MONOGRAPHS AND TEXTBOOKS IN ALGEBRA

at the turn of the 20th century

1 Introduction

Algebra as a *mathematical discipline* was initially concerned with equation solving. It originated in ancient Egypt and Mesopotamia four thousand years ago, and somewhat later also in ancient China and India. At that time, *equations* in the present sense, i.e., formal expressions based on a certain, perhaps very primitive notation, did not exist yet. The ancient arithmeticians were able to solve word problems leading to equations or their systems by means of meticulously memorized procedures, which can be nowadays aptly designated as algorithms. They have successfully tackled a number of problems, which often correspond to present-day problems of school mathematics, sometimes being much more difficult. Their methods of calculation largely correspond to our procedures used for solving equations or their systems.

Problems leading to linear equations were being solved already in ancient Egypt and Mesopotamia, and problems leading to systems of linear equations in ancient China. Contemporary arithmeticians in Mesopotamia were also able to deal with problems leading to the solution of quadratic equations, or certain special types of higher-order equations. Later development of algebra was significantly influenced mainly by Arab mathematicians; the word algebra derives from the name of a treatise by Al-Khwarizmi (about 780 – about 850). In the first half of the 16th century, Italian mathematicians discovered algebraic solutions of the cubic and quartic equations; the results were summarized by Girolamo Cardano (1501–1576) in his 1545 book *Ars Magna* (containing, among other results, the Cardano formula). For almost three centuries since then, a number of mathematicians were unsuccessfully trying to find similar formulas for the roots of fifth-order algebraic equations.[1]

The importance of algebra in geometry was realized during the 17th century – analytic geometry was born.[2] Its basic idea is to express curves and surfaces by means of algebraic equations, which provide relations between the coordinates of points on these geometric objects in a certain coordinate system. This idea was soon taken over by algebraic geometry, which studies algebraic curves and surfaces. Analytic geometry provided impetus for the development of

[1] We remark that the problem of finding an algebraic solution to an algebraic equation (i.e., finding the solution *in radicals*) requires expressing a root of the equation by means of a formula involving only the coefficients and the arithmetic operations $+, -, \cdot, :, \sqrt[n]{\ }$.

[2] R. Descartes (1596–1650), P. de Fermat (1601–1665).

differential and integral calculus,[3] which was also referred to as the "algebraic analysis" (*analyse algébrique*).

Essential new results in algebra appeared throughout the 19th century. Shortly after the proof of the *fundamental theorem of algebra*,[4] the problem of finding algebraic solutions to algebraic equations (solutions in radicals) has been completely settled. The proof of the algebraic unsolvability of a general fifth-order equation was followed by a theory encompassing all possible cases (Galois theory).[5]

The second half of the 19th century saw the beginning of significant changes in algebra. Mathematicians began more and more to deal with new collections of objects, whose structure was similar to the structure of classes of numbers. Like with ordinary numbers, it was possible to perform "calculations" with the elements of these collections.

Such investigations have led to the study of new structures: linear associative algebras (e.g., quaternions), nonassociative algebras (e.g., octonions), vector spaces, groups, fields, rings, integral domains, etc. The development of Galois theory, as well as investigations of solvability of various geometric problems using straightedge and compass constructions, provided motivation for the study of groups, solvable groups, field extensions, algebraic and transcendental numbers, etc. On the other hand, the study of congruences, residue classes, integral domains and finite fields was closely related to number theory. The algebra of abstract structures began to emerge.[6] Moreover, algebra has permeated geometry in a new way, by means of Klein's Erlangen program.[7]

The essential change in the character of algebra that took place during several decades at the turn of the 20th century was caused mainly by new discoveries, which shifted the focus from the study of algebraic and numeric solution of equations to investigations of algebraic structures and their homomorphisms, above all groups and fields, algebras, but also rings, integral domains, their ideals, etc.[8] At the same time, the world of objects that behave

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[3] J. Kepler (1571–1630), R. Descartes, P. de Fermat, B. Cavalieri (1598–1647), E. Torricelli (1608–1647), Ch. Huygens (1629–1695), I. Barrow (1630–1677) and others, later I. Newton (1643–1727) and G.W. Leibniz (1646–1716).

[4] J.-B. le R. d'Alembert (1717–1855), J.R. Argand (1768–1822), C.F. Gauss (1777–1855), later L. Kronecker (1823–1891).

[5] J.L. Lagrange (1736–1813), P. Ruffini (1765–1822), N.H. Abel (1802–1829), E. Galois (1811–1832).

[6] C.F. Gauss, A.-L. Cauchy (1789–1857), W.R. Hamilton (1805–1865), J.T. Graves (1806–1870), H.G. Grassmann (1809–1877), B. Peirce (1809–1880), E. Kummer (1810–1893), J.J. Sylvester (1814–1897), A. Cayley (1821–1895), L. Kronecker (1823–1891), R. Dedekind (1831–1916), L. Sylow (1832–1918), P.G. Tait (1831–1901), C. Jordan (1838–1922), H. Weber (1842–1913), W.K. Clifford (1845–1879), G. Frobenius (1849–1917), G. Peano (1858–1939), L.O. Hölder (1859–1937), and others.

[7] F. Klein (1849–1925).

[8] D. Hilbert (1862–1943), E. Noether (1882–1935), E. Artin (1898–1962), E. Steinitz (1871–1928), H. Hasse (1898–1979), W. Krull (1899–1971), O. Schreier (1901–1929), B.L. van der Waerden (1903–1996).

like ordinary numbers has grown substantially (substitutions, transformations, permutations, matrices, vectors, hypercomplex numbers, residue classes, etc.).[9]

The character of algebra has been substantially influenced by set theory, which began to permeate other mathematical disciplines at the turn of the 20th century.[10] The process of axiomatization of mathematics, which began in the 1880s and was becoming ever stronger in the following decades, had an enormous impact on the conception of algebra.[11] The emergence of set theory and the progress in axiomatization were in a natural way related to an all-embracing strengthening of abstraction.[12]

As of today, algebra is largely a theory of algebraic structures and their homomorphisms, or representations of certain structures by another ones. The term algebraic structure refers to an abstract set (a set of arbitrary objects, whose nature is left unspecified) with a collection of operations (and possibly relations), which satisfy given axioms. This conception of algebra rests on the notion of a set, and on the axiomatic approach.

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[9] Following the opinions of Thomas Samuel Kuhn (1922–1996), we may talk about the change of paradigm. See T.S. Kuhn: *The Structure of Scientific Revolutions*, The University of Chicago Press, Chicago, London, 1962, 1970, 1996, 4th ed. with an introductory essay by Ian Hacking, 2012, 264 pages. See also Ziauddin Sardar: *Thomas Kuhn and the Science Wars. Postmodern Encounters*, Icon Books, Ltd., 2000, 76 pages, or H. Mehrtens: *T.S. Kuhn's theories and mathematics: A discussion paper on the "new historiography" of mathematics*, Historia mathematica 3(1976), pp. 297–320.

[10] G. Cantor (1845–1918).

[11] The main credit for the axiomatization of mathematics throughout the 19th century belongs to M. Pasch (1843–1930), G. Frege (1848–1925), G. Peano, R. Dedekind, and D. Hilbert.

[12] Readers interested in the development of algebra in the 19th and 20th century are referred to the following monographs:
H. Wussing: *Die Genesis des abstrakten Gruppenbegriffes. Ein Beitrag zur Entstehung der abstrakten Gruppentheorie*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1969, 258 pages. English translation (Abe Shenitzer): *The Genesis of the Abstract Group Concept. A Contribution to the History of the Origin of Abstract Group Theory*, MIT Press, Cambridge, Massachusetts, London, England, 1984, 331 pages, reprint: Dover Publications, Mineola, New York, 2007, 331 pages.
A.N. Kolmogorov, A.P. Juškevič (red.): *Matematika XIX veka. Matematičeskaja logika, Algebra, Teorija čisel, Teorija verojatnostej*, Nauka, Moskva, 1978, 255 pages. English translation: *Mathematics of the 19th Century: Mathematical Logic, Algebra, Number Theory, Probability Theory*, Birkhäuser, Basel, 1992, 308 pages.
E. Scholz (Hrsg.): *Geschichte der Algebra: eine Einführung*, BI-Wissenschaftsverlag, Mannheim, Wien, Zürich, 1990, viii + 506 pages.
See also [Al], [Co1], [GP], [Gy], [K], [Ko], [KP], [No] and [Wa5].
We also recall the following journal articles: B.M. Kiernan: *The Development of Galois Theory from Lagrange to Artin*, Archive for History of Exact Sciences 8(1971), pp. 40–154.
H.M. Edwards: *The Genesis Ideal Theory*, Archive for History of Exact Sciences 23(1980), pp. 321–378.
W. Purkert: *Zur Genesis des abstrakten Körperbegriffes*, Schriftenreihe für Geschichte der Naturwissenschaften, Technik und Medizin 10(1971), No. 1, pp. 23–37, No. 2, pp. 8–20.

The first goal of the following text is to foreshadow the spirit of algebra of the second half of the 19th century by means of several textbooks, which were written and widely used at that time. The next part is devoted to Heinrich Weber and his extensive algebra textbook, which to a large extent characterized algebra in the end of the 19th century. The final part discusses algebra textbooks from the first three decades of the 20th century. The text contains a number of excerpts from the forewords and introductions to particular textbooks, which document changing views of their authors on the content and character of algebra. Excerpts from the tables of contents of particular textbooks make it possible to compare their scope. Finally, excerpts containing definitions of fundamental notions (field, group, ring, etc.) provide an opportunity to see the transformation from the algebra of equations into the algebra of structures, and to a certain extent document the changing nature of this discipline, in particular the process of axiomatization and the increase of abstraction.

2 Some algebra textbooks of the second half of the 19th century

2.1 Joseph-Alfred Serret

The French mathematician and astronomer Joseph-Alfred Serret (1819–1885) graduated from the École Polytechnique in Paris in 1840, and subsequently served in artillery. Since 1848 he had a position at the École Polytechnique, since 1849 at the Sorbonne, and since 1856 he taught astronomy at the Collège de France. He became a member of the Académie des Sciences in 1860. One year later, he was appointed professor of celestial mechanics at the Collège de France, and in the period 1863–1871 he lectured as a professor of differential and integral calculus at the Sorbonne. Since 1873 he was a member of the Bureau des longitudes.[13]

Serret was a universal mathematician. He worked mainly in analysis, differential geometry (well known are the Frenet-Serret formulas for curves in the three-dimensional space), algebra, arithmetic, number theory, mechanics, and astronomy. He was the first to lecture on Galois theory at the Sorbonne, and contributed to the development of group theory.[14] He was the author of numerous popular, frequently published and translated textbooks, e.g., *Traité de trigonométrie*,[15] *Traité d'arithmétique*,[16] *Éléments de trigonométrie*

[13] Literally "The Office of Longitudes". A scientific institution whose original goal was the improvement of maritime navigation (determination of the longitude) and timekeeping, and later in the 19th century the synchronization of time across the whole Earth. It was founded in 1795, and its members included e.g. Lagrange, Laplace, Arago, and Poincaré.

[14] See G. Pacheco, F. Javier: *Galois Theory in the Work of Serret* (Spanish), Epsilon 16(3)(48)(2000), pp. 377–408.

[15] Bachelier, Paris, 1850, viii + 215 pages, 2nd ed. 1857, xi + 240 pages, 3rd ed. 1862, 4th ed. 1868, 6th ed. 1880, x + 336 pages, 10th ed. 1916; reprint of the 6th ed.: 1992.

[16] Bachelier, Paris, 1852, xii + 408 pages, 6th ed. 1875, xii + 325 pages, 7th ed. 1883, xii + 336 pages. Russian translation: 1895.

rectilingne, à l'usage des arpenteurs,[17] *Leçons sur les applications pratiques de la géométrie et de la trigonométrie*,[18] *Éléments d'arithmétique, à l'usage des candidats au baccalauréat ès-sciences*,[19] *Théorie nouvelle géométrique et mécanique des lignes a double courbure*,[20] and the two-volume extensive *Cours de calcul différentiel et intégral.*[21]

Serret edited the fourteen volumes of *Œuvres de Lagrange*,[22] and the third edition of Langrange's *Théorie des fonctions analytiques.*[23] Jointly with Ch. Hermite (1822–1901), he edited the sixth edition of the two-volume treatise *Traité élémentaire de calcul différentiel et de calcul intégral*[24] by S.-F. Lacroix (1765–1843).[25]

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Serret's very popular *Cours d'algèbre supérieure* [Se] served as a textbook for over eight decades. It was first published in 1849 as a single volume comprising 400 pages. Its third edition from 1866 had two volumes comprising more than 1300 pages, and the seventh French edition of the same extent appeared in 1928. A German translation of the third French edition, published under the title *Handbuch der höheren Algebra*, was translated and compiled by Georg Wertheim (1857–1939), and appeared also in two volumes – for the first time in 1868, and for the second time in 1878 and 1879.

Let us quote an interesting passage from the introduction to the 1868 German edition, which characterizes algebra as it was perceived in the second half of the 19th century.

Die Algebra hat es wesentlich mit der Analysis der Gleichungen zu thun; die verschiedenen besonderen Theorien, die sie enthält, beziehen sich sämmtlich mehr oder weniger gerade auf diesen Gegenstand.[26] *Von diesem Gesichtspunkte aus lassen sich drei ganz verschiedene Theile der Algebra unterscheiden:*

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[17] Bachelier, Paris, 1853, iv + 52 pages.

[18] Bachelier, Paris, 1851, viii + 82 pages.

[19] Mallet-Bachelier, Paris, 1855, viii + 135 pages, 5th ed. Gauthier-Villars, Paris, 1868, viii + 242 pages.

[20] Mallet-Bachelier, Paris, 1860, xix + 276 pages.

[21] Gauthier-Villars, Paris, 1868, xiii + 618, xii + 731 pages, 6th ed. 1911. German translation (G. Scheffers, 3 vols.): *Lehrbuch der Differential- und Integralrechnung*, 1884, 1885; there exists a series of subsequent editions until 1924.

[22] Gauthier-Villars, Paris, 1867–1892, reprint: Olms, Hildesheim, New York, 1973.

[23] Bachelier, Paris, 1847, xii + 399 pages.

[24] Mallet-Bachelier, Paris, 1861–1862, xv + 460 pages, 9th ed. 1881.

[25] *Funérailles de M. Serret*, Bulletin des sciences mathématiques 9(1885), pp. 123–132. D.J. Struik: *Serret, Joseph Alfred*, [Gi]-XII, pp. 318–319.

[26] *L'Algèbre est, à proprement parler, l'Analyse des équations; les diverses théories partielles qu'elle comprend se rattachent toutes, plus ou moins à cet objet principial.* ([Se]-1849, p. 1)

1. Die allgemeine Theorie der Gleichungen, d. i. die Gesammtheit der allen Gleichungen gemeinsamen Eigenschaften.

2. Die Auflösung der numerischen Gleichungen, d. h. die Bestimmung der exacten oder der Näherungswerthe der Wurzeln einer Gleichung, deren Coefficienten bestimmte Zahlen sind.

3. Die algebraische Auflösung der Gleichungen, d. h. die Bestimmung eines aus den Coefficienten einer gegebenen Gleichung zusammengesetzten Ausdrucks, welcher, für die Unbekannte substituirt, der Gleichung identisch genügt, mögen nun ihre Coefficienten numerisch gegeben sein, oder mögen sie, einfach als bekannt angesehen, unbestimmt gelassen und durch Buchstaben bezeichnet werden. ([Se]-I-1868, p. 1)

The two volumes of Serret's textbook (in paricular, the German editions from 1868 and 1878/79, and the French edition from 1866) have five parts, which are further divided into chapters. The first volume contains two parts, and the second volume three parts. The contents and organization of the textbook are well characterized by the titles of the five parts, and by the corresponding page ranges:

1. *Die allgemeinen Eigenschaften und die numerische Auflösung der Gleichungen* (pp. 5–294, or 5–304, or 1–368, respectively).
2. *Die symmetrischen Functionen* (pp. 295–508, or 305–528, or 369–643, respectively).
3. *Die Eigenschaften der ganzen Zahlen* (pp. 3–172, or 3–201, or 3–216, respectively).
4. *Die Substitutionen* (pp. 173–340, or 203–372, or 217–420, respectively).
5. *Die algebraische Auflösung der Gleichungen* (pp. 341–540, or 373–574, or 421–664, respectively).[27]

The final fifth part, which is the highlight of Serret's textbook, deals with algebraic equations of the third and fourth order, algebraic solution of algebraic equations, nonexistence of an algebraic solution to a general higher-order equation, and solvability of some particular higher-order equations. Included are the elements of the Galois theory; the author made use of certain results from the earlier chapters, in which the concept of a group slowly emerged.

The fourth part ends with the following passage:

Die Substitutionen $1, S_1, S_2, \ldots, S_{\nu-1}$, *durch welche man von der Permutation* (6) *der Wurzeln* $x_0, x_1, \ldots, x_{n-1}$ *zu den* ν *Permutationen* (7) *übergeht, bilden ein conjugirtes System. Mit andern Worten: die* ν *Permutationen* (7) *machen eine Gruppe aus.* ([Se]-II-1868, p. 340, or [Se]-II-1879, p. 372, or [Se]-II-1866, p. 420, respectively)

[27] Unfortunately, neither the French nor the German version has an index, but the tables of contents are quite similar: 7 + 8 pages in the French version, 6 + 6 pages in German.

2.2 Camille Marie Ennemond Jordan

The French mathematician Camille Jordan (1838–1922) studied at the École Polytechnique from 1855 till 1859, and at the École des Mines in Paris, where he graduated in 1861. He was a pupil of Serret, whom we held in high esteem. He obtained his doctoral degree in 1861. He worked as a mining engineer from 1861 till 1873, became a repetitor/examiner at the École Polytechnique in 1873, and was appointed professor three years later; he was pensioned in 1912. He also taught at the Collège de France since 1883.

He became a corresponding member of the Academy of Sciences in Göttingen in 1869, succeded Michel Chasles (1809–1880) as a member of the Académie des Sciences in 1881 (he was vicepresident in 1915 and president in 1916),[28] became president of the Société mathématique de France (founded 1872) in 1880, a member of the Saint Petersburg Academy of Sciences in 1895, and member of the National Academy of Sciences (USA) in 1920. He was awarded the Legion of Honour (*Ordre national de la Légion d'honneur*) in 1890.[29] He was the Honorary President of the International Congress of Mathematicians in Strasbourg in 1920. From 1855 till 1922, he was the editor of the *Journal de mathématiques pures et appliquées*, a leading mathematical journal founded by Joseph Liouville (1809–1882) in 1836.

