presented here and Kruskal's "constructions" are identical. (...) [Loberman & Weinberger 1957]

KRUSKAL's description is used in this paper: **Construction A**[3]

Step 1 Sort edges according to their lengths from the smallest to the largest.

Step 2 Take edges from the shortest to the longest and decide whether the particular edge is part of the minimum spanning tree or not.

Criterion: Does the edge form a cycle with the previously chosen edges?

Yes: Discard the edge

No: Add the edge to the minimum spanning tree.

Repeat Step 2 as long as it is possible.

For our specific example, the algorithm adds these edges to the minimum spanning tree: *15 - CF, 18 - BD, 19 - AD, 21 - AE,* (*23 - AB* forms a cycle and is not added), *25 - BC.* The other edges all form cycles with the edges previously chosen.

[3] There is also Construction A' and Construction B. The beginning is the same for all.

Conclusion

The above-described algorithms all deal with the same problem - the minimum spanning tree - and for those graphs where the minimum spanning tree is uniquely determined, they arrive at the same result. However, it is interesting to look at the complexity results for the algorithms. According to DONALD KNUTH[4], BORŮVKA's algorithm gave the best results when tested on a series of graphs. On the other hand, the computer programmers now employ more elaborate methods for implementing algorithms. Using *heaps*, or even *Fibonacci heaps*, they make the algorithms run in the same time.

It is also interesting to observe the development of new language for the problem. What BORŮVKA says with matrices, JARNÍK translates into geometry, until finally KRUSKAL, PRIM, DIJKSTRA, and LOBERMAN and WEINBERGER use some kind of graph terminology.

Bibliography

Primary Sources: Algorithms

BORŮVKA, OTAKAR: O jistém problému minimálním (About a certain minimum problem). *Práce moravské prírodovedecké spolecnosti v Brne* **III**, 3 (1926), pp. 37-58.

BORŮVKA, OTAKAR.: Príspevek k rešení otázky ekonomické stavby elektrovodných sítí (A contribution to the solution of a problem of economical construction of electrical networks). *Elektrotechnický obzor* **15** (1926), pp. 153-154.

DIJKSTRA, EDSGER WYBE: A note on two problems in connection with graphs. *Numerische Mathematik* **1** (1959), pp. 269-271.

JARNÍK, VOJTĚCH: O jistém problému minimálním. (About a certain minimum problem). *Práce moravské prírodovedecké spolecnosti v Brne* **VI**, 4 (1930), pp. 57-63.

[4] Reported in [Šišma 1996].

KRUSKAL, JOSEPH B.: On the shortest spanning subtree of a graph and the travelling salesman problem. *Proceedings of the American Mathematical Society* 7 (1956), pp. 48-50.

LOBERMAN, H. & WEINBERGER, A.: Formal procedures for connecting terminals with a minimum total wire length. *Journal of the Association for Computing Machinery,* **4** (1957), pp. 428-438.

Secondary Sources

GRAHAM, R.L. and HELL, P.: On the history of the minimum spanning tree problem. *Annals of the History of Computing* 7 (1985), pp. 43-57.

KRUSKAL, Joseph B.: A reminiscence about shortest spanning subtrees. *Archivum Mathematicum*, Tomus **33** (1997), pp. 13-14.

ŠIŠMA, PAVEL: *Teorie grafu 1736-1963 (Graph Theory 1736-1963).* Prometheus, Praha, 1997.

TREŠNÁK, Z., ŠARMANOVÁ, P., PUŽA, B.: *Otakar Boruvka.* Universitas Masarykiana, Brno, 1996.

Mgr. Helena Durnová, Department of Mathematics, Faculty of Science of Masaryk University in Brno, Janáckovo nám. 2a, CZ-662 95 Brno