He worked mainly in algebra (groups, linear algebra), number theory, real functions, differential equations, geometry, topology, mechanics, and crystallography. He wrote the three-volume textbook *Cours d'analyse de l'école Polytechnique*,[30] which received a lot of praise. His name is remembered in the Jordan-Hölder theorem on the composition series, Jordan canonical form, Dirichlet-Jordan convergence test, Jordan measure, Jordan curve, etc.[31]

A complete collection of Jordan's works entitled *Oeuvres de Camille Jordan I.–IV.* was published by Jean Dieudonné (1906–1992) and Gaston Julia (1893–1978) in Paris from 1961 till 1964.

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[28] The history of the French Académie des Sciences goes back to a group of thinkers, whose meetings took place with Marin Mersenne (1588–1648); it was officially established in Paris, 1666, by Jean-Baptiste Colbert (1619–1683), a minister of Louis XIV (1638–1715, King of France from 1643).

[29] This order was established in 1802 by Napoleon Bonaparte (1769–1821).

[30] Gauthier-Villars, Paris, 1882–1887, xvi + 377, xiii + 432, xiv + 617 pages, 2nd ed. 1893–1896, xviii + 612, xviii + 627, xi + 542 pages, 3rd ed. 1909–1915, xv + 620, 705, 631, reprint: 1959.

[31] H. Villat: *Camille Jordan*, Journal de mathématiques pures et appliquées 1(1922), pp. 0–iv. A. Buhl: *Camille Jordan*, L'Enseignement Mathématique 22(1922), pp. 214–218. H.H.: *Camille Jordan*, Proceeding of the London Mathematical Society 21(1923), pp. xliii–xlv. H.B.: *Camille Jordan*, Revue des questiones scientifiques 9(1926), pp. 165–166. H. Lebesgue: *Notice sur la vie et les travaux de Camille Jordan (1838–1922)*, Mémoires de l'Académie des sciences de l'Institut de France 58(1923), pp. xxxix–lxvi, also in *Oeuvres de Camille Jordan IV.*, pp. x–xxxiii. J. Dieudonné: *Jordan, Camille*, [Gi]-VII, pp. 167–169.

Jordan's extensive monograph *Traité des substitutions et des équations algébriques* [Jo] was published in Paris in 1870, and was awarded the Poncelet prize of the Académie des Sciences. It has four parts, which are divided (except the first part) into thirteen chapters, some of which are further split into paragraphs (their total number is 48) comprising of short numbered passages (their total number is 850). We present only the titles of the four main parts and the corresponding chapters (a complete table of contents spans eight pages).

1. *Des congruences* (pp. 1–18).
2. *Des substitutions* (*Des substitutions en général, Des substitutions linéaires*) (pp. 19–249).
3. *Des irrationnelles* (*Généralités, Applications algébriques, Applications géométriques, Applications à la théorie des transcendantes*) (pp. 251–382).
4. *De la résolution par radicaux* (*Conditions de résolubilité, Réduction du problème A, Réduction du problème B, Réduction du problème C, Résumé, Groupes a exclure, Indépendance des groupes restants*) (pp. 383–662).[32]

Camille Jordan wrote in the foreword to his monograph:

Le problème de la résolution algébrique des équations est l'un des premiers qui se soient imposés aux recherches des géomètres. Dès les débuts de l'Algèbre moderne, plusieurs procédés ont été mis en avant pour résoudre les équations des quatre premiers degrés: mais ces diverses méthodes, isolées les unes des autres et fondées sur des artifices de calculs, constituaient des faits plutôt qu'une théorie, jusqu'au jour où Lagrange, les soumettant à une analyse approfondie, sut démêler le fondement commun sur lequel elles reposent et les ramener à une même méthode véritablement analytique, et prenant son point de départ dans la théorie des substitutions. ([Jo], p. v)

Jordan's entire book is soaked with groups; this was unusual in the given time period.[33] The definition of a group of substitutions appears already in the beginning of the second part, followed a few pages later by lucid statements of Lagrange's[34] and Cauchy's theorem:

On dira qu'un système de substitutions forme un groupe (ou un faisceau) si le produit de deux substitutions quelconques du système appartient lui-même au système. ([Jo], p. 22)

Si le groupe H est contenu dans le groupe G, son ordre n est un diviseur de N, ordre de G. ([Jo], p. 25)

[32] Pages 663 to 667 contain *Notes*.

[33] Jordan's interest in groups is also documented by his works *Commentaire sur Galois*, Mathematische Annalen 1(1869), pp. 141–160, and *Mémoire sur les groupes de mouvements*, Annali Matematica pura ed applicata, 2(1868/69), pp. 167–215, 327–345.

[34] The frontispiece of Jordan's monograph features a nice portrait of Lagrange.

Réciproquement, si p est un nombre premier, tout groupe dont l'ordre est divisible par p contiendra une substitution d'ordre p. ([Jo], p. 26)

Let us quote an interesting excerpt – two theorems providing equivalent conditions for the solvability of algebraic equations in radicals. They can be found in the fourth part of Jordan's monograph:

Pour qu'une équation soit résoluble par radicaux, il faut et il suffit que sa résolution se ramène à celle d'une suite d'équations abéliennes de degré premier. ([Jo], p. 386)

Pour qu'une équation soit résoluble par radicaux, il faut et il suffit que ses facteurs de composition soient tous premiers. ([Jo], p. 387)

Jordan's monograph was enormously helpful in spreading and acknowledging the ideas of Galois contained in his work *Mémoire sur les conditions de résolubilité des équations par radicaux*, which was published by Joseph Liouville in 1846, i.e., fourteen years after Galois's death, in the eleventh volume of the Journal de mathématiques pures et appliquées.[35] It contained the first systematic and clear exposition of Galois's theory, contributed to the study of groups of substitutions (permutations), as well as abstract (even infinite) groups. It was a radical step toward the 20th century, toward the algebra of structures.

2.3 Heinrich Friedrich Ludwig Matthiessen

The German physicist Ludwig Matthiessen (1830–1906) studied from 1852 at the University of Kiel, where he graduated in 1857, and obtained his doctoral degree as well as habilitation in the same year. From 1855 till 1859 he worked as an assistant at the institute of physics in Kiel. From 1859 till 1873, he was a gymnasium teacher in Jever and Husum.[36] Since 1874 he was an ordinary professor of physics at the University of Rostock, since 1875 the director of its institute of physics as well as physics seminar. He was twice the dean of the philosophical faculty, and once the rector. He was pensioned in 1905.[37]

Besides a number of works and books in physics,[38] he revised and published the two-volume text *Schlüssel zur Sammlung von Beispielen und Aufgaben aus*

[35] Liouville was the editor of the journal until 1874. He took care to publish essential and topical works by the best mathematicians.

[36] Both cities are located in northern Germany near the German Bight (Deutsche Bucht).

[37] R. Mahnke: *Ludwig Matthiessen, der erste ordentliche Professor der Physik an der Universität Rostock, 1874–1905*, Wissenschaftliche Zeitschrift der Universität Rostock 34(1985), No. 1, pp. 74–86. R. Mahnke: *Ludwig Matthiessen – erster ordentlicher Professor der Physik an der Universität Rostock*, in *Zu Entwicklung der Physik an der Rostocker Universität*, Beiträge zur Geschichte der Universität Rostock, Heft 17, Universität Rostock, 1991, pp. 19–33.

[38] E.g., *Grundriss der Dioptrik geschichteter Linsensysteme*, Teubner, Leipzig, 1877, viii + 276 pages.

der allgemeinen Arithmetik und Algebra von Prof. Dr. Eduard Heis. Praktischer Leitfaden für Studirende.[39]

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In 1878, Matthiessen published in Leipzig the extensive monograph entitled *Grundzüge der antiken und modernen Algebra der litteralen Gleichungen* [Ma], whose second edition appeared in 1896.[40] It consists of seven parts, which are divided into 42 chapters, and further into 376 paragraphs. The adjoined eighth part provides an extensive bibliography in chronological ordering. A detailed table of contents spans 12 pages. Their titles (as well as page ranges) characterize the contents and scope of the book:

1. *Allgemeine Eigenschaften der algebraischen Gleichungen mit einer Unbekannten* (pp. 1–23).
2. *Von den Transformationen der Gleichungen und den symmetrischen Functionen der Wurzeln* (pp. 24–153).
3. *Directe Auflösung particulärer Gleichungen* (pp. 154–236).
4. *Directe Auflösung der Gleichungen von den ersten vier Graden durch Substitution* (pp. 237–788).
5. *Directe Auflösung der Gleichungen von den ersten vier Graden durch Combination* (pp. 789–878).
6. *Von der Auflösung der Gleichungen der ersten vier Grade mit Hülfe goniometrischer Functionen* (pp. 879–920).
7. *Von den geometrischen Constructionen der Wurzeln der algebraischen Gleichungen* (pp. 921–963).
8. *Die Gesammtlitteratur der Algebra der Gleichungen* (pp. 964–1001).[41]

Matthiessen's book focuses on the theory of algebraic equations and its historical development. It contains an enormous amount of material. In a very detailed manner, it presents a number of results, procedures and methods discovered in the previous centuries. To a large extent, it characterizes the conception of algebra in the 1870s. It is of considerable historical importance.

2.4 Diedrich August Klempt

In 1880, Diedrich August Klempt, a school teacher in Rostock, published in Leipzig the textbook *Lehrbuch zur Einführung in die moderne Algebra. Mit einigen hundert Beispielen* [Kl]. He wrote in the foreword:

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[39] M. DuMont-Schauberg, Köln, 1873, xii + 514, viii + 563 pages, later ed. 1886, xvi + 610, vi + 546 pages. English translation: Arkose Press, 2015, 572 pages.

[40] The monograph was a follow-up to Matthiessen's treatise *Die algebraischen Methoden der Auflösung der litteralen quadratischen, cubischen und biquadratischen Gleichungen. Nach ihren Principien und ihrem innern Zusammenhange*, Teubner, Leipzig, 1866, vii + 46 pages.

[41] A detailed table of contents is on pages v–xvi, corrections on page 1002. The book has no index.

Während die ältere Algebra sich vorzugsweise mit der Aufgabe beschäftigt, diejenigen Werthe einer Funktion zu finden, für welche dieselbe verschwindet, sucht die neuere Algebra Eigenschaften der Funktionen zu entdecken und betrachtet die Kenntniss der Zahlwerthe der Wurzeln als etwas Nebensächliches. ... Im Laufe weniger Jahrzehnte eroberten unsere ersten Forscher ein sehr ausgedehntes Reich für die Algebra und ihre Bedeutung ist in stetem Wachsthum begriffen. Deshalb ist es für den Studirenden der Mathematik nothwendig geworden, sich möglichst bald mit den wichtigsten Principien derselben vertraut zu machen, ja es ist sogar wünschenswerth, dass den Studirenden bereits in dem ersten Semester Gelegenheit geboten werde, die Hauptsätze und die vorzüglichsten Denkoperationen der modernen Algebra kennen zu lernen, um dann mit besserem Verständniss an die übrigen Disciplinen zu gehen. ([Kl], p. iii)

The book consists of nine parts, which are further divided into 149 paragraphs; a detailed table of contents spans six pages. The titles of the nine parts outline the content and organization of Klempt's entire book:

1. *Combinatorik* (pp. 1–47).
2. *Determinanten* (pp. 47–147).
3. *Lineare Gleichungen und lineare Funktionen* (pp. 148–183).
4. *Ganze und homogene Funktionen des zweiten Grades* (pp. 183–205).
5. *Allgemeine Sätze über ganze algebraische Funktionen* n*ten Grades mit einer Veränderlichen* (pp. 205–228).
6. *Symmetrische Funktionen der Wurzeln* (pp. 229–244).
7. *Die Elimination* (pp. 244–248).
8. *Die Discriminante* (pp. 249–253).
9. *Kanonische Formen* (pp. 254–260).[42]

The titles and page ranges make it obvious that the topics within the book are not well balanced. An exaggerated attention is paid to the theory of determinants, which was a very popular discipline throughout the 19th century. A major advantage of the textbook are the numerous examples and problems, which supplement the exposition and facilitate good understanding of the material. The experience of the author, a high-school teacher who wrote the book for beginners, is evident. The textbook does not manifest new trends in algebra, and the reference to *modern* algebra in the title of the textbook is quite exaggerated.

The notions of a group and subgroup (*Gruppe*, *Untergruppe*) appear early in the book in the context of permutations, but more or less just in the sense of a collection:

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[42] There are some subtle differences between the titles of the chapters in the table of contents and in the text itself. The book has no index.

Man schreibe erst alle Permutationen hin, die das Element a_1 als e r s t e s *Element enthalten, dadurch bekommt man eine Gruppe von so viel Permutationen, als die $(n-1)$ Elemente $a_2 \dots a_n$ überhaupt zulassen. Dann schreibe man alle Permutationen hin, die a_2 als Anfangselement enthalten. Es entsteht eine Gruppe von so viel Permutationen, als die $(n-1)$ Elemente $a_1 a_3 \dots a_n$ zulassen. So fahre man fort bis a_n incl. Man erhält so n Gruppen.* ([Kl], p. 2)

A similar usage of the word "group" appears in the part dealing with symmetric functions:

... Bedienen wir uns des Ausdruckes G r u p p e *für ein beliebiges der Aggregate $b_1 \sum x_1$, $b_{12} \sum x_1^2 x_2$ ..., so bemerken wir leicht, dass jede Gruppe wieder eine symmetrische Funktion der Wurzeln ist und dass die Aufgabe, Φ zu berechnen, auf die Aufgabe zurückgeführt ist, jede Gruppe zu berechnen.* ([Kl], p. 237)

3 Heinrich Martin Weber and his textbooks

Heinrich Weber was born in Heidelberg on March 5, 1842. He passed the graduation exam at a lyceum in his hometown, since 1860 he studied at the universities of Heidelberg and Leipzig, and obtained his doctoral degree from Heidelberg in 1863. Then he left for Königsberg (now Kaliningrad), where he spent time broadening his knowledge of mathematical physics and analysis with Franz Ernst Neumann (1798–1895) and Friedrich Julius Richelot (1808–1875), respectively. He finished his habilitation thesis at the same place.

He obtained habilitation in 1866 from the University of Heidelberg, where he spent several years as a private docent and later extraordinary professor (1869). In the following years, he held the position of an ordinary professor at the Zurich polytechnic institute (1869–1875), University of Königsberg (1875–1883, rector 1880), where his pupils included Hermann Minkowski (1864–1909) and David Hilbert, then at the Berlin-Charlottenburg polytechnic (1883–1884), University of Marburg (1884–1892, rector 1890/91), and University of Göttingen (1892–1895). Since 1895 he worked at the University of Strasbourg (rector 1900/91), which was then a part of Germany.[43] He was a member of the academies in Göttingen, Munich, Stockholm, Uppsala, and of the Accademia dei Lincei in Rome. Since 1893, he was the editor of the Mathematische Annalen.[44] In 1895 and 1904, he was the president of the German Mathematical Society. In 1904, he presided the third International Congress of Mathematicians in Heidelberg. He died in Strasbourg on May 17, 1913.[45]

[43] Until the Franco-Prussian War (1870–1871) and since the end of World War I, it was a part of France.

[44] One of the most prestigious mathematics journals. It was founded in 1868 by Alfred Clebsch (1833–1872) and Carl Neumann (1832–1925), son of Franz Ernst Neumann.

[45] The publication *Festschrift Heinrich Weber zu seinem siebzigsten Geburtstag am 5. März 1912*, Gewidmet von Freunden und Schülern, Teubner, Leipzig, Berlin, 1912,

Heinrich Weber was a universal mathematician. He worked mainly in algebra, number theory, algebraic geometry, analysis and its applications in physics. He published the book *Theorie der Abelschen Functionen von Geschlecht 3* in 1876, and the textbook *Elliptische Functionen und algebraische Zahlen. Akademische Vorlesungen*[46] in 1891.

Together with Richard Dedekind, they wrote the joint paper *Theorie der algebraischen Funktionen einer Veränderlichen*,[47] and jointly published the *Gesammelte mathematische Werke* of Bernhard Riemann (1826–1866).[48] Weber revised Riemann's lectures in mathematical physics and published them in two volumes (as their fourth edition) entitled *Die partiellen Differential-Gleichungen der mathematischen Physik. Nach Riemann's Vorlesungen.*[49] He prepared the German edition of Poincaré's treatise *Der Wert der Wissenschaft*,[50] which was translated by his daughter Emilie (1882–1911).

Heinrich Weber took part in editing a new version of the works of Leonhard Euler (1707–1783). He published *Volständige Anleitung zur Algebra. Mit Zusätzen von Joseph-Louis Lagrange.*[51]

Weber's broad interests led to his engagement in the book series *Ostwald's Klassiker der exakten Wissenschaften*, in which he edited several volumes:

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viii + 500 pages, which appeared on the occassion of Weber's 70th birthday, contains a photograph of Weber at the age about seventy. Review (R.C. Archibald): Bulletin of the American Mathematical Society 20(1913), No. 3, pp. 152–155.
F. Rudio: *Nekrolog. Heinrich Weber* (*1842–1913, Mitglied der Gesellschaft seit 1870. Ehrenmitglied seit 1896*), Naturforschende Gesellschaft (Zürich) 58(1913), pp. 437–453. A. Voss: *Heinrich Weber*, Jahresbericht der Deutschen Mathematiker-Vereinigung 23(1914), pp. 431–444 (Weber's portrait is on the frontispiece of this volume). G. Frei: *Heinrich Weber and the Emergence of Class Field Theory*, pp. 424–450, in J. McCleary, D.E. Rowe (ed.): *The History of Modern Mathematics*, Vol. I. *Ideas and their Reception*, Academic Press, MA, Boston, 1989, 453 pages. G. Frei: *Heinrich Weber* (*1842–1913*), in D. Rauschning et al. (eds.): *Die Albertus-Universität zu Königsberg und ihre Professoren. Aus Anlass der Gründung der Albertus-Universität vor 450 Jahren*, Duncker & Humblot, Berlin, 1995, pp. 509–520. N. Schappacher, K. Volkert: *Heinrich Weber; un mathématicien à Strasbourg, 1895–1913*, L'Ouvert 89(1997), pp. 1–18. B. Schoeneberg: *Weber, Heinrich*, [Gi]-XIV, pp. 202–203.

[46] Reimer, Berlin, 1876, iv + 182 pages. Vieweg, Braunschweig, 1891, xiii + 504 pages; in 1908, a revised version of this text was published as the third volume of Weber's textbook *Lehrbuch der Algebra.*

[47] Journal für die reine und angewandte Mathematik 92(1882), pp. 181–290.
K. Scheel, T. Sonar (eds.): *Der Briefwechsel Richard Dedekind – Heinrich Weber*, unter Mitarbeit von Karin Reich, De Gruyter Akad. Forschung, München, München, Boston, 2014, xx + 490 pages.

[48] Teubner, Leipzig, 1876, viii + 526 pages, 2nd ed. 1892, x + 558 pages; new ed. Springer, Berlin, 1990, x + 911 pages.

[49] Vieweg, Braunschweig, 1900, 1901, xvii + 506, xi + 527 pages, 5th ed. 1910, 1912, xviii + 527, xiv + 575 pages.

[50] Teubner, Leipzig, 1906, iv + 252 pages, later editions 1910, 1921.

[51] Leonhardi Euleri Opera Omnia, Series Prima, Band I., Teubner, Lipsiae, 1911, xcv + 651 pages; later ed. Birkhäuser, Basel, 2009.

- C.G.J. Jacobi: *Ueber die vierfach periodischen Functionen zweiter Variablen, auf die sich die Theorie de Abel'schen Transcendenten stützt* (1834).[52]
- G. Rosenhain: *Abhandlungen über die Functionen mit vier Perioden, welche die Inversen sind der ultra-elliptischen Integrale erster Klasse* (1851).[53]
- A. Göpel: *Entwurf einer Theorie der Abel'schen Transcendenten erster Ordnung* (1847).[54]
- J.L. Lagrange: *Zusätze zu Eulers Elementen der Algebra: Unbestimmte Analysis* (1771).[55]
- C.F. Gauss: *Allgemeine Grundlagen einer Theorie der Gestalt von Flüssigkeiten im Zustand des Gleichgewichts* (1830).[56]

Another evidence of Weber's broad interests is provided in his inaugural lecture as the newly appointed rector *Über Kausalität in den Naturwissenschaften.*[57]

Together with his pupil Joseph Wellstein (1869–1919) and other collaborators, he compiled an extensive three-volume work entitled *Encyklopädie der Elementar-Mathematik* [WW], which was first published in the period 1903–1907. It was widely used, reprinted in several subsequent editions, and the third volume was substantially enlarged – it was published in two volumes in 1910 and 1912.

In 1900, Heinrich Weber wrote the paper *Komplexe multiplikation*, which appeared in the Encyklopädie der Mathematischen Wissenschaften.[58] In 1906, he published the paper *Elementare Mengenlehre*;[59] it is remarkable that he was able to compile a treatise on a new, rapidly developing subject at the age of 64.

* * *

In 1893, Weber published in the Mathematische Annalen an important work entitled *Die allgemeinen Grundlagen der Galois'schen Gleichungstheorie* [We]. It contained a general exposition of the Galois theory, which is considered to be

[52] Vol. 64, Engelmann, Leipzig, 1895, 40 pages, translation from Latin by A. Witting.
[53] Vol. 65, Engelmann, Leipzig, 1895, 94 pages, translation from French by A. Witting.
[54] Vol. 67, Engelmann, Leipzig, 1895, 60 pages, translation from Latin by A. Witting.
[55] Vol. 103, Engelmann, Leipzig, 1898, 171 pages, translation from French by A.J. von Oettingen.
[56] Vol. 135, Engelmann, Leipzig, 1903, 73 pages, translation from Latin by Weber's son Rudolph Heinrich (1874–1920).
[57] Rede, gehalten bei der öffentlichen Feier der Übergabe des Prorektorats der Albertus-Universität zu Königsberg, Engelmann, Leipzig, 1881, 30 pages.
[58] Bd. I-2, Arithmetik und Algebra, 1900–1904, pp. 716–732.
[59] Jahresbericht der Deutschen Mathematiker-Vereinigung 15(1906), pp. 173–184.

its first modern presentation.[60] This work of Weber is also regarded as the first treatise devoted to both finite and infinite abstract groups. The introduction says:

Im Folgenden ist der Versuch gemacht, die Galois'sche Theorie der algebraischen Gleichungen in einer Weise zu begründen, die soweit möglich alle Fälle umfasst, in denen diese Theorie angewandt worden ist. Sie ergiebt sich hier als eine unmittelbare Consequenz des zum Körperbegriff erweiterten Gruppenbegriffs, als ein formales Gesetz ganz ohne Rücksicht auf die Zahlenbedeutung der verwendeten Elemente. Diese Begründung ist hiernach also auch ganz unabhängig von dem Fundamentalsatz der Algebra über die Wurzelexistenz. Die Theorie erscheint bei dieser Auffassung freilich als ein reiner Formalismus, der durch Belegung der einzelnen Elemente mit Zahlwerthen erst Inhalt und Leben gewinnt. ...

Ich beginne, um vollständig klar zu sein, mit einer genauen Begriffsbestimmung des Gruppen- und Körperbegriffs ... ([We], p. 521)

Weber divided the exposition into seven paragraphs: *Gruppen*, *Körper*, *Formenkörper*, *Congruenzkörper*, *Der Satz von Lagrange*, *Die Galois'sche Theorie*, *Endliche und unendliche Körper*.

As an illustration, let us quote the passage of Weber's paper containing an axiomatic definition of a group.

Ein System $\mathfrak{G}$ von Dingen (Elementen) irgend welcher Art in endlicher oder unendlicher Anzahl wird zur Gruppe, wenn folgende Voraussetzungen erfüllt sind.

1) *Es ist eine Vorschrift gegeben, nach der aus einem ersten und einem zweiten Element des Systems ein ganz bestimmtes drittes Element desselben Systems abgeleitet wird. ...*
2) *das associative Gesetz vorausgesetzt, ...*
3) *Es wird vorasgesetzt, dass, wenn $AB = AB'$ oder $AB = A'B$ ist, nothwendig $B = B'$ oder $A = A'$ sein muss. ...*
4) *Wenn von den drei Elementen A, B, C zwei beliebig aus $\mathfrak{G}$ genommen werden, so kann man das dritte immer und nur auf eine Weise so bestimmen, dass $AB = C$ ist.* ([We], p. 522)

Weber emphasized that for finite groups, the requirement 4) follows from the previous three, but this is no longer the case for infinite groups:

Für unendliche Gruppen ist dieser Schluss nicht mehr zwingend. Für unendliche Gruppen wollen wir also die Eigenschaft 4) *noch als Forderung in die Begriffsbestimmung mit aufnehmen.* ([We], p. 523)

* * *

[60] In this context, we recall the paper by O. Hölder entitled *Galois'sche Theorie mit Anwendungen*, Encyklopädie der Mathematischen Wissenschaften, Bd. I-1, Arithmetik und Algebra, 1898–1904, pp. 480–520.

The first volume of Weber's extensive textbook *Lehrbuch der Algebra* [W] was published in 1895. In the foreword, the author referred to the development of algebra in the previous decades, and thus explained the need for a more recent and modern textbook in comparison with the then used excellent book by Serret.

Es war meine Absicht, ein Lehrbuch zu geben, das, ohne viel Vorkenntnisse vorauszusetzen, den Leser in die moderne Algebra einführen und auch zu den höheren und schwierigeren Partien hinführen sollte, in denen das Interesse an dem Gegenstande erst recht lebendig wird. Dabei sollten die erforderlichen Hülfsmittel, die elementaren sowohl als die höheren, aus dem Gange der Entwickelung selbst abgeleitet werden, um die Darstellung von anderen Lehrbüchern möglichst unabhängig zu machen.

Zwei Dinge sind es, die für die neueste Entwickelung der Algebra ganz besonders von Bedeutung geworden sind; das ist auf der einen Seite die immer mehr zur Herrschaft gelangende Gruppentheorie, deren ordnender und klärender Einfluss überall zu spüren ist, und sodann das Eingreifen der Zahlentheorie. ([W]-1895, p. v)

Heinrich Weber briefly described the scope of both volumes of his textbook:

Der erste Band enthält den elementaren Theil der Algebra, den man mit einem hergebrachten Ausdruck als B u c h s t a b e n r e c h n u n g bezeichnen kann, sodann die Vorschriften über die numerische Berechnung der Gleichungswurzeln und die Anfänge der G a l o i s'schen Theorie.

Der zweite Band ... soll die allgemeine Theorie der endlichen Gruppen, die Theorie der linearen Substitutionsgruppen und Anwendungen auf verschiedene einzelne Probleme bringen und soll abschliessen mit der Theorie der algebraischen Zahlen, wo der Versuch gemacht ist, die verschiedenen Gesichtspunkte, unter denen diese Theorie bisher betrachtet worden ist, zu vereinigen. ([W]-1895, p. vi)

The first volume of Weber's textbook (in the first or the second edition) consists of three parts (which are further subdivided into 18 sections; the first edition has 188 paragraphs, while the second has 196 paragraphs):

Einleitung (pp. 1–20, or 1–22, respectively).

1. *Die Grundlagen* (*Rationale Functionen, Determinanten, Die Wurzeln algebraischer Gleichungen, Symmetrische Functionen, Lineare Transformation. Invarianten, Tschirnhausen-Transformation*) (pp. 21–237, or 23–267, respectively).
2. *Die Wurzeln* (*Realität der Wurzeln, Der Sturm'sche Lehrsatz, Abschätzung der Wurzeln, Genäherte Berechnung der Wurzeln, Kettenbrüche, Theorie der Einheitswurzeln*) (pp. 239–445, or 269–488, respectively).

3. *Algebraische Grössen* (*Die Galois'sche Theorie, Anwendung der Permutationsgruppen auf Gleichungen, Cyklische Gleichungen, Kreistheilung, Algebraische Auflösung von Gleichungen, Wurzeln metacyklischer Gleichungen*) (pp. 447–653, or 489–703, respectively).[61]

The third part of the first volume begins with the section *Die Galois'sche Theorie*. The introductory paragraph presents the notion of a field:

Ein System von Zahlen wird ein Zahlkörper genannt, wenn es so in sich vollendet und abgeschlossen ist, dass die vier fundamentalen Rechenoperationen (die vier Species), die Addition, die Subtraction, die Multiplication und die Division, ausgeführt mit irgend welchen Zahlen des Systems, ausgenommen die Division durch Null, immer auf Zahlen führen, die demselben System angehören. ...

Der Begriff des Zahlkörpers kann erweitert und auf alle Grössen übertragen werden, mit denen nach den Regeln der vier Species gerechnet werden kann, insbesondere also auf rationale Functionen irgend welcher Veränderlichen. ([W]-1895, p. 449, [W]-1898, p. 491)

Ein Körper ist dann also ein System von Grössen von der Vollständigkeit, dass in ihm die Grössen addirt, subtrahirt, multiplicirt und dividirt werden können. ([W]-1895, p. 450, [W]-1898, p. 492)

In the introduction to the 17th section entitled *Algebraische Auflösungen von Gleichungen*, Heinrich Weber wrote:

Eine der ältesten Fragen, an der sich vorzugsweise die neuere Algebra entwickelt hat, ist die nach der sogenannten algebraischen Auflösung der Gleichungen, worunter man eine Darstellung der Wurzeln einer Gleichung durch eine Reihe von Radicalen, oder die Berechnung durch eine endliche Kette von Wurzelziehungen versteht. Auf diese Frage fällt von der Gruppentheorie das hellste Licht. ([W]-1895, p. 595, [W]-1898, p. 644)

For example, he showed here that the alternating group is simple; this result is needed for the proof of unsolvability of the fifth-order equation. The proof was preceded by the following paragraph:

Wir wollen nun nachweisen, dass, wenn n grösser als 4 ist, die alternirende Gruppe ausser der Einheitsgruppe überhaupt keine normalen Theiler hat, oder nach der früher eingeführten Bezeichnung einfach ist. Daraus folgt dann, dass die Bedingung, die wir für die algebraische Auflösbarkeit einer Gleichung als nothwendig gefunden haben, für die Gleichungen von höherem als dem vierten Grade, deren Gruppe die symmetrische oder die alternirende ist, nicht erfüllt ist, und dass also Gleichungen von höherem als dem vierten Grade, so lange die Coëfficienten unabhängige Variable sind, nicht mehr algebraisch lösbar sind. ([W]-1895, p. 600, [W]-1898, p. 649)

[61] A detailed table of contents of the first volume spans seven pages.

The second volume of Weber's algebra textbook is literally permeated with groups; its four parts (in the first or the second edition) are further subdivided into 25 sections. The subdivision in the second edition is slightly different from the first edition (the first edition has 207 paragraphs, the second has 228 paragraphs):

1. *Gruppen* (*Allgemeine Gruppentheorie, Abel'sche Gruppen, Die Gruppe der Kreistheilungskörper, Cubische und biquadratische Abel'sche Körper, Constitution der allgemeinen Gruppen*) (pp. 1–148, or 1–160, respectively).
2. *Lineare Gruppen* (*Gruppen linearer Substitutionen, Gruppeninvarianten,*[62] *Gruppen binärer linearer Substitutionen, Die Polyëdergruppen, Congruenzgruppen*) (pp. 149–286, or 161–347, respectively).
3. *Anwendungen der Gruppentheorie* (*Allgemeine Theorie der metacyklischen Gleichungen, Die Wendepunkte einer Curve dritter Ordnung, Die Doppeltangenten einer Curve vierter Ordnung, Allgemeine Theorie der Gleichung fünften Grades, Gruppen linearer ternärer Substitutionen, Das Formenproblem der Gruppe* G_{168} *und die Theorie der Gleichungen siebenten Grades*) (pp. 287–486, or 349–550, respectively).
4. *Algebraische Zahlen*[63] (*Zahlen und Functionale eines algebraischen Körpers, Theorie der algebraischen Körper, Discriminanten, Beziehung eines Körpers auf seine Theiler, Quadratische Körper, Kreistheilungskörper, Abel'sche Körper und Kreistheilungskörper, Classenzahl, Classenzahl der Kreistheilungskörper, Transcendente Zahlen* (pp. 487–767), *Nachträge* (pp. 768–784)).[64]

The second volume of the textbook begins with the following words, which explain the meaning of the notion of a group throughout the entire volume:

Wir haben im ersten Bande bei den Permutationen den Begriff einer Gruppe kennen gelernt und wichtige algebraische Anwendungen von ihm gemacht. Es muss nun unsere nächste Aufgabe sein, diesen in der ganzen neueren Mathematik so überaus wichtigen Begriff allgemeiner zu fassen und die dabei herrschenden Gesetze kennen zu lernen. ([W]-1896, p. 3)

What follows is the definition of an abstract group in a form similar to the one given in the 1893 paper [We].

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[62] The section *Gruppeninvariante* is missing from the first edition.

[63] In the second edition, the fourth part is subdivided as follows: *Zahlen und Functionale eines algebraischen Körpers, Theorie der algebraischen Körper, Beziehungen eines Körpers zu seinen Theilern, Das Punktgitter, Classenzahlen, Kreistheilungskörper, Abel'sche Körper und Kreistheilungskörper, Classenzahl der Kreistheilungskörper, Transcendente Zahlen* (pp. 551–844).

[64] A detailed table of contents of the second volume has eight pages. A joint index for both volumes is on pages 785–794, or 845–855, respectively. Corrections are on pages 795–796, or 856, respectively.

Weber was well aware that it is necessary to survey the world of finite groups. Let us document it by means of the following excerpt.

Die allgemeine Definition der Gruppe ... lässt über die Natur dieses Begriffes noch manches im Dunkel ... In der Definition der Gruppe ist mehr enthalten, als es auf den ersten Blick den Anschein hat, und die Zahl der möglichen Gruppen, die aus einer gegebenen Anzahl von Elementen zusammengesetzt werden können, ist eine sehr beschränkte. Die allgemeinen Gesetze, die hier herrschen, sind erst zum kleinsten Theile erkannt, so dass jede neue specielle Gruppe, namentlich bei kleinerer Gliederzahl, ein neues Interesse bietet und zu eingehendem Studium auffordert.

Welche Gruppen sind zwischen einer gegebenen Zahl von Elementen, d. h. bei gegebenem Grade überhaupt möglich? Das ist die allgemeine Frage, um die es sich handelt, von deren vollständiger Lösung wir aber noch weit entfernt sind. Cayley hat diese Aufgabe zuerst für die niedrigsten Gradzahlen in Angriff genommen. ([W]-1896, p. 114, [W]-1899, p. 121)

The fourth part of the second volume contains even the notion of an ideal.

Das System aller ganzen Zahlen eines algebraischen Zahlkörpers Ω *soll ... mit* $\mathfrak{o}$ *bezeichnet werden. Ein in* $\mathfrak{o}$ *enthaltenes Zahlensystem* $\mathfrak{a}$ *wird ein Ideal genannt, wenn es den beiden Forderungen genügt:*

I. Summe und Differenz irgend zweier Zahlen in $\mathfrak{a}$ *geben immer wieder Zahlen in* $\mathfrak{a}$.

II. Das Produkt irgend einer Zahl in $\mathfrak{a}$ *und einer Zahl in* $\mathfrak{o}$ *gehört dem System* $\mathfrak{a}$ *an.* ([W]-1896, p. 547–548, [W]-1899, p. 620)

Weber's textbook [W] was quite successful, and its second edition appeared as early as 1898 and 1899. As the third volume of the second edition published in 1908 served Weber's monograph *Elliptische Funktionen und algebraische Zahlen* [Wel], which was published independently for the first time in 1891.[65]

The third volume of Weber's monograph (from 1908) has five parts, which are subdivided into 27 sections and 201 paragraphs:

1. *Analytischer Teil* (pp. 1–317).
2. *Quadratische Körper* (pp. 319–410).
3. *Komplexe Multiplikation* (pp. 411–560).
4. *Klassenkörper* (pp. 561–620).
5. *Algebraische Funktionen* (pp. 621–707).

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[65] The third edition, or the reprint of all three volumes, appeared in 1961, or 2001/02, respectively.

Tabellen (pp. 709–726).[66]

We remark that in 1912, a reduced version of Weber's algebra textbook was published in a single volume entitled *Lehrbuch der Algebra. Kleine Ausgabe in einem Bande* [Wkl] (x + 528 pages), whose second edition appeared in 1921, and third in 1928.

Weber's book [W] is the last major textbook, which to a certain extent reflects the scope of algebra in the end of the 19th century. Although it already presented new concepts of the emerging algebra of structures (group, field, etc.) as well as numerous results of "modern algebra", it remained faithful to the key topics of the past, real and complex polynomials, and solvability of algebraic equations. For Weber, groups and fields were only tools for dealing with the classical topics, instead of objects that ought to be studied independently, or subjects of new theories.

Weber's textbook became a standard treatise on groups. Together with the monograph *Theory of groups of finite order*[67] published in 1897 by William Burnside (1852–1927),[68] it provided inspiration for mathematicians of the next generations for the study of group theory, one of the most important algebraic theories of the 20th century.[69] It also contributed to the creation and extension of a suitable terminology. It dominated for over thirty years, and its contents and conception were only surpassed by van der Waerden's two-volume book *Moderne Algebra* from 1930 and 1931, which presented a completely new view of algebra.

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[66] A detailed table of contents has eight pages. An index is on pages 727–731. Pages 732–733 contain corrections to all volumes. Page v contains the following dedication: *Richard Dedekind, David Hilbert, Hermann Minkowski in herzlicher Freundschaft gewidmet.*
The first edition from 1891 has only three parts, which are divided into 16 sections and 120 paragraphs: *Analytischer Theil* (pp. 1–169), *Algebraischer Theil* (pp. 171–324), *Zahlentheoretischer Theil* (pp. 325–498), *Anhang* (pp. 499–504).

[67] Cambridge University Press, Cambridge, 1897, xxi + 388 pages, 2nd ed. 1911, xxiv + 512 pages, reprint: Dover, New York, 1955.

[68] A.R. Forsyth: *William Burnside*, Journal of the London Mathematical Society 3(1928), pp. 64–80.

[69] The increase of interest in group theory at the end of the 19th century can be documented by two papers published in the Encyklopädie der Mathematischen Wissenschaften, Bd. I-1, Arithmetik und Algebra, 1898–1904: Heinrich Burkhardt (1861–1914): *Endliche discrete Gruppen*, pp. 208–226, Anders Wiman (1865–1959): *Endliche Gruppen linearer Substitutionen*, pp. 522–554.
Let us also recall two additional texts by Luigi Bianchi (1856–1928): *Teoria dei gruppi di sostituzioni e delle equazioni algebrique*, Gozani, Pisa, 1897, 376 pages, *Lezioni sulla teoria dei gruppi di sostituzioni e delle equazioni algebriche secondo Gallois*, Spoerri, Pisa, 1901, iv + 283 pages.
A little book by Ludwig Baumgartner entitled *Gruppentheorie*, W. de Gruyter, Leipzig, Berlin, 1921, 120 pages, 5th ed. 1972, appeared as volume 837 of the popular book series Sammlung Göschen.

4 Algebra textbooks of the first three decades of the 20th century

4.1 Gustav Conrad Bauer, Ludwig Bieberbach

Gustav Bauer (1820–1906) studied at the universities of Erlangen and Berlin. He spent the rest of his life at the University of Munich; he was a private docent since 1857, extraordinary professor since 1865, and ordinary professor since 1869.[70]

* * *

On the occasion of Bauer's 80th birthday, it was decided to publish his textbook *Vorlesungen über Algebra* [Ba].[71] The printing took place in the Teubner publishing house in Leipzig, 1903, and was arranged by the Mathematischer Verein München; the main credit goes to Bauer's pupil Karl Doehlemann (1864–1926).[72] The second and third edition appeared posthumously in 1910 and 1921, respectively.

Gustav Bauer's textbook was based on his lectures for first- and second-year university students in Munich, which took place in the period from 1870 till 1897. It has four parts divided into 31 chapters:

1. *Allgemeine Eigenschaften der algebraischen Gleichungen* (pp. 1–105).
2. *Algebraische Auflösung der Gleichungen* (pp. 106–199).
3. *Numerische Auflösung der Gleichungen* (pp. 200–256).
4. *Theorie und Anwendung der Determinanten* (pp. 257–373).[73]

The book deals with the "traditional algebra", and contains almost no indications of the changing nature of this discipline. For example, let us note that the notion of a group was discussed only tangentially, and the notions of a field, ring and integral domain are completely absent. A pleasant fact is the presence of numerous historical remarks.

After Doehlemann's death, Bauer's textbook was revised and modernized by Ludwig Bieberbach (1886–1982).

He studied at the universities of Heidelberg and Göttingen. In the period 1913–1915 he was a professor at the University of Basel, in 1915–1921 at the

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[70] A. Voss: *Zur Erinnerung an Gustav Bauer*, Jahresbericht der Deutschen Mathematiker-Vereinigung 16(1907), pp. 54–75. C. von Voit: *Gustav Bauer*, Sitzungsberichte der mathematisch-naturwissenschaftlichen Klasse der Bayerischen Akademie der Wissenschaften zu München 37(1907), pp. 249–257.

[71] The book contains a nice portrait of Bauer.

[72] Karl Doehlemann was the author of several textbooks: *Projektive Geometrie in synthetischen Behandlung I., II.* (1901, 176 pages, 9th ed. 1937, 131 pages; Russian translation: 1908), *Geometrische Transformationen I., II.* (1902, vii + 322, viii + 328, 2nd ed. 1930, xvi + 254 pages), *Grundzüge der Perspektive nebst Anwendungen* (1916, iv + 104 pages). Let us also recall his paper *Über dekorative Malerei*, Zeitschrift für Ästhetik und allgemeine Kunstwissenschaft 9(1914), pp. 387–391.

[73] The index is on pages 374 and 375, corrections on page 376.

University of Frankfurt, and in 1921–1945 at the University of Berlin. He was a strong supporter of the fascist regime; he served as the editor of the infamous "Aryan" journal *Deutsche Mathematik* from 1936 till 1942. He was living in seclusion after the war.[74] He worked mainly in analysis and geometry, and was the author of numerous successful and repeatedly published textbooks.[75]

Bieberbach's revision of Bauer's textbook was published in 1928 as the fourth edition of the original textbook, but with the names of both authors. Bieberbach changed the original structure of the textbook. Five parts are divided into 24 chapters, which consist of paragraphs:

1. *Grundlegende Eigenschaften der algebraischen Gleichungen* (pp. 1–40).
2. *Theorie und Anwendung der Determinanten* (pp. 41–98).
3. *Haupteigenschaften der algebraischen Gleichungen* (pp. 99–128).
4. *Numerische Auflösung der Gleichungen* (pp. 129–202).
5. *Algebraische Auflösung der Gleichungen* (pp. 203–316).

   Anhang: Kettenbrüche (317–332).[76]

In the introduction to the fifth edition from 1933, Bieberbach commented on his revision of Bauer's textbook as follows:

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[74] H. Grunsky: *Ludwig Bieberbach zum Gedächtnis*, Jahresbericht der Deutschen Mathematiker-Vereinigung 88(1986), pp. 190–205. H. Mehrtens: *Ludwig Bieberbach and Deutsche Mathematik*, in E.R. Phillips (ed.): *Studies in the history of mathematics*, Washington, D.C., 1987, pp. 195–241.

[75] *Zur Theorie der automorphen Funktionen* (1910, 42 pages), *Differential- und Integralrechnung I., II.* (3rd ed. 1928, vi + 142, vi + 150 pages, 5th ed. 1944), *Funktionentheorie* (1922, iv + 118 pages), *Einführung in die Funktionentheorie* (2nd ed. 1952, 220 pages, 4th ed. 1966), *Lehrbuch der Funktionentheorie I. Elemente der Funktionentheorie* (1921, vi + 314, 3rd ed. 1930), *Lehrbuch der Funktionentheorie II. Moderne Funktionentheorie* (1927, vii + 366 pages, 2nd ed. 1931, reprint: 1945), *Einleitung in die höhere Geometrie* (1933, viii + 128 pages), (*Einführung in die*) *Analytische Geometrie* (1930, iv + 120 pages, 6th ed. 1962), *Differentialgeometrie* (1932, vi + 140 pages), *Projektive Geometrie* (1931, vi + 190 pages), *Einführung in die konforme Abbildung* (2nd ed. 1927, 131 pages, 6th ed. 1966, 180 pages), *Theorie der Differentialgleichungen* (1923, viii + 317, 3rd ed. 1930, 1944, xiii + 399 pages, reprint: 1979), *Theorie der gewöhnlichen Differentialgleichungen: auf fuktionentheoretische Grundlage dargestellt* (1953, ix + 338 pages, 2nd ed. 1965), *Einführung in die Theorie der Differentialgleichungen im reellen Gebiet* (1956, viii + 281 pages, reprint: 1979), *Analytische Fortsetzung* (1955, 167 pages), *Theorie der geometrischen Konstruktionen* (1952, 162 pages), *Galilei und die Inquisition* (1938, iii + 143 pages, 2nd ed. 1942), *Carl Friedrich Gauss* (1938, iii + 178 pages). Bieberbach prepared the second edition of the book *Die Determinanten* (1925, vii + 123 pages) by Eugen Netto (1846–1919), and translated the book *Zur Geschichte der Logik* (1927, v + 240 pages) by Federigo Enriques (1871–1946). For the second volume of the Encyklopädie der mathematischen Wissenschften, he wrote in 1921 the paper *Neuere Untersuchungen über Funktionen von komplexen Variablen*, Bd. II-3-1, Analysis, 1909–1921, pp. 379–532.

[76] A detailed table of contents has six pages. An index is on pages 333 and 334.
The fifth slightly revised edition appeared in 1933. The title of the third part was changed to *Symmetrische Funktionen*, the page ranges of the individual parts have also changed: pp. 1–57, 58–116, 117–146, 147–220, 221–340, 341–356. The index is on pages 357 and 358.

An der Tendenz des Buches, das eine Darstellung der Theorie der algebraischen Gleichungen geben will, habe ich nichts geändert. Die Belange der abstrakten Gruppen- und Körpertheorie habe ich etwas deutlicher hervortreten lassen, die numerischen Methoden etwas weiterverfolgt, und manchen schönen modernen Satz über die Lage der Gleichungswurzeln aufgenommen. Die Determinanten und die quadratischen Formen, die lineare Algebra habe ich, ihrer besonderen Bedeutung wegen, etwas eingehender behandelt, als es dem unmittelbaren Bedürfnis der Gleichungstheorie entspricht. ([Ba]-1933, p. iv)

Bieberbach's modernization of the text is also apparent from the fact that the notions of a ring and a field appear at the very beginning of the book:

Man sagt, eine Menge $\mathfrak{M}$ *von Elementen ... bildeten einen Körper, wenn die folgenden Tatsachen, Körperaxiome genannt, gelten. Die Körperaxiome zerfallen in zwei Sorten, die Ringaxiome und die eigentlichen Körperaxiome. Gelten nur die Ringaxiome, so heißt* $\mathfrak{M}$ *ein Ring. Ein Ring heißt somit ein Körper, wenn auch die an zweiter Stelle aufgeführten eigentlichen Körperaxiome gelten.* ([Ba]-1933, p. 6)

This passage is followed by a detailed description of the axioms for rings and fields (viz [Ba]-1933, pp. 6–8).

The notion of a group was introduced in the context of permutations as follows:

Eine Menge von Substitutionen von n *Elementen heißt eine Gruppe, wenn mit irgend zwei (nicht notwendig verschiedenen) Substitutionen* S, T *der Menge stets auch die Produkte* ST *und* TS *der Menge angehören. Die Anzahl der Substitutionen heißt die Ordnung der Gruppe.* ([Ba]-1933, p. 307)

Die Substitutionsgruppen sind ein Beispiel zum allgemeinen Gruppenbegriff. Wir nennen eine Menge $\mathfrak{G}$ *von Gegenständen (Elementen)* $a, b \ldots$ *eine Gruppe, wenn jedem geordneten Paar* a, b *von Elementen der Menge eindeutig ein weiteres Element* ab *zugeordnet ist, derart, daß für dieses Produkt* ab *die folgenden Gruppenpostulate erfüllt sind.*

1. ab *ist ein Element von* G.

2. Es gilt das assoziative Gesetz

$$(ab)c = a(bc).$$

3. Sind a *und* b *Elemente aus* $\mathfrak{G}$, *so gibt es genau ein Element* x *und genau ein Element* y, *beide aus* $\mathfrak{G}$, *so daß die Gleichungen*

$$xa = b$$
$$ay = b$$

erfüllt sind. ([Ba]-1933, p. 309)

Even though the book contains some basic facts on groups and fields (simple groups, composition series, field extensions, etc. – see pp. 307–340) in connection with the solvability of algebraic equations in radicals, it cannot be regarded as a major impetus toward the study of algebraic structures.

4.2 Maxime Bôcher

The American mathematician Maxime Bôcher (1867–1918) studied at Harvard (University of Cambridge) from 1883 till 1888, and was a pupil of Felix Klein in Göttingen from 1888 till 1891. There he obtained his doctoral degree, and his dissertation appeared in print.[77] Then he returned to Harvard, where he was originally appointed as an instructor, and since 1904 as a professor.

He worked in analysis, potential theory, paid considerable attention to differential and integral equations, harmonic functions, trigonometric series, mathematical physics, but also algebra and geometry. He was the author of several popular textbooks, both elementary and advanced ones, e.g., *An Introduction to the Study of Integral Equations*,[78] *Über die Reihenentwickelungen der Potentialtheorie. Mit einem Vorwort von Felix Klein*,[79] remembered is also his extensive text *Introduction to Theory of Fourier's Series*.[80] His lectures on linear differential equations, which he held at the Sorbonne in the academic year 1913/14, were published by Gaston Julia.[81]

Maxime Bôcher deserves major credit for the development of American mathematics. For many years, he was the editor of the Annals of Mathematics,[82] and was involved in the creation of the Transactions of the American Mathematical Society (1910).

He was a member of the American Mathematical Society, its vicepresident in 1902, and president in the years 1909 and 1910. He was also a member of the American Philosophical Society, American Academy of Arts and Sciences, and National Academy of Sciences. His name is remembered in the Bôcher Memorial Prize, which is awarded by the American Mathematical Society every five years for important results in analysis. He was an invited speaker at the fifth International Congress of Mathematicians held in Cambridge, 1912.[83]

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[77] *Über die Reihenentwickelung der Potentialtheorie*, Gekrönte Preisschrift, Göttingen, 1891, iv + 66 pages.

[78] Cambridge University Press, 1909, v + 72 pages, 2nd ed. 1914, reprint: 1971.

[79] Teubner, Leipzig, 1894, viii + 258 pages.

[80] Annals of Mathematics 7(1906), pp. 81–152.

[81] *Leçons sur les méthodes de Sturm ...*, Paris, 1917, vi + 118 pages.

[82] One of the most prestigious journals in mathematics. It was founded in 1884 by the American astronomer and mathematician Ormond Stone (1847–1933). The precursor of the Annals of Mathematics was the journal The Analyst (1874–1883).

[83] –: *Maxime Bôcher*, Science 48(1918), 29. Nov. 1918, No. 1248, pp. 534–535. W.F. Osgood: *The Life and Services of Maxime Bôcher*, Bulletin of the American Mathematical Society 25(1919), pp. 337–350. W.F. Osgood: *Maxime Bôcher*, ibid. 35(1929), pp. 205–217. G.D. Birkhoff: *The Scientific Work of Maxime Bôcher*, ibid. 25(1919), pp. 197–215, reprint in P. Duren, R.A. Askey, U.C. Merzbach (eds.): *A Century of Mathematics in America*, Part II., AMS, Providence, RI, 1989, x + 585 pages, pp. 59–78. C. Eisele: *Bôcher, Maxime*, [Gi]-II, pp. 217–218.

In 1907, Bôcher published the textbook *Introduction to Higher Algebra* [Bô]. It was quite successful and appeared in later editions. The German translation by Hans Beck with a foreword by Eduard Study (1862–1930) was published as early as 1910, the Russian translation in 1933. From the present viewpoint, it might be included among the first linear algebra textbooks, as it pays a lot of attention to matrices, determinants, linear dependence, linear equations, linear transformations, invariants, bilinear and quadratic forms, and their geometric aspects. It also treats polynomials and their factorization, symmetric polynomials, multivariate polynomials, theory of elementary divisors, equivalence of polynomial matrices, and equivalence and classification of pairs of bilinear and quadratic forms.[84]

Still, the book brings numerous results from the so-called universal algebra. The notion of a group is introduced in an efficient and modern manner in the 26th paragraph entitled *Sets, Systems, and Groups*, which begins as follows:

These three words are the technical names for conceptions which are to be met with in all branches of mathematics. In fact the first two are of such generality that they may be said to form the logical foundation on which all mathematics rests. ([Bô], p. 80)[85]

The definition of an abstract group is modern and based on the axiomatic approach:

A system consisting of a set of elements and one rule of combination, which we will denote by $\circ$, is called a group if the following conditions are satisfied:

(1) If a and b are any elements of the set, whether distinct or not, $a \circ b$ is also an element of the set.

(2) The associative law holds; that is, if a, b, c are any elements of the set,

$$(a \circ b) \circ c = a \circ (b \circ c).$$

(3) The set contains an element, i, called the identical element, which is such that every element is unchanged when combined with it,

$$i \circ a = a \circ i = a.$$

(4) If a is any element, the set also contains an element a', called the inverse of a, such that

$$a' \circ a = a \circ a' = i.$$

([Bô], p. 82)[86]

[84] A detailed table of contents (22 chapters, 106 paragraphs) spans five pages; an index spanning five pages facilitates the orientation within the book.

[85] In this context, Bôcher refers the readers to his paper *The Fundamental Conceptions and Methods of Mathematics*, Bulletin of the American Mathematical Society 11(1904), pp. 115–135.

[86] See also R. Franci: *On the Axiomatization of Group Theory by American Mathematicians: 1902–1905*, in S.S. Demidov (ed.): *Amphora. Festschrift für Hans Wußing zu seinem 65. Geburtstag*, Birkhäuser, Basel, 1992, pp. 261–277.

This elegant, concise axiomatic definition of a (finite or infinite) group surpasses the definitions in older and even some more recent textbooks (e.g., in the textbooks of Fricke, Beck, and Perron).

Bôcher's textbook does not refer to the notion of an *abstract field*, but nevertheless deals with number fields. The beginning of chapter 14 contains the following definition:

A set of numbers is said to form a domain of rationality if, when a *and* b *are any numbers of the set,* $a + b$, $a - b$, ab, *and far as* $b \neq 0$, a/b *are also numbers of the set.* ([Bô], p. 175)

An interesting review of Bôcher's book written by Robert Thompson (1931–1995) was published almost eight decades after the book itself.[87] Let us quote its introductory and final words:

Reviewing a book written about 75 years ago surely is a challenging task: what new can be said about old mathematics? Well, there is one thing: in spite of its age, Bocher's book is alive, interesting, and a delight to read! Of course, the reader must fit the book into the temper of its time, but even so, the "classical" (by necessity) theorems are still important, and the neatness of their proofs compares well with the best of modern exposition. ...

Today's student who wishes some exposure to the classical style (that's a synonym for the style of the 1800's), and this should be every student, probably cannot do better than to read Bocher. It is for this audience that Bocher's book is highly recommended.

4.3 Robert Karl Emanuel Fricke

Robert Fricke (1861–1930) studied since 1880 in Göttingen, Berlin, Strasbourg, and Leipzig. He graduated in Leipzig in 1885 as a pupil of Felix Klein, with whom he kept collaborating. Since 1886, he was a teacher at the court of Albrecht, the Prince Regent of the Duchy of Braunschweig, and since 1890 he taught at a gymnasium in Braunschweig. After his habilitation in 1892 he became a private docent at the University of Göttingen, and two years later a professor at the polytechnic in Braunschweig.

Together with Felix Klein, he coauthored the two-volume monograph *Vorlesungen über die Theorie der elliptischen Modulfunktionen.* I. *Grundlegung der Theorie*, II. *Fortbildung und Anwendungen der Theorie.*[88] Also with Klein as a coauthor, he published additional two volumes *Vorlesungen über die Theorie der automorphen Functionen*, I. *Die gruppentheoretischen Grundlagen*, II. *Die funktionentheoretischen Ausführungen und die Anwendungen.*[89]

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[87] The American Mathematical Monthly 91(1984), pp. 147–150.

[88] Teubner, Leipzig, 1890, 1892, xix + 764, xv + 712 pages. Reprint: Johnson, New York, 1966.

[89] Teubner, Leipzig, Berlin, 1897, 1912, xiv + 634, xiv + 668 pages. Reprint: Johnson, New York, 1965.

In 1916 and 1922, Fricke published the two-volume book *Die elliptischen Funktionen und ihre Anwendungen*, I. *Die funktionentheoretischen und analytischen Grundlagen*, II. *Die algebraischen Ausführungen*,[90] which contained, among other things, a short exposition of the Galois theory. Later he stated that he would like to publish a detailed version of this admirable theory. In 1913, he wrote two long papers on this topic, which he dealt with for several years, for the second volume of the Encyklopädie der mathematischen Wissenschaften.[91]

Fricke also wrote popular textbooks of mathematical analysis and analytic geometry, which were published repeatedly. For example, let us recall his texts *Analytische Geometrie*,[92] *Lehrbuch der Differential- und Integralrechnung und ihre Anwendungen*,[93] and the three-volume *Hauptsätze der Differential- und Integral-Rechnung als Leitfaden zum Gebrauch bei Vorlesungen zusammengestellt*.[94]

In the 1920s, Fricke was one of the editors of Klein's collected works, and a few years later also one of the editors of Dedekind's collected works.[95]

Although he worked mainly in analysis, he was well aware of the progress in group theory. This fact is evidenced not only by the first volume of the above-mentioned textbook *Vorlesungen über die Theorie der automorphen Functionen*, but also by his paper *Zur gruppentheoretischen Grundlegung der automorphen Functionen*.[96]

* * *

In 1922, Fricke learned from the publisher Friedrich Vieweg that Weber's book *Lehrbuch der Algebra* [W] will be soon out of print, and there is the need for a new algebra textbook. Fricke thus wrote a three-volume algebra textbook and published it in the period 1924–1928 under the title *Lehrbuch der Algebra verfaßt mit Benutzung von Heinrich Webers gleichnamigem Buche* [Fr]. In this way, he made it clear that he appreciated Weber's textbook and was inspired by its conception.[97] In the foreword he wrote:

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[90] Teubner, Leipzig, Berlin, 1916, 1922, x + 500, viii + 546 pages. The second volume contains the following dedication: *Dem Andenken Richard Dedekind's gewidmet.* Reprint: Springer, Berlin, Heidelberg, 2011, xviii + 500, xiv + 546 pages. The next volume was printed in 2012: C. Adelmann, J. Elstrod, E. Klimenko (eds.): III. *Anwendungen. Aus dem Nachlass*, xvii + 323 pages.

[91] Bd. II-2, Analysis, 1901–1921: *Elliptische Funktionen*, pp. 177–348, *Automorphe Funktionen mit Einschluß der elliptischen Modulfunktionen*, pp. 349–470.

[92] Teubner, Leipzig, 1915, vi + 135 pages, 2nd ed. 1922.

[93] Teubner, Leipzig, Berlin, 1918, 1921, xii + 399, v + 413 pages.

[94] Vieweg, Braunschweig, 1897, ix + 80, viii + 66, viii + 38 pages, 9th ed. 1923, xii + 219 pages.

[95] F. Klein: *Gesammelte mathematische Abhandlungen I., II., III.*, Springer, Berlin, Heidelberg, 1921, 1922, 1923, xii + 612, vi + 713, ix + 774 + 35 pages. R. Dedekind: *Gesammelte mathematische Werke I., II., III.*, Vieweg, Braunschweig, 1930–1932, 397, 442, 508 pages.

[96] Mathematische Annalen 42(1893), pp. 564–594.

[97] Let us note that Fricke helped in preparing Weber's textbook for publication: *Aber der*

Im ersten Bande wird man die Einwirkung von Webers Buch vielfach bemerken. Wenn dies auch bei den folgenden Bänden kaum im gleichen Maße der Fall sein wird, so habe ich doch im Andenken an die verehrungswürdige Persönlichkeit Heinrich Webers und in Dankbarkeit für die vielfältige Belehrung, die ich seinem Buche schulde, kein Bedenken getragen, Webers Namen in den Titel des vorliegenden Werkes aufzunehmen.

Der jetzt abgeschlossene erste Band bringt die Grundlagen der Theorie der algebraischen Gleichungen unter Einschluß der Galoisschen Theorie. ... Der zweite Band wendet sich zu den niedersten, nicht mehr „algebraisch" lösbaren Gleichungen. ... Der dritte Band soll die Theorie der algebraischen Zahlen auf Dedekindscher Grundlage behandeln. ([Fr]-I, pp. iii–iv)

Fricke's textbook has three volumes, which are subdivided into sections (consisting of chapters and paragraphs). We provide the titles of all volumes and their sections:

1. *Allgemeine Theorie der algebraischen Gleichungen* (viii + 468 pages). Sections: *Grundlegende Entwicklungen* (pp. 4–185). *Einzelsätze über reelle Gleichungen* (pp. 186–264). *Galoissche Gleichungstheorie* (pp. 265–461).

2. *Ausführungen über Gleichungen niederen Grades* (viii + 418 pages). Sections: *Endliche Gruppen binärer Substitutionen und Gleichungen fünften Grades* (pp. 2–181). *Endliche Gruppen ternärer Substitutionen und zugehörige Gleichungen* (pp. 182–329). *Geometrische Anwendungen der Gruppentheorie* (pp. 330–414).

3. *Algebraischen Zahlen* (vii + 506 pages). Sections: *Allgemeine Theorie der Zahlkörper* (pp. 2–189). *Ausführungen über besondere Zahlkörper* (pp. 190–502).[98]

The third section of the first volume entitled *Galoissche Gleichungstheorie* begins with basic results of group theory. The first chapter deals with finite groups (and includes the Sylow theorems), the second with Abelian groups, and the third with permutation groups. The fourth chapter discusses the Galois theory, and the fifth one algebraic solvability of equations. Let us present Fricke's definition of a group from the beginning of the third section.

Die m verschiedenen Elemente $E_0, E_1, E_2, \ldots, E_{m-1}$ *bilden eine „Gruppe"* $\mathfrak{G}_m$ *der „Ordnung"* m*, wenn folgende drei Bedingungen erfüllt sind:*

I. Das Ergebnis der Zusammensetzung $E_b \cdot E_a$ *irgend zweier Elemente* E_a *und* E_b *ist wieder eines der Elemente;*

Herren E. Hess in Marburg, Fr. Meyer in Clausthal, R. Fricke in Braunschweig, die durch kundige und sorgfältige Ausführung der mühevollen Correctur der Druckbogen Genauigkeit und Richtigkeit des Textes gefördert haben, muss ich hier noch gedenken. ([W]-1895, p. vii)

[98] Detailed tables of contents (4 + 4 + 3 pages) and indexes (pp. 463–468, 415–418, 503–506) facilitate the orientation within the book.

II. Für die nach I herstellbaren symbolischen Produkte von drei Faktoren gilt das assoziative Gesetz:

$$E_c \cdot (E_b \cdot E_a) = (E_c \cdot E_b) \cdot E_a;$$

III. Ist $E_b \neq E_c$, d. h. sind E_b und E_c irgend zwei verschiedene Elemente, und ist E_a irgend ein Element, so gilt auch:

$$E_b \cdot E_a \neq E_c \cdot E_a \qquad \text{und} \qquad E_a \cdot E_b \neq E_a \cdot E_c.$$

([Fr]-I, p. 268)

The notion of a field is postponed until the beginning of the fourth chapter on Galois theory.

Für die Begründung der Galoisschen Gleichungstheorie bedarf man neben dem Gruppenbegriffe noch des von Dedekind eingeführten Körperbegriffs. Ein System konstanter Zahlen wird ein „Zahlenkörper“ oder „Zahlkörper“ oder kurz „Körper“ $\mathfrak{K}$ genannt, wenn mit irgend zwei Zahlen a, b des Systems auch $(a + b)$, $(a - b)$, $a \cdot b$ und, falls $b \neq 0$ gilt, auch $a : b$ im System enthalten ist. Irgend eine rationale Rechnung, auf Zahlen von $\mathfrak{K}$ angewandt, ergibt stets wieder eine Zahl von $\mathfrak{K}$, wenn man nur allemal die Division durch 0 vermeidet. ([Fr]-I, p. 354)

The definition of a group is more general – it is based on axioms, while the notion of a field is derived from the properties of (rational, real, complex) numbers. This corresponds to applications in Galois theory, where we need permutation groups and numeric fields. The reader meets groups throughout the entire second volume of Fricke's textbook (groups of linear substitutions, cyclic groups, applications in geometry, etc.).

The definition of a field is presented almost in the same form at the beginning of the third volume (see [Fr]-III, p. 2). The definition of an ideal follows a few pages later:

Als ein Ideal $\mathfrak{a}$ des Körpers $\mathfrak{K}$ wird jedes nicht nur aus der Zahl 0 bestehende System von ganzen Zahlen des Körpers $\mathfrak{K}$ bezeichnet, das folgende Eigenschaft hat: Mit den Zahlen α und β gehört auch jede Zahl $(\lambda\alpha + \mu\beta)$ dem System $\mathfrak{a}$ an, wenn λ und μ irgendwelche ganze Zahlen aus $\mathfrak{K}$ sind. ([Fr]-III, p. 15)

Thus, an ideal of a (numeric) field $\mathfrak{K}$ is an ideal (in the present meaning of the word) of the integral domain $\mathbb{Z}$, which is contained in the field $\mathfrak{K}$.

We see that Fricke's definitions of a group, ring, or field are still not completely abstract. The author had in mind classes of numbers, or sets of permutations.

4.4 Helmut Hasse

The German mathematician Helmut Hasse (1898–1979) studied since 1917 at the universities of Kiel, Göttingen, and Marburg, where he graduated in 1921 as a pupil of Kurt Hensel (1861–1941), and obtained his habilitation one year later. Since 1922, he spent three years as a private docent in Kiel. He was a university professor in Halle since 1925, in Marburg since 1930, and in Göttingen since 1934. During World War II, he was the director of the Research Institute of the Imperial Naval Office in Berlin. Since 1945, he had a position at the German Academy of Sciences in Berlin, since 1949 at the Humboldt University of Berlin, and in the years 1950–1966 at the University of Hamburg.[99]

Helmut Hasse worked mainly in algebra and number theory, and is considered a member of the school of Emmy Noether and Emil Artin. His name is linked to numerous concepts and results in algebraic number theory and theory of commutative fields.[100] Hasse's mathematical works *Mathematische Abhandlungen I., II., III.*[101] were published in 1975. Interesting is his correspondence with Emmy Noether, Emil Artin, Arnold Scholz (1906–1941) and Olga Taussky-Todd (1906–1995), which has been published not long ago.[102]

Helmut Hasse is the author of the books *Vorlesungen über Zahlentheorie*[103] and *Zahlentheorie.*[104]

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[99] G. Frei: *Helmut Hasse (1898–1979). A biographical sketch dealing with Hasse's fundamental contributions to mathematics, with explicit references to the relevant mathematical literature*, Expositiones mathematicae 3(1985), pp. 55–69. H.W. Leopoldt: *Obituary. Helmut Hasse (August 25, 1898 – December 26, 1979)*, Journal of Number Theory 14(1982), pp. 118–120. H.W. Leopoldt: *Zum wissenschaftlichen Werk von Helmut Hasse*, Journal für die reine und angewandte Mathematik 262/263(1973), pp. 1–17. H.M. Edwards: *Hasse, Helmut*, [Gi]-XVII, pp. 385–387.
An interesting article on the rise of fascism – S.L. Segal: *Helmut Hasse in 1934*, Historia Mathematica 7(1980), pp. 46–56.

[100] For example his work *Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper, I. Klassenkörpertheorie, II. Reziprozitätsgesetz*, Jahresbericht der Deutschen Mathematiker-Vereinigung 35(1926), pp. 1–55, 36(1927), pp. 233–311, Ergänzungsband VI., 1930, iv + 204 pages, 2nd ed. Physica-Verlag, Würzburg, 1965, 135 + 204 pages. *Über die Klassenzahl Abelscher Zahlkörper*, Akademie-Verlag, Berlin, 1952, xii + 190 pages.

[101] *Mathematische Abhandlungen I., II., III.*, W. de Gruter, Berlin, New York, 1975, xv + 535, xv + 525, x + 532 pages.

[102] F. Lemmermeyer, P. Roquette (eds.): *Helmut Hasse und Emmy Noether. Die Korrespondenz 1925–1935*, Universitätsverlag Göttingen, Göttingen, 2006, 301 pages. G. Frei, P. Roquette (eds.): *Emil Artin und Helmut Hasse. Die Korrespondenz 1923–1934*, Universitätsverlag Göttingen, Göttingen, 2008, 499 pages. K. Reich: *Der Briefwechsel Emil Artin – Helmut Hasse (1937/38 und 1953 bis 1958). Die Freundschaft der beiden Gelehrten im historischen Kontext.* Edition am Gutenbergplatz, Leipzig, 2018, 254 pages.

[103] Springer, Berlin, Göttingen, Heidelberg, 1950, xii + 474 pages, 2nd ed. 1964, xv + 504 pages.

[104] Akademie-Verlag, Berlin, 1949, xii + 468 pages, Springer, Berlin, Heidelberg, 1980, xvii + 638 pages. 2nd ed. 1963, xvi + 611 pages, 3rd ed. 1969, xvii + 611 pages. English translation: *Number Theory*, Springer, Berlin, Heidelberg, 1980, xvii + 638 pages.

Let us also recall his interesting little book *Die Grundlagenkrisis der Griechischen Mathematik*[105] written in collaboration with the philosopher Heinrich Scholz (1884–1956), who devoted himself, among other topics, to the foundations of mathematics and history of logic.

* * *

The two volumes of Hasse's textbook *Höhere Algebra* [H] appeared in 1926 and 1927 as volumes 931 and 932 of the popular book series *Sammlung Göschen.* They are slender books of a small size (160 + 160 pages), whose content or purpose cannot be compared to that of Weber's monumental textbook. Still, to a certain extent, they reflect contemporary views of the scope and goals of algebra, characterized by Hasse in the introduction to the first volume as follows:

> *Das Wort A l g e b r a stammt aus dem Arabischen und bedeutet wörtlich das Hinüberschaffen eines Gliedes von einer Seite einer Gleichung auf die andere. Späterhin versteht man unter Algebra allgemein die Lehre von der Auflösung von Gleichungen (und zwar ausschließlich von solchen, die zu ihrer Bildung nur die vier sog. elementaren Rechenoperationen erfordern) mit einer Anzahl unbekannter Größen nach diesen. Dieser Aufgabe sind die beiden vorliegenden Bändchen gewidmet.* ([H]-I, p. 5)

Hasse delineated the main objective of algebra (*Grundaufgabe der Algebra*) in the following way:

> *E s s o l l e n a l l g e m e i n e, f o r m a l e M e t h o d e n e n t w i c k e l t w e r d e n, n a c h d e n e n m a n m i t t e l s d e r v i e r e l e m e n t a r e n R e c h e n o p e r a t i o n e n g e b i l d e t e G l e i c h u n g e n z w i s c h e n b e k a n n t e n u n d u n b e k a n n t e n E l e m e n t e n e i n e s K ö r p e r s n a c h d e n u n b e k a n n t e n a u f l ö s e n k a n n.* ([H]-I, pp. 6–7)

The first volume of Hasse's textbook is entitled *Lineare Gleichungen.* The introduction and table of contents are followed by four chapters:

1. *Ringe, Körper, Integritätsbereiche* (pp. 7–50).
2. *Gruppen* (pp. 50–70).
3. *Determinantenfreie lineare Algebra* (pp. 71–109).
4. *Lineare Algebra mit Determinanten* (pp. 109–156).
   Schluß (pp. 157–158).[106]

The second volume of Hasse's textbook is entitled *Gleichungen höheren Grades.* It consists of five chapters:[107]

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[105] Charlottenburg, 1928, 72 pages.

[106] Table of contents, references and introduction are on pages 3–7, index on pages 159 and 160. There is an embedded leaflet with corrections.

[107] As a preparatory reading for the second volume, Hasse recommended volume 930 of the same book series – P.B. Fischer: *Elementare Algebra*, 1926, 149 pages. Its first part

1. *Die linken Seiten algebraischer Gleichungen* (pp. 8–40).
2. *Die Wurzeln algebraischer Gleichungen* (pp. 41–52).
3. *Die Körper der Wurzeln algebraischer Gleichungen* (pp. 52–81).
4. *Die Struktur der Wurzelkörper algebraischer Gleichungen* (pp. 81–129).
5. *Auflösbarkeit algebraischer Gleichungen durch Wurzelzeichen* (pp. 129–157).[108]

The previous brief tables of contents characterize the scope and focus of Hasse's two-volume textbook. Detailed indexes facilitate quick orientation within the book.

The presentation of basic concepts and results of structural algebra contained in the first two chapters of the first volume (*Ringe, Körper, Integritätsbereiche* and *Gruppen*) serves only as a preparation for the material in subsequent chapters of both volumes, i.e., solvability of equations. In the second volume, the exposition concludes with the Galois theory and solvability of algebraic equations in radicals.

In 1934, Hasse published a supplement to his two-volume textbook – a problem book entitled *Aufgabensammlung zur höheren Algebra* [H-A] (175 pages), which appeared as volume 1082 of the book series Sammlung Göschen. Additional two editions of this book (1952, 1961) were prepared with the help of Walter Klobe, assistant master at the Lessing-Oberschule in Erfurt.

The third edition of the first volume of Hasse's *Höhere Algebra* from 1951 contains the following interesting passage, which to a certain extent documents the change in the interpretation of algebra since the first edition:

Es ist für die m o d e r n e Entwicklung der Algebra charakteristisch, daß die oben genannten Hilfsmittel zu selbstständigen umfangreichen Theorien Anlaß gegeben haben, die gegenüber der vorstehend angeführten Grundaufgabe der k l a s s i s c h e n Algebra immer mehr in den Mittelpunkt des Interesses getreten sind. So ist denn in moderner Auffassung die Algebra nicht mehr bloß die Lehre von der Auflösung der Gleichungen, sondern die L e h r e v o n d e n f o r m a l e n R e c h e n b e r e i c h e n, wie Körpern, Gruppen u. a., und ihre Hauptaufgabe ist die Gewinnung von Einsichten in die S t r u k t u r s o l c h e r B e r e i c h e ... Im beschränkten Rahmen der vorliegenden Bändchen ist es uns jedoch nicht möglich, diesen allgemeineren, modernen Gesichtspunkt in den Vordergrund zu stellen. Wir nehmen daher die vorstehend ausgesprochene Grundaufgabe der klassischen Algebra als wegweisenden Leitfaden und abgrenzenden Rahmen für unsere Darlegungen, werden aber dabei in der Tat, vor

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is entitled *Allgemeine Theorie der algebraischen Gleichungen*, the second one *Besondere Gleichungen und besondere Lösungsverfahren.* In 1939, the same volume number in this book series was assigned to a little book by Wolfgang Krull entitled *Elementare Algebra vom höheren Standpunkt* (143 pages). It should have probably served as a substitute for Fischer's book.

[108] Table of contents and introduction are on pages 3–8, index on pages 157–159, corrections on pages 159 and 160.

allem in 2, auch zu strukturellen Aussagen im Sinne der modernen Algebra geführt werden. ([H]-I-1951, p. 7)

Hasse's two-volume textbook and the supplementary textbook remained in print until the end of the 1960s. In 1954, there appeared a single-volume English translation of the third edition from 1951, as well as an English translation of the second edition of the problem book from 1952.

4.5 Leonard Eugene Dickson

Leonard Eugene Dickson (1874–1954) studied at the University of Texas at Austin. After obtaining his doctorate in Chicago in 1896, he went for a study tour to Leipzig and Paris. He was instructor in mathematics at the University of California from 1897 till 1899, when he was appointed associate professor at the University of Texas at Austin, but left for the University of Chicago. There he served as assistant professor from 1900, associate professor from 1907, and full professor from 1910.[109] Dickson's collected works were published by Abraham Adrian Albert (1905–1972) in 1975.[110]

Dickson worked mainly in algebra, number theory, algebraic geometry, and invariant theory. His celebrated three-volume monograph *History of the Theory of Numbers* was published in 1919, 1920, 1923.[111] Number theory was also the subject of his textbooks *Introduction to the Theory of Numbers*,[112] *Modern Elementary Theory of Numbers*,[113] and the monograph *Researches on Waring's Problem*.[114]

Dickson is the author of numerous texts on linear and abstract algebra: *Introduction to the Theory of Algebraic Equations*,[115] *Elementary Theory of Equations*,[116] *A First Course in the Theory of Equations*,[117] *New First Course*

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[109] A.A. Albert: *Leonard Eugene Dickson. 1874–1954*, Bulletin of the American Mathematical Society 61(1955), pp. 331–345. R.S. Calinger: *Dickson, Leonard Eugene*, [Gi]-IV, pp. 82–83. D. Fenster: *Leonard Eugene Dickson (1874–1954): An American Legacy in Mathematics*, The Mathematical Intelligencer 21(1999), No. 4, pp. 54–59. D. Dumbaugh Fenster: *Research in Algebra at the University of Chicago: Leonard Eugene Dickson and A. Adrian Albert*, [GP], pp. 179–197. D. Dumbaugh, A. Shell-Gellasch: *The „Wide“ Influence of Leonard Eugene Dickson*, Notices of the American Mathematical Society 64(2017), No. 7, pp. 772–776.

[110] *The Collected Mathematical Papers*, Chelsea, Bronx, New York, 1975, xvii + 680, 766, 580, 636, 644 pages.

[111] Carnegie Institution, Washington, iv + 313, xxv + 803, xii + 486 pages. Reprints: 1966, 2005.

[112] Chicago University Press, Chicago, 1929, viii + 183 pages, reprint: 1957. German translation: *Einfuhrung in die Zahlentheorie*, 1931.

[113] Cambridge University Press, London, 1939, vii + 309 pages.

[114] Carnegie Institution, Washington, Publication No. 464, 1935, v + 257 pages.

[115] Wiley, New York, 1903, v + 104 pages.

[116] Wiley, New York, 1914, iv + 184 pages, later editions.

[117] Wiley, New York, 1922, vi + 168 pages.

in the Theory of Equations,[118] *College Algebra*,[119] *Algebraic Invariants*,[120] *On Invariants and the Theory of Numbers*.[121] He did intense research on algebras, as evidenced by his books *Linear Algebras*[122] and *Algebras and their Arithmetics*,[123] whose German translation entitled *Algebren und ihre Zahlentheorie* appeared four years later together with the supplement *Idealtheorie in rationalen Algebren* by Andreas Speiser (1885–1970).[124]

One of the first books that paid greater attention to groups was Dickson's *Linear Groups. With an Exposition of the Galois Field Theory*.[125] Together with George Abram Miller (1863–1951) and Hans Frederik Blichtfeldt (1873–1945) as coauthors, Dickson wrote the monograph *Theory and Applications of Finite Groups*.[126] Let us also recall his paper *Definitions of a Group and a Field by Independent Postulates*.[127]

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In 1926, Dickson published the book *Modern Algebraic Theories* [Di] comprising ix + 276 pages, which has 14 chapters subdivided into 147 articles; it contains a subject index as well as a name index. It was based on his lectures from the previous years. A German translation entitled *Höhere Algebra* by Ewald Bodewig (1902–?) appeared three years later. A reprint of the original book with the shorter title *Algebraic Theories* appeared in 1959. In the introduction to his textbook, Dickson wrote:

This book ... presupposes calculus and elementary theory of algebraic equations. Its aim is to provide a simple introduction to the essentials of each of the branches of modern algebra, with the exception of the advanced part treated in the author's Algebras and Their Arithmetics. The book develops the theories which center around matrices, invariants, and groups, which are among the most important concepts in mathematics.

The book provides adequate introductory courses in (i) higher algebra, (ii) the Galois theory of algebraic equations, (iii) finite linear groups, inclu-

[118] Wiley, New York, 1939, ix + 185 pages.

[119] Wiley, New York, 1902, vi + 214 pages, 2nd ed. 1912.

[120] Wiley, New York, 1914, x + 100 pages.

[121] AMS, New York, 1914, iv + 110 pages.

[122] Cambridge University Press, 1914, viii + 73 pages, reprint: 1960. Russian translation: Charkov, 1935.

[123] University of Chicago Press, Chicago, 1923, xii + 241, reprint: Stechert, New York, 1938, reprint: 1960.

[124] Orell Füssli Verlag, Zürich und Leipzig, 1927, viii + 308 pages. Speiser's treatise is in Chapter XIII (pp. 269–303).

[125] Teubner, Leipzig, 1901, x + 312 pages, reprint: Dover, New York, 1958. The book has two parts: *Introduction to the Galois Field Theory* (pp. 1–71) and *Theory of Linear Groups in a Galois Field* (pp. 73–310).

[126] Wiley, New York, xviii + 390 pages.

[127] Transactions of the American Mathematical Society 6(1905), pp. 198–204. See also E.V. Huntington: *Note on the Definition of Abstract Groups and Fields by Sets of Independent Postupates*, ibid. 6(1905), pp. 181–197.

ding Klein's "icosahedron" and theory of equations of the fifth degree, and (iv) algebraic invariants. ([Di], p. iii)

Dickson was familiar with Serret's textbook [Se], Weber's textbook [W], as well as Jordan's monograph *Traité des substitutions et des équations algébriques* [Jo], all of which are referred to in his book.

Today, we would classify a number of topics from Dickson's book as belonging to linear algebra (matrices, determinants, linear transformations, bilinear, quadratic, Hermitian forms, canonical forms, elementary divisors, linear dependence and independence, linear equations, pairs of bilinear, quadratic, and Hermitian forms, ...). Nevertheless, other topics definitely belong to algebra proper (groups, fields, solvability of algebraic equations in radicals, straightedge and compass constructions, higher-order equations, representation of finite groups by linear groups, ...).

Dickson began by introducing the notion of a group of substitutions, and only later (in a paragraph printed in a smaller font) provided the general definition of an abstract group:

... An abstract group is a system composed of a set of elements $a, b, \ldots$ and a rule of combining any two of them to produce their "product", such that (i) every product of two of the elements and the square of each element are elements of the set, (ii) the associative law holds, (iii) the set contains an identity element I such that $Ia = aI = a$ for every element a of the set, and (iv) each element a of the set has an inverse a^{-1} belonging to the set, such that $aa^{-1} = a^{-1}a = I$. ([Di], p. 144)

On the other hand, he introduced the notion of a field only in connection with complex numbers ([Di], p. 54, 150). He referred readers interested in a deeper understanding of the general notion of a field to his book *Algebras and their Arithmetics*.

4.6 Rudolf Hans Heinrich Beck

The German mathematician Hans Beck (1876–1942) studied at high school and university in Greifswald, spent some time working at the Hannover Polytechnic, and later served as mathematics professor at the University of Bonn. He completed a German translation of Bôcher's book *Introduction to higher algebra* [Bô] entitled *Einführung in die höhere Algebra* and printed by the Teubner publishing house in Leipzig and Berlin in 1910. His book *Koordinatengeometrie. Die Ebene*[128] was published in 1919 in Berlin, and the two-volume textbook *Elementargeometrie* in 1929 and 1930 in Leipzig.[129]

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[128] Springer, Berlin, 1919, x + 432 pages.

[129] Akademische Verlagsgesellschaft, 1929, 1930, xii + 112, x + 184 pages.
For more on Beck's life and work, see E. Salkowski: *Hans Beck zum Gedächtnis*, Jahresbericht der Deutschen Mathematiker-Vereinigung 53(1943), pp. 91–103.

Beck's book *Einführung in die Axiomatik der Algebra* [Be] was published in the book series Göschens Lehrbücherei[130] in 1926. It was based on his introductory four-hour classes at the University of Bonn. He stated in the foreword:

Die Axiomatik der Algebra ist heute keineswegs abgeschlossen, aber immerhin soweit entwickelt, daß der Universitätsunterricht zu ihr hat Stellung nehmen müssen. ([Be], p. v)

In the beginning of the first chapter entitled *Zahlen und Verknüpfungen*, he provided the following characterization of algebra:

Algebra ist die Lehre von den vier Spezies, das ist von Verknüpfungen der Addition, Subtraktion, Multiplikation und Division in endlichmaliger Wiederholung.

Nach dieser landläufigen Definition haben wir es in der Algebra nicht mit unendlichen Prozessen zu tun. Die gehören in die Analysis.

Objekte der Algebra oder vielmehr ihrer Verknüpfungen sind (zunächst wenigstens) die Zahlen schlechthin; beschränkt man sich auf die ganzen Zahlen, so treibt man Zahlentheorie oder Arithmetik. Danach ist diese ein Ausschnitt aus der Algebra.

Was sind nun Zahlen? Diese Frage lassen wir insofern unbeantwortet ... mehr als diese Objekte interessieren uns die mit ihnen vorzunehmenden Operationen. ([Be], p. 1)

Beck's book is rather slender. It has twelve chapters, most of which fall within the scope of linear algebra:

1. *Zahlen und Verknüpfungen* (pp. 1–14).
2. *Punktmengen* (pp. 15–22).
3. *Zahlenpaare* (pp. 23–30).
4. *Matrizes* (pp. 31–40).
5. *Vektoren* (pp. 41–47).
6. *Lineare Gleichungen* (pp. 48–73).
7. *Lineare Vektorgebilde* (pp. 74–93).
8. *Bilineare und quadratische Formen* (pp. 94–110).
9. *Proportionalität der Matrizes* (pp. 111–126).
10. *Determinanten* (pp. 127–164).
11. *Unabhängigkeit und Widerspruchslosigkeit* (pp. 165–178).
12. *Der genetische Aufbau der Algebra* (pp. 179–194).[131]

[130] 1. Gruppe. Reine Mathematik, Band 6.
[131] A detailed table of contents is on pages vii–x, an index is in the end of the book on pages 195–197.

Beck's textbook is significantly built on set-theoretic foundations and axiomatic approach. For the record, we cite the axioms of a group, which are given in chapter 11, paragraph 115 entitled *Gruppenaxiome*.

G 1. Zu zwei Elementen A und B des Systems gibt es stets ein einziges Element $A \circ B$ *des Systems.*

G 2. Die Komposition ist assoziativ $(A \circ B) \circ C = A \circ (B \circ C)$.

G 3. Aus $A \circ C = B \circ C$ *und ebenso aus* $C \circ A = C \circ B$ *folgt stets* $A = B$.

G 4. Zu zwei Elementen A und B des Systems gibt es stets ein einziges X und ein einziges Y im System so, daß $B \circ X = A$ *und* $Y \circ B = A$ *ist.*

Es wird also für die Komposition nicht die kommutative Eigenschaft gefordert. ...

E 34. Ein System von Elementen bildet gegenüber einer Komposition, die den Forderungen G 1 bis G 4 genügt, eine Gruppe. ([Be], pp. 165–166)

It is strange that Beck did not draw inspiration for these axioms from Bôcher's book [Bô], which he translated in 1910. Axiom G3 in his list is redundant. In general, Beck's axiomatization of algebra is not quite successful.

Beck subsequently presented several examples of groups, derived their basic properties (existence of an identity element and inverse elements, etc.), stated Euler's theorem $a^{\varphi(n)} \equiv 1 \pmod{n}$, and discussed the consistency and independence of the axioms of a group. Then, in a similar vein, he dealt with the notion of a field (axiomatic definition, consequences, etc.). In the last chapter he constructed the domain of integers (Peano axioms, addition and subtraction, definition of integers, etc.).

4.7 Oskar Perron

The German mathematician Oskar Perron (1880–1975) studied in Munich, where he graduated in 1902 as a pupil of Ferdinand Lindemann (1852–1939). He continued his studies in Tübingen and Göttingen, and obtained habilitation at the University of Munich in 1906. In the following years, he worked in Tübingen, and since 1914 as an ordinary professor in Heidelberg. He served on the front in World War I, and worked at the University of Munich from 1922 till 1950. In the era of fascism, he bravely opposed the Nazi mathematicians Ludwig Bieberbach and Theodor Vahlen (1869–1945).[132]

He worked mainly in analysis (ordinary and partial differential equations, infinite series), integration theory (the Perron integral), foundations of geometry,

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[132] J. Heinhold: *Oskar Perron*, Jahresbericht der Deutschen Mathematiker-Vereinigung 90(1988), pp. 184–199. E. Hlawka: *Das Werk Perrons auf dem Gebiete der diophantischen Approximationen*, ibid. 80(1978), pp. 1–12. J. Heinhold: *Oskar Perron*, Jahrbuch Überblicke Mathematik 1980, pp. 121–139. E. Frank: *Oskar Perron (1880–1975)*, Journal of Number Theory 14(1982), pp. 281–291. F. Litten: *Oskar Perron. Ein Beispiel für Zivilcourage im Dritten Reich*, Mitteilungen der Deutsche Mathematiker-Vereinigung 2(1994), No. 3, pp. 11–12.

potential theory, and mathematical physics. He is the author of the extensive monograph *Die Lehre von den Kettenbrüchen*,[133] the book *Irrationalzahlen*,[134] as well as the textbook *Nichteuklidische Elementargeometrie der Ebene*.[135]

* * *

Perron's two-volume *Algebra* [Pe] was published in 1927 in the book series Göschens Lehrbücherei.[136] The first volume bears the title *Die Grundlagen*, the second one is entitled *Theorie der algebraischen Gleichungen.* The first volume has six chapters (54 paragraphs), the second one only five (44 paragraphs); the following page ranges correspond to the first or second edition, respectively:

1. *Grundbegriffe* (pp. 1–37, or 1–42, respectively).
2. *Polynomischer und Taylorscher Satz* (pp. 38–81, or 43–80, respectively).
3. *Determinanten* (pp. 82–146, or 81–140, respectively).
4. *Symmetrische Funktionen* (pp. 147–178, or 141–168, respectively).
5. *Teilbarkeit* (pp. 179–246, or 169–230, respectively).
6. *Existenz der Wurzeln* (pp. 247–304, or 231–297, respectively).[137]

1. *Numerische Auflösung von Gleichungen* (pp. 1–40, or 1–56, respectively).
2. *Gleichungen bis zum vierten Grad und reziproke Gleichungen* (pp. 41–83, or 57–95, respectively).
3. *Substitutionsgruppen* (pp. 84–138, or 96–157, respectively).
4. *Die Galoissche Gleichungstheorie* (pp. 139–205, or 158–225, respectively).
5. *Die Gleichungen fünften Grades* (pp. 206–240, or 226–257, respectively).[138]

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[133] Teubner, Leipzig, Berlin, 1913, xii + 520 pages, 2nd ed. 1929, xii + 524 pages, 3rd ed. (two vols.): Teubner, Stuttgart, 1954, 1957, vi + 193, vi + 314 pages. Reprint: Chelsea, New York, 1950.

[134] W. de Gruyter, Berlin, Leipzig, 1921, viii + 185 pages, 2nd ed. 1939, viii + 199 pages, 3rd ed. 1947, viii + 199 pages, 4th ed. 1960, viii + 202 pages. Reprint: Chelsea, New York, 1951.

[135] Teubner, Stuttgart, 1962, 134 pages.

[136] 1. Gruppe. Reine Mathematik, Band 8, 9. The second edition appeared in 1932 and 1933, third edition in 1951.

[137] A subject index and a name index follow on pages 305–307 (or 298–300, respectively), a list of 141 theorems on page 307 (*Verzeichnis der Sätze*), or 151 theorems on page 301, respectively. In the second edition, the title of the sixth chapter was expanded: *Existenz der Wurzeln von Gleichungen und Gleichungssystemen.*

[138] Pages 241–242 (or 258–260, respectively) contain a subject index and a name index, a list of 99 theorems is provided on page 243 (or 118 theorems on page 261, respectively). In the second edition, the title of the third chapter was altered: *Substitutionen und Gruppen.*

From the titles and contents of both volumes, it is obvious that the first volume presents introductory material required later on (number classes, fields, rings, properties of polynomials, symmetric functions, fundamental theorem of algebra, systems of higher-order equations, etc.), as well as some parts of linear algebra (determinants, matrices, linear systems of equations, bilinear, quadratic, and Hermitian forms, etc.). The main topic of the second volume are algebraic equations, their numeric and algebraic solution, Galois theory and its consequences.

Oskar Perron was well aware of the transformation of algebra witnessed by his own generation. In the foreword to the first volume, he wrote:

Was Algebra ist, läßt sich heute nicht so einfach definieren. Man hat in neuerer Zeit das Wort auch in die Mehrzahl gesetzt; es gibt bereits eine ganze Reihe verschiedener Algebren. Von diesen soll aber nur eine einzige in dem vorliegenden zweibändigen Werk behandelt werden, und zwar sozusagen die traditionelle Algebra, d. i. diejenige mathematische Disziplin, die man seit jeher mit diesem Namen belegt hat und deren Endziel die Theorie der algebraischen Gleichungen ist. ...

Im Mittelpunkt der modernen Algebra muß, da es sich immer um die vier Grundrechnungsarten handelt, der K ö r p e r b e g r i f f stehen, und deshalb habe ich meine Leser mit diesem Begriff, der in den meisten anderen Darstellungen viel zu spät eingeführt wird und dann fast als notwendiges Übel erscheint, von Anfang an vertraut gemacht. Allerdings die Theorie der a b s t r a k t e n Körper, die vom Standpunkt des Axiomatikers an die Spitze zu stellen wäre, wird man in dem Buch vergebens suchen. ([Pe]-I, pp. v–vi)

The beginning of the first paragraph of the first volume is somewhat odd:

Unter Algebra versteht man im wesentlichen die Theorie der rationalen Funktionen. Eine solche Funktion wird mit Hilfe der vier Grundrechnungsarten (Addition, Subtraktion, Multiplikation, Division) aus einer oder mehreren Variabeln und aus Zahlen aufgebaut. Dabei werden als Zahlen die reellen und komplexen zugelassen, wenigstens im allgemeinen und in diesem Buche ausschließlich. ([Pe]-I, p. 1)

As far as we are concerned with structural algebra, the textbook discusses the notions of a field, ring, and group. Fields and rings are introduced in the beginning of the first volume, firstly in the connection with number classes. A few pages later, the definitions are carried over to functions.

Eine Gesamtheit von mehreren Zahlen, die so beschaffen ist, daß die Summe, die Differenz, das Produkt und der Quotient von je zwei (gleichen oder verschiedenen) Zahlen der Gesamtheit stets wieder dieser Gesamtheit angehören, heißt ein **Zahlenkörper***, auch kurz* **Körper** *oder* **Rationalitätsbereich***.* ([Pe]-I, p. 17)

Eine Gesamtheit von mehreren (= mehr als eine) Zahlen, die so beschaffen ist, daß die Summe, die Differenz und das Produkt von je zwei (gleichen *oder verschiedenen) Zahlen der Gesamtheit stets wieder dieser Gesamtheit angehören, heißt ein* **Zahlenring** *oder kurz* **Ring**. ([Pe]-I, p. 21)

Auch die Begriffe „Körper“ und „Ring“ lassen sich von Zahlen auf Funktionen übertragen. Mann nennt eine Gesamtheit von mehreren rationalen Funktionen irgendwelcher Variabeln einen K ö r p e r, wenn Summe, Differenz, Produkt und Quotient von je zwei (gleichen oder verschiedenen) wieder der Gesamtheit angehören, wobei natürlich Quotienten mit dem Nenner 0 *auszuschließen sind; man nennt sie einen Ring, wenn wenigstens Summe, Differenz und Produkt wieder der Gesamtheit angehören. Man spricht hier speziell von einem F u n k t i o n e n k ö r p e r oder F u n k t i o n e n r i n g.* ([Pe]-I, p. 36)

Let us note that the definitions are constructed – besides inevitable differences – in an identical way. There is no mention of the properties of operations, since we deal with objects satisfying the usual laws of arithmetic. The properties of real and complex numbers (axioms) were already summarized on the previous pages.

Let us also remark that Oskar Perron introduced the notion of a group only for substitutions (permutations), as late as in the second volume:

Ein Komplex $\mathfrak{G}$ *heißt eine* **Gruppe**, *wenn das Produkt* $\mathfrak{G}\mathfrak{G}$ *nur Substitutionen aus* $\mathfrak{G}$ *enthält; oder ausführlicher: wenn das Produkt von zwei beliebigen (nicht notwendig verschiedenen) Substitutionen des Komplexes stets wieder dem Komplex angehört.* ([Pe]-II, p. 95)

We emphasize that Perron introduced only those group-theoretic notions and results that he immediately needed for dealing with the solvability of algebraic equations in radicals (symmetric, alternating, simple, cyclic, and Abelian groups, etc.). In the foreword to the second volume, he wrote:

Dabei habe ich aus der Gruppentheorie nur das gebracht, was für die Algebra unmittelbar gebraucht wird. Insbesondere habe ich den allgemeinen abstrakten Gruppenbegriff gänzlich vermieden und ausschließlich S u b s t i t u t i o n s g r u p p e n in den Kreis der Betrachtung gezogen. ([Pe]-II, p. v)

4.8 Otto Haupt

Otto Haupt (1887–1988) studied in Würzburg and Berlin from 1906, and obtained his doctorate from the University of Würzburg in 1911. After a year of military service, he visited the universities of Munich and Breslau (Wrocław) in the academic year 1911/12, and obtained his habilitation from the Karlsruhe Polytechnic in 1913. He was fighting in World War I in the period 1914–1918. He was appointed ordinary professor at the University of Rostock in 1920, and served as ordinary professor at the University of Erlangen since 1921 till 1953. He died at the blessed age of 101.[139]

[139] M. Barner, F. Flohr: *Otto Haupt zum 100. Geburtstag*, Jahresbericht der Deutschen Mathematiker-Vereinigung 89(1987), pp. 61–80. H. Bauer: *Otto Haupt. Zu Person und*

He worked in real analysis, measure theory, geometry, algebra, and was also interested in postgraduate learning of high-school teachers. In 1938, he published the three-volume analysis textbook entitled *Differential- und Integralrechnung*, which was written together with Georg Auman (1906–1980) as a coauthor, and appeared in the book series Göschens Lehrbücherei.[140] In 1967, he published the book *Geometrische Ordnungen*[141] written together with Hermann Künneth (1892–1975).

* * *

Haupt's two-volume algebra textbook entitled *Einführung in die Algebra* [Ha] was published in Leipzig in 1929. Its second edition appeared in 1952 and 1954, and the third edition of the first volume in 1956. The textbook consists of six parts divided into 23 chapters.[142]

1. *Grundbegriffe* (*Körper*, *Integritätsbereich und Quotientenkörper*, *Gruppen*, *Teilbarkeit*, *Restklassenringe*, *Adjunktion und Erweiterung. Ergänzungen*) (pp. 1–138).
2. *Transzendente Elemente* (*Einfache und mehrfache transzendente Erweiterung*, *Symmetrische Funktionen*, *Lineare Gleichungen*, *Teilbarkeit von Polynomen in einer Unbestimmten über einem Körper*, *Teilbarkeit von Polynomen über einem Integritätsbereich*) (pp. 139–253).
3. *Nullstellen von Polynomen* (*Existenz der Wurzeln. Eindeutigkeitssatz*, *Vielfachheit der Wurzeln und Reduzibilität*, *Resolventen eines Polynoms. Auflösung der Gleichungen 3. und 4. Grades*, *Einheitswurzeln und reine Gleichungen*, *Anhang zum Kapitel 1*) (pp. 254–325).[143]
4. *Polynome über Zahlkörpern* (pp. 369–427).
5. *Galoissche Theorie* (*Basis und Grad endlicher Erweiterungen*, *Auflösung von Gleichungen durch Radikale*, *Normalkörper*, *Gruppe eines Normalkörpers*, *Auflösung einer Gleichung gemäß der Galoisschen Theorie mit endlich vielen Schritten*, *Permutationsgruppe eines Polynoms*) (pp. 428–573).
6. *Ergänzungen zur Körpertheorie* (pp. 574–629).[144]

Werk, ibid. 92(1990), pp. 169–181. K. Jacobs: *Otto Haupt. Centenarian mathematician*, The Mathematical Intelligencer 9(1987), No. 4, pp. 50–51. H. Bauer: *Otto Haupt – Zum 100. Geburtstag*, Aequationes mathematicae 32(1987), pp. 1–18. *List of papers by Otto Haupt in chronological order*, ibid. 35(1988), pp. 125–131.

[140] W. de Gruyter, Berlin, 1938, 196 + 168 + 183 pages. A second revised edition (prepared in collaboration with Christian Y. Pauc) was published in the years 1948, 1950, and 1955. Third edition appeared in the years 1974, 1979, 1983 under the title *Einführung in die reelle Analysis*, 320 + 314 + 298 pages.

[141] Springer, Berlin, 1967, vii + 429 pages.

[142] Pages viii–ix of both volumes contain *Wichtige Abkürzungen und Zeichen*, followed by detailed tables of contents spanning 6 + 4 pages.

[143] The third edition also contains *Anhang zum 2. Kapitel* (4 pages) entitled *Boolesche Ringe*.

[144] Pages 613–629 contain the *Anhang von W. Krull*: *Galoissche Theorie der abzählbar*

In the foreword, Haupt stated that his book is aimed primarily at students, but he also had in mind teachers of higher schools. The book is also suitable for self study.

Es beschränkt sich daher auf die Elemente, unter Berücksichtigung auch der neueren Algebra. ...

Einerseits war ich bestrebt, möglichst an Bekanntes anzuknüpfen, also von Konkretem auszugehen ...

Andererseits forderten die großen Fortschritte der Algebra in den letzten Jahrzehnten zum Versuch heraus, die modernen Methoden und Ergebnisse für die Darstellung nutzbar zu machen, weil und soweit dadurch ein Gewinn an Einfachheit und zugleich Verständlichkeit zu erhoffen war. Dieser Versuch bedingt einen der Unterschiede der vorliegenden Einführung gegenüber fast allen bereits vorhandenen Lehrbüchern. [In this place, he remarked in a footnote: *Eine Ausnahme macht, soweit mir bekannt, nur H. Hasse, Höhere Algebra I und II, Leipzig 1926/27, Sammlung Göschen Nr. 931/32.*] *Demgemäß ist das vorliegende Buch durchweg beeinflußt von der bahnbrechenden „Algebraischen Theorie der Körper" von Herrn E. Steinitz, was hier ein für allemal hervorgehoben sei.*[145] ([Ha]-I, pp. v–vii)

In comparison with earlier textbooks, Haupt's book focuses more significantly on the algebra of structures, as evidenced by the brief table of contents presented above. A substantial part of Haupt's book is devoted to the study of algebraic structures, although the problems of solvability and solution of algebraic equations still remain an important topic. This is evident from the introductory sentence of the second part of the book:

Das ursprüngliche Problem der Algebra war die „Auflösung algebraischer Gleichungen". ([Ha]-I, p. 139)

The first paragraphs of the first chapter provide a detailed exposition of the notions of *equality*, *smaller* and *greater*, followed by a passage on the operations of addition and multiplication and their properties (laws of arithmetic, powers, multiples), the derived operations of subtraction and division and their properties, etc. Only then comes the notion of a field:

Die Gesamtheit der Postulate der Gleichheit, Addition, Subtraktion, Multiplikation und Division nebst dem Distributivpostulat nennen wir die **Körperpostulate**.

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unendlichen Normalkörper, Verallgemeinerte Abelsche Gruppen, Elementarteilertheorie der Matrizen.
In the second edition, the original part six was appended to part five as chapter 23 with the altered title *Weitergehende Untersuchungen zur Körpertheorie.*

[145] E. Steinitz: *Algebraische Theorie der Körper*, Journal für die reine und angewandte Mathematik 137(1910), pp. 167–309; book edition: *Algebraische Theorie der Körper. Neu herausgegeben, mit Erläuterungen und einem Anhang: Abriß der Galoisschen Theorie versehen von Reinhold Baer und Helmut Hasse*, W. de Gruyter, Berlin, Leipzig, 1930, 150 + 27 pages; later ed. Chelsea, New York, 1950, 176 pages.

Ein System von Elementen, dessen Elemente sich gemäß den Körperpostulaten verknüpfen lassen und das nicht nur aus dem einzigen Element Null besteht, heißt ein **Körper** (*auch „Rationalitätsbereich"*). ([Ha]-I, p. 27)

The notions of a ring and integral domain appear in the second chapter. The topic of rings becomes a matter of interest in several chapters, for example, field extensions are studied in the sixth part of the book. Haupt's textbook also contains material that is now part of linear algebra (matrices, linear systems of equations, bilinear forms, etc.).

Haupt provided an axiomatic definition of an abstract group in the first part of the book, namely in the beginning of the third chapter. Surprisingly, one of his axioms of a group reads as follows:

Aus $R = T$ *folgt* $R \circ S = T \circ S$ *und* $S \circ R = S \circ T$. ([Ha]-I, p. 57)

The notion of an ideal appeared in the textbook in connection with the divisibility of integers:

Unter einem **Ideal** *von ganzen rationalen Zahlen versteht man eine Gesamtheit von ganzen Zahlen* $b_1, b_2, \ldots$ *von folgender Eigenschaft: Mit* 2 *ganzen Zahlen* c *und* d *gehört auch deren Differenz* $c - d$ *zum Ideal; mit einer Zahl* d *gehört auch jedes Vielfache* kd *zum Ideal* (k *beliebige ganze Zahl*). *Es besteht nun folgende Zuordnung zwischen den ganzen Zahlen und den Idealen:*

Satz: *Jedes Ideal von ganzen rationalen Zahlen ist ein* **Hauptideal**, *d. h. identisch mit der Gesamtheit der Vielfachen einer ganzen rationalen Zahl* t ... ([Ha]-I, p. 101)

A major advantage of Haupt's textbook is a collection of problems and exercises; their solutions are printed in the end of both volumes (pages 331–367, 630–648), while page 649 contains an alternative proof of the fundamental theorem of algebra. A detailed index for both volumes is on pages 650–663. The book also contains historical notes and extensive bibliographical information.

5 Conclusion

The entire second half of the 19th century was dominated by Serret's textbook, whose size grew from four hundred pages (1849) to more than 1300 pages (1866). It was translated to German (1868) and Russian (1897). It was used until the end of the 1920s, perhaps even longer.

Jordan's monograph is not a textbook in a proper sense. However, it helped to spread the knowledge of groups and their applications in algebra, and popularized the Galois theory.

Matthiesen's monograph brought an overview of all possible methods of classical algebra.

Despite its title, there is no doubt that Klempt's imbalanced book (1880) is not a textbook of modern algebra.

Although Weber's two-volume textbook (1895, 1896) is largely devoted to the classical topic of algebra, i.e., algebraic equations, it also brought new topics (groups and their applications, fields and their significance). It was used until the end of the 1920s.

Bauer's 1903 textbook is not as extensive as Weber's one. Its primary goal was to serve for introductory courses of mathematics at the university level. Its 1928 revision by Bieberbach, an extraordinarily able mathematician who was two generations younger than Bauer, already reflected the recent development of algebra (fields, rings, groups) in a significant way. It was used until the 1940s.

Fricke's starting point was Weber's textbook, which he tried to modernize.

Bôcher's textbook (1907) was primarily focused on the emerging linear algebra. It represented a major step toward the modern axiomatic conception.

The most significant progress toward the algebra of structures was due to Haupt.

K.-H. Schlote underlined the qualitative shift from the textbooks of Dickson, Hasse, Perron and Haupt to the monograph *Moderne Algebra* [Wa] written by Bartel Leendert van der Waerden by these words:

But none of these authors succeded in describing the substance of the new conception of algebra as clearly as van der Waerden, or in interweaving it so elegantly with the body of classical results. In all their efforts to describe clearly and systematically the progress in algebra, they maintained a more or less strong commitment to the classical theory, and this hampered their exposition of the material. We note in passing that the first textbook in English to present algebra in the style of van der Waerden was A survey of modern algebra by Birkhoff and Saunders Maclane (*b. 1909*)*, which was published in 1941.* ([Sl], p. 913)

See also [Co2] and [Co3].

* * *

In 1939, the Czech universities were closed by the German occupants. In consequence, three professors of the Charles University, mathematicians Bohumil Bydžovský (1880–1969), Vojtěch Jarník (1897–1970), and Vladimír Kořínek (1899–1981), compiled brief self-study guidelines for geometry, analysis and algebra, and published them in the Časopis pro pěstování matematiky a fysiky [Journal for the cultivation of mathematics and physics]. In his paper *Návod ke studiu algebry pro začátečníky* [A Beginner's Guide to the Study of Algebra] [Kv], Kořínek outlined a study program that more or less corresponds to the structure of his later textbook *Základy algebry* [Elements of Algebra] published in 1953.[146] First of all, he recommended to study from Bieberbach's and Bauer's book [Ba] (fourth or fifth edition), and Perron's two-volume textbook *Algebra* [Pe]; he also mentioned the advantages and disadvantages of both textbooks. Among other things, he wrote:

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[146] Nakladatelství ČSAV, Praha, 1953, 488 pages, 2nd ed. 1956, 520 pages.

Our program bears closer relation the book of Bieberbach ... In particular, the fifth edition of Bieberbach's Vorlesungen is a modernly-written book ...

Perron's Algebra is not as modernly written, the algebraic viewpoint is not as much stressed, the proofs are often unnecessarily complicated. Perron prefers proofs by calculation, sometimes very complicated, instead of logical reasoning. On the other hand, everything is carried out very meticulously up to the smallest detail, thus the reader is never left puzzled. ([Kv], p. D26)

Vladimír Kořínek also referred to the first volume of Weber's textbook [W], first volume of Fricke's textbook [Fr], and first volume of the book *Einführung in die analytische Geometrie und Algebra*, which was published by Otto Schreier and Emanuel Sperner (1905–1980) in 1931.[147] He also commented on the suitability of additional textbooks for the self-study of algebra:

Textbooks of modern abstract algebra, such as the books of Haupt, Hasse or van der Waerden, are ill-suited for our purposes. The goal of these books is an abstract, purely algebraic development of theory, only a small part of which belongs to our program, and on the other hand leaves aside many things which from the viewpoint of abstract algebra belong to the theory of polynomials as functions, e.g., the fundamental theorem of algebra, estimates for the absolute values of roots in terms of coefficients, separation of roots, numerical solution of equations, trigonometric solutions of certain equations, and related topics. But for a beginner, precisely these things are important. On one hand, a beginner needs to learn practical calculations, and on the other hand to become familiar with the properties of the field of complex numbers as well as polynomials over this field because, if we disregard a detailed study of algebra itself, these are the most necessary things for further mathematical studies.[148] ([Kv], p. D27)

6 Algebra textbooks from the middle of the 19th century until 1930s

1849 Serret: *Cours d'algèbre*, 1st ed.

1854 Serret: *Cours d'algèbre*, 2nd ed.

1866 Serret: *Cours d'algèbre I., II.*, 3rd ed.

1868 Serret: *Handbuch der höheren Algebra I., II.*, 1st ed.

1870 Jordan: *Traité des substitutions*

1877 Serret: *Cours d'algèbre I.*, 4th ed.

1878 Matthiessen: *Grundzüge der antiken und modernen Algebra*, 1st ed.

[147] Teubner, Leipzig, Berlin, Vol. I: 1931, 238 pages, Vol. II: 1935, 308 pages.

[148] See Z. Kohoutová, J. Bečvář: *Vladimír Kořínek* (*1899–1981*), book series Dějiny matematiky [History of mathematics], vol. 27, Výzkumné centrum pro dějiny vědy, Praha, 2005. A short treatise on Kořínek's *Návod ke studiu algebry pro začátečníky* is on pages 139–142. Kořínek's textbook *Základy algebry* is discussed on pages 101–128.

1878 Serret: *Handbuch der höheren Algebra I.*, 2nd ed.
1879 Serret: *Handbuch der höheren Algebra II.*, 2nd ed.
1879 Serret: *Cours d'algèbre II.*, 4th ed.
1880 Klempt: *Lehrbuch zur Einführung in die moderne Algebra*
1885 Serret: *Cours d'algèbre I., II.*, 5th ed.
1895 Weber: *Lehrbuch der Algebra I.*, 1st ed.
1896 Weber: *Lehrbuch der Algebra II.*, 1st ed.
1896 Matthiessen: *Grundzüge der antiken und modernen Algebra*, 2nd ed.
1897 Serret: *Kurs vyššej algebry*, 1st ed.
1898 Weber: *Traité d'algèbre supérieure*,
1898 Weber: *Lehrbuch der Algebra I.*, 2nd ed.
1899 Weber: *Lehrbuch der Algebra II.*, 2nd ed.
1903 Bauer: *Vorlesungen über Algebra*, 1st ed.
1907 Bôcher: *Introduction to Higher Algebra*, 1st ed.
1908 Weber: *Lehrbuch der Algebra III.*, 2nd ed.
1909 Serret: *Cours d'algèbre I., II.*, 6th ed.
1910 Serret: *Kurs vyššej algebry*, 2nd ed.
1910 Bauer: *Vorlesungen über Algebra*, 2nd ed.
1910 Bôcher: *Einführung in die höhere Algebra*, 1st ed.
1912 Weber: *Lehrbuch der Algebra. Kleine Ausgabe*, 1st ed.
1921 Bauer: *Vorlesungen über Algebra*, 3rd ed.
1921 Weber: *Lehrbuch der Algebra. Kleine Ausgabe*, 2nd ed.
1922 Bôcher: *Introduction to Higher Algebra*, 2nd ed.
1924 Bôcher: *Introduction to Higher Algebra*, reprint
1924 Fricke: *Lehrbuch der Algebra I.*
1925 Bôcher: *Einführung in die höhere Algebra*, 2nd ed.
1926 Fricke: *Lehrbuch der Algebra II.*
1926 Beck: *Einführung in die Axiomatik der Algebra*
1926 Dickson: *Modern Algebraic Theories*, 1st ed.
1926 Hasse: *Höhere Algebra I.*, 1st ed.
1927 Hasse: *Höhere Algebra II.*, 1st ed.
1927 Perron: *Algebra I., II.*, 1st ed.
1928 Fricke: *Lehrbuch der Algebra III.*
1928 Bauer, Bieberbach: *Vorlesungen über Algebra*, 4th ed.
1928 Serret: *Cours d'algèbre I., II.*, 7th ed.
1929 Haupt: *Einführung in die Algebra I., II.*, 1st ed.
1929 Dickson: *Höhere Algebra*, 1st ed.
1930 **van der Waerden:** *Moderne Algebra I.*, 1st ed.
1931 **van der Waerden:** *Moderne Algebra I.*, 1st ed.

1932 Bôcher: *Einführung in die höhere Algebra*, reprint of 2nd ed.
1932 Perron: *Algebra I.*, 2nd ed.
1933 Perron: *Algebra II.*, 2nd ed.
1933 Bauer, Bieberbach: *Vorlesungen über Algebra*, 5th ed.
1933 Bôcher: *Vvedenie v vysšuju algebru*
1933 Bôcher: *Introduction to Higher Algebra*, reprint
1933 Hasse: *Höhere Algebra I.*, 2nd ed.
1934 Hasse: *Aufgabensammlung*, 1st ed.
1937 Hasse: *Höhere Algebra II.*, 2nd ed.

REFERENCES

[Al] Alten H.-W., Djafari Naini A., Folkerts M., Schlosser H., Schlote K.-H., Wußing H., *4000 Jahre Algebra*, Springer-Verlag, Berlin, Heidelberg, 2003, 2. korrigierter Nachdruck 2008, xiv + 655 pages.

[Ba] Bauer G., *Vorlesungen über Algebra*, B.G. Teubner, Leipzig, 1903, vi + 376 pages, 2nd ed. 1910, vi + 366 pages, 3rd ed. 1921, v + 366 pages, 4th ed. (L. Bieberbach): B.G. Teubner, Leipzig und Berlin, 1928, x + 334 pages, 5th ed. 1933, x + 358 pages. Review: The Mathematical Gazette 2(1904), No. 43, p. 391, 3(1904), No. 44, p. 21, 14(1929), No. 201, pp. 469–470, 18(1934), No. 228, p. 114.

[Be] Beck H., *Einführung in die Axiomatik der Algebra*, Göschens Lehrbücherei 6, W. de Gruyter & Co., Berlin, Leipzig, 1926, x + 197 pages. Review: The American Mathematical Monthly 35(1928), pp. 89–91.

[Bô] Bôcher M., *Introduction to higher algebra*, Prepared for publication with the cooperation of E.P.R. Duval, The Macmillan Company, New York, London, 1907, xi + 321 pages; 2nd ed. 1922, later editions: 1924, 1933, 1938, 1949, reprint: Dover, New York, 1964, 2004, xi + 321 pages. German translation (H. Beck): *Einführung in die höhere Algebra*, B.G. Teubner, Leipzig und Berlin, 1910, xii + 348 pages, 2nd ed. 1925, later print: 1932, …, 1952. Russian translation: *Vvedenie v vysšuju algebru*, GTTI, Moskva, Leningrad, 1933, 291 pages, 2nd ed. LKI, Moskva, 2007. Review: The Journal of Education 67(1908), No. 9(1669), p. 245, Science. New Series 27(1908), No. 692, pp. 535–536, The Mathematical Gazette 4(1908), No. 72, pp. 291–293, 50(1966), No. 371, p. 82, Monatshefte für Mathematik und Physik 20(1909), pp. A57–A58, 22(1911), pp. A31–A32, The American Mathematical Monthly 15(1908), pp. 23–24, 91(1984), pp. 147–150.

[Co1] Corry L., *Modern Algebra and the Rise of Mathematical Structures*, Science Networks – Historical Studies, Vol. 17, Birkhäuser Verlag, Basel, Boston, Berlin, 1996, 460 pages, 2nd ed. 2004, xv + 451 pages, Review: The Bulletin of Symbolic Logic 13(2007), No. 1, pp. 102–104.

[Co2] Corry L., *Heinrich Weber, Lehrbuch der Algebra* (*1895–1896*), pp. 690–699, in I. Grattan-Guinness (ed.): *Landmark Writings in Western Mathematics 1640–1940*, Elsevier, Amsterdam, Boston, Heidelberg, London, New York, Oxford, Paris, San Diego, San Francisco, Singapore, Sydney, Tokyo, 2005, xvii + 1022 pages.

[Co3] Corry L., *From Algebra (1895) to Modern Algebra (1930): Changing conception of a discipline – a guided tours using the Jahrbuch über die Fortschritte der Mathematik*, in [GP], pp. 221–243.

[Di] Dickson L.E., *Modern Algebraic Theories*, B.H. Sanborn & Co., Chicago, New York, Boston, 1926, ix + 276 pages, reprint: *Algebraic Theories*, Dover Publications, Inc., New York, 1959. German translation: *Höhere Algebra*, Autorisierte deutsche Ausgabe von L.E. Dickson "Modern algebraic theories" herausgegeben von Ewald Bodewig, B.G. Teubner, Leipzig und Berlin, 1929, vii + 242 pages. Review: The American Mathematical Monthly 34(1927), pp. 532–535, The Mathematical Gazette 13(1927), No. 186, pp. 288–289.

[Fr] Fricke R., *Lehrbuch der Algebra verfaßt mit Benutzung von Heinrich Webers gleichnamigem Buche I., II., III.*, F. Vieweg & Sohn, Braunschweig, 1924, 1926, 1928, viii + 468, viii + 418, vii + 506 pages. Review: Bulletin of the American Mathematical Society 31(1925), No. 7, p. 372–373, 34(1928), No. 4, p. 531, 36(1930), No. 1, p. 31.

[Gi] Gillispie Ch.C. (ed.), *Dictionary of Scientific Biography*, Charles Scribner's Sons, New York, 1970–1990.

[Gy] Gray J.J., *A History of Abstract Algebra. From Algebraic Equations to Modern Algebra*, Springer Undergraduate Mathematics Series, Springer Nature Switzerland, 2018, xxiv + 415 pages.

[GP] Gray J.J, Parshall K.H. (ed.), *Episodes in the History of Modern Algebra (1800–1950)*, History of Mathematics, Vol. 32, American Mathematical Society, London Mathematical Society, 2007, viii + 335 pages.

[H] Hasse H., *Höhere Algebra I. Lineare Gleichungen, II. Gleichungen höheren Grades*, Sammlung Göschen 931, 932, W. de Gruyter & Co., Berlin und Leipzig, 1926, 1927, 160 + 160 pages, 2nd ed. 1933, 1937, 158 + 158 pages, 3rd ed. 1951, 1952, 152 + 158 pages, 4th ed. 1957, 1958, 5th ed. 1963, 1967, 150 + 158 pages, 6th ed. 1969. English translation (T.J. Benac): *Higher Algebra, I. Linear Equations, II. Equations of Higher Degree*, F. Ungar Publishing Co., New York, 1954, 336 pages. Review: The Mathematical Gazette 48(1964), No. 366, p. 468.

[H-A] Hasse H., *Aufgabensammlung zur höheren Algebra*, Sammlung Göschen 1082, W. de Gruyter, Berlin, 1934, 175 pages; later editions with W. Klobe: 2nd ed. 1952, 181 pages, 3rd ed. 1961, 183 pages. English translation (T.J. Benac): *Exercises to Higher Algebra*, F. Ungar Publishing Co., New York, 1954, 212 pages. Review: Zeitschrift für angewandte Mathematik und Mechanik 17(1937), p. 376, The Mathematical Gazette 36(1952), No. 318, pp. 295–296.

[Ha] Haupt O., *Einführung in die Algebra I., II.*, Mathematik und ihre Anwendungen in Monographien und Lehrbüchern 5, Akademische Verlagsgesellschaft M. B. H., Leipzig, 1929, xv + xiii + 663 pages, 2nd ed. Akademische Verlagsgesellschaft, Geest & Portig K.-G., 1952, 1954, viii + xvi + 673 pages, 3rd ed. 1956 (only 1st vol.), xviii + 370 pages.

[Jo] Jordan C., *Traité des substitutions et des équations algébriques*, Gauthier-Villars, Paris, 1870, xviii + 667 pages, reprint: 1957.

[KP] Katz V.J., Parshall K.H., *Taming the Unknown. A History of Algebra from Antiquity to Early Twentieth Century*, Princeton University Press, Princeton and Oxford, 2014, xiii + 485 pages.

[K] Kleiner I., *A History of Abstract Algebra*, Birkhäuser, Boston, Basel, Berlin, 2007, xv + 168 pages.

[Kl] Klempt D. A., *Lehrbuch zur Einführung in die moderne Algebra. Mit einigen hundert Beispielen*, B.G. Teubner, Leipzig, 1880, xii + 260 pages.

[Ko] Koreuber M., *Emmy Noether, die Noether-Schule und die moderne Algebra. Zur Geschichte einer kulturellen Bewegung*, Mathematik im Kontext, Springer Spektrum, Springer-Verlag, Berlin, Heidelberg, 2015, xv + 368 pages.

[Kv] Kořínek V., *Návod ke studiu algebry pro začátečníky*, Časopis pro pěstování matematiky a fysiky **70** (1941–42), pp. D25–D39 (Czech).

[Ma] Matthiessen L., *Grundzüge der antiken und modernen Algebra der litteralen Gleichungen*, B.G. Teubner, Leipzig, 1878, xvi + 1002 pages, 2nd ed. 1896.

[No] Nový L., *Origins of modern Algebra*, Academia, Prague, 1973, viii + 252 pages.

[Pe] Perron O., *Algebra I. Die Grundlagen, II. Theorie der algebraischen Gleichungen*, Göschens Lehrbücherei, I. Gruppe. Reine Mathematik 8, 9, W. de Gruyter & Co., Berlin, Leipzig, 1927, viii + 307, viii + 243 pages, 2nd ed. 1932, 1933, viii + 301, viii + 261 pages, 3rd ed. 1951, reprint: W. de Gruyter, Berlin, 2011.

[Sl] Schlote K.-H., *B.L. van der Waerden, Moderne Algebra, First Edition* (*1930–1931*), pp. 901–916, in I. Grattan-Guinness (ed.): *Landmark Writings in Western Mathematics 1640–1940*, Elsevier, Amsterdam, Boston, Heidelberg, London, New York, Oxford, Paris, San Diego, San Francisco, Singapore, Sydney, Tokyo, 2005, xvii + 1022 pages.

[Se] Serret J.-A., *Cours d'algèbre supérieure*, Bachelier, Paris, 1849, ix + 400 pages, 2nd ed. Mallet-Bachelier, Paris, 1854, xvi + 600 pages, other editions in two volumes: 3rd ed. Gauthier-Villars, Paris, 1866, xv + 643, xii + 664 pages, 4th ed. 1877, 1879, xiii + 647, xii + 694 pages, reprint: J. Gabay, Sceaux, 1992, 5th ed. Gauthier-Villars, 1885, xiii + 647, xii + 694 pages, 6th ed. 1909, 7th ed. 1928. German translation (G. Wertheim): *Handbuch der höheren Algebra*, B.G. Teubner, Leipzig, 1868, viii + 508, viii + 540 pages, 2nd ed. 1878, 1879, viii + 528, viii + 574 pages. Russian translation: *Kurs vysšej algebry*, Sankt-Petersburg, 1897, 605 pages, 2nd ed. 1910, 573 pages.

[Wa] van der Waerden B.L., *Moderne Algebra I., II.*, Springer-Verlag, Berlin, 1930, 1931, viii + 243, vii + 216 pages.

[Wa5] van der Waerden B.L., *A History of Algebra. From al-Khwārizmī to Emmy Noether*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985, pp. xii + 271 pages.

[We] Weber H., *Die allgemeinen Grundlagen der Galois'schen Gleichungstheorie*, Mathematische Annalen **43** (1893), 521–549.

[W] Weber H., *Lehrbuch der Algebra I., II.*, F. Vieweg und Sohn, Braunschweig, 1895, 1896, xv + 654, xiv + 796 pages, 2nd ed. 1898, 1899, xvi + 704, xvi + 856 pages, 3rd ed. Chelsea Publishing C., New York, 1961, reprint: AMS Chelsea, Providence, RI, 2002. French translation (1st Vol., 2nd ed., J. Griess): *Traité d'algèbre supérieure. Principes, racines des équations, grandeurs algébriques, théorie de Galois*, Gauthier-Villars, 1898, xi + 764 pages. Review: Bulletin of the American Mathematical Society 3(1897), No. 2, pp. 174–195, 4(1898), pp. 200–234, 5(1899), pp. 480–482, Science. New Series 38(1913), No. 981, pp. 550–551.

[Wel] Weber H., *Elliptische Functionen und algebraische Zahlen*, F. Vieweg und Sohn, Braunschweig, 1891, xiv + 504 pages, 2nd ed. *Lehrbuch der Algebra III. Elliptische Funktionen und algebraische Zahlen*, F. Vieweg und Sohn, Braunschweig, 1908, xvi + 733 pages, 3rd ed. Chelsea Publishing Co., New York, 1961, reprint: AMS Chelsea, Providence, 2001.

[Wkl] Weber H., *Lehrbuch der Algebra. Kleine Ausgabe in einem Bande*, F. Vieweg, Braunschweig, 1912, x + 528 pages, 2nd ed. 1921, later print: 1928. Review: Science. New Series 38(1913), No. 981, pp. 550–551.

[WW] Weber H., Wellstein J. et all., *Encyklopädie der Elementar-Mathematik. Ein Handbuch für Lehrer und Studierende*, B.G. Teubner, Leipzig, 1903–1907, I. *Elementare Algebra und Analysis*, 1903, xiv + 447 pages, 2nd ed. 1906, xvii + 539 pages, 3rd ed. 1909, xviii + 531 pages, 4th ed. 1922, xvi + 568 5th ed. 1934; II. *Elemente der Geometrie*, 1905, xii + 604 pages, 2nd ed. 1907, xii + 596 pages, 3rd ed. 1915, xiv + 594 pages; III. *Angewandte Elementar-Mathematik*, 1907, xiii + 666 pages, 2nd ed. III.1 *Mathematische Physik*, 1910, xii + 536 pages, 3rd ed. 1923; x + 536 pages, 2nd ed. III.2. *Darstellende Geometrie, graphische Statik, Warscheinlichkeitsrechnung, politische Arithmetik und Astronomie*, 1912, xiv + 671 pages, 3rd ed. 1924; x + 671 pages, reprint: University of Michigan, An Arbor, Michigan, 2005. Russian translation (V.F. Kagan): 1906–1910, 1911, 1913, 1914, 1927 Review: The Mathematical Gazette 4(1907), No. 67, p. 163, 19(233)(1935), pp. 159–160, Bulletin of the American Mathematical Society 10(1904), pp. 200–204, 14(1908), pp. 499–501, 17(1911), p. 546, 19(1912), pp. 87–88, 19(1913), pp. 422–423, 29(1923), p. 480–481.