Point Sets

Chapter V: Local connectedness

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Chapter V

LOCAL CONNECTEDNESS

§ 22. General theorems concerning local connectedness

22.1. Let P be a metric space. Let $a \in P$. We say that P is *locally connected at the point* a, if, for every neighborhood U of a, a is an interior point (see 8.6) of that component (see 18.2.1) K of U which contains a.

We say that P is *locally connected* (without further determination), if it is locally connected at every point $a \in P$.

Local connectedness is a topological property (see 9.3).

22.1.1. *Let $a \in P$. P is locally connected at a if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $x \in P$ with $\varrho(a, x) < \delta$ there is a connected $S \subset P$ with $a \in S$, $x \in S$, $d(S) < \varepsilon$.*

Proof: I. Let the condition be satisfied. Let U be a given neighborhood of a. There is an $\varepsilon > 0$ such that $\Omega(a, \varepsilon) \subset U$. Choose an appropriate $\delta > 0$. Let K be the component of U containing a. If $\varrho(a, x) < \delta$ there is a connected $S \subset P$ with $a \in S$, $x \in S$, $d(S) < \varepsilon$. Since $a \in S$, $d(S) < \varepsilon$, we have $S \subset U$, so that (see 18.2.5) S is contained in a component of U. As $a \in K \cap S$, we have $S \subset K$, so that $x \in K$. Thus, $\varrho(a, x) < \delta$ implies $x \in K$, i.e. $\Omega(a, \delta) \subset K$, so that a is an interior point of K.

II. Let P be locally connected at a point a. Let $\varepsilon > 0$. Then $\Omega(a, \frac{1}{3}\varepsilon)$ is a neighborhood of a. If K is the component of $\Omega(a, \frac{1}{3}\varepsilon)$ containing a, then a is an interior point of K, i.e. there is a $\delta > 0$ such that $\varrho(a, x) < \delta$ implies $x \in K$. On the other hand, the set K is connected and evidently $K \subset \Omega(a, \frac{1}{3}\varepsilon)$ implies $d(K) < \varepsilon$.

The following theorems are evident:

22.1.2. *Let P be locally connected at a point a. Let a be an interior point of $Q \subset P$. Then Q is locally connected at the point a.*

22.1.3. *Let P be locally connected. Then every open $Q \subset P$ is locally connected.*

22.1.4. *P is locally connected if and only if the components of open sets are open sets.*

Proof: I. Let the condition be satisfied. Let U be a neighborhood of a point $a \in P$. Let K be the component of U, containing a. As U is open, K is also open. Thus, a is an interior point of K, and hence P is locally connected at a.

II. Let P be locally connected and let K be a component of an open set G. For every $a \in K$, G is a neighborhood of a and K is the component of G containing a. Thus, every $a \in K$ is an interior point of K, i.e., K is open (see 8.6.1).

22.1.5. *In locally connected spaces the quasicomponents are identical with the components.*

Proof: Let $a \in P$. Let K be the component (see 18.2.1) and Q the quasicomponent (see18.3.4.) containing a. We have to prove that $K = Q$. By 18.3.9, $K \subset Q$. We have to prove that $Q \subset K$. By 18.3.5 it suffices to prove that $P = K \cup (P - K)$ with separated summands. By 18.2.2. K is closed. By 22.1.4 also $P - K$ is closed. Thus, K and $P - K$ are separated.

22.1.6. *All components of a locally connected space are locally connected.*

This follows by 22.1.3 and 22.1.4.

22.1.7. *Let P and Q be locally connected spaces. Then $P \times Q$ is a locally connected space.*

Proof: Let us take a point $(a, b) \in P \times Q$ and a number $\varepsilon > 0$. Since P and Q are locally connected, by 22.1.1 there is a $\delta > 0$ such that: [1] if $x \in P$, $\varrho(a, x) < \delta$, there is a connected $S_1 \subset Q$ with $a \in S_1$, $x \in S_1$, $d(S_1) < \frac{1}{2}\varepsilon$, [2] if $y \in Q$, $\varrho(b, y) < \delta$, there is a connected $S_2 \subset Q$ with $b \in S_2$, $y \in S_2$, $d(S_2) < \frac{1}{2}\varepsilon$. If $(x, y) \in P \times Q$, $\varrho[(a, b), (x, y)] < \delta$, we have $\varrho(a, b) < \delta$, $\varrho(b, y) < \delta$, so that there exist sets S_1, S_2 satisfying the conditions above. We have then $(a, b) \in S_1 \times S_2$, $(x, y) \in S_1 \times S_2$, $d(S_1 \times S_2) < \varepsilon$ and $S_1 \times S_2$ is connected by 18.1.13. Thus, $P \times Q$ is locally connected by 22.1.1.

22.1.8. *The euclidean space $\mathbf{E}_m$ $(m = 1, 2, 3, \ldots)$ is locally connected.*

Proof: It follows easily by 22.1.1 and 19.2.2 that $\mathbf{E}_1$ is locally connected. As $\mathbf{E}_{m+1} = \mathbf{E}_m \times \mathbf{E}_1$, we learn from 22.1.7 by induction that every $\mathbf{E}_m$ is locally connected.

22.1.9. *Let P be a locally connected space. Let $G \subset P$ be an open set; let $K \subset G$ be a connected set. K is a component of G if and only if $B(K) \subset P - G$.*

Proof: I. Let the condition be satisfied. By 10.5.1, $B_G(K) = \emptyset$, so that, for every $x \in G$, we have either $\varrho(x, K) > 0$ or $\varrho(x, G - K) > 0$. Thus (see 10.2.3), $G = K \cup (G - K)$ with separated summands. If $H \supset K$, $H \subset G$ and if H is connected, then $H = K$ by 18.1.2. Thus, K is a component of G.

II. Let K be a component of G. Then K is closed in G by 18.2.2 and K is open in P by 22.1.4. Thus, $G \cap \overline{K} = K$ (see 8.7.1) and $\overline{P - K} = P - K$, so that $B(K) = \overline{K} \cap \overline{P - K} = \overline{K} - K \subset P - G$.

22.1.10. *Let P be a locally connected space. Let $a \in P$, $b \in P$, $Q \subset P$. Q is an irreducible cut of P between the points a, b, if and only if there exist two distinct connected sets G_1, G_2 such that $a \in G_1$, $b \in G_2$, $G_1 \cup G_2 \subset P - Q$, $B(G_1) = B(G_2) = Q$.*

Proof: I. Let the condition be satisfied. Q is closed by 10.3.1. Thus, by 22.1.9, G_1 and G_2 are components of $P - Q$, so that $G_1 \cap G_2 = \emptyset$. G_1 is closed in $P - Q$ by 18.2.2 and open in $P - Q$ by 22.1.4 (see also 8.7.6), so that $P - Q = G_1 \cup \cup [(P - Q) - G_1]$ with separated summands. We have $a \in G_1$, $b \in G_2 \subset (P - Q) - - G_1$. Thus, Q separates a from b in P. Let $R \subset Q \neq R$. We have to prove that R does not separate a from b in P. We have $Q - R \subset Q = B(G_1) \subset \overline{G}_1$, so that $G_1 \cup (Q - R)$ is connected by 18.1.7. Similarly, $G_2 \cup (Q - R)$ is also connected. Since $Q - R \neq \emptyset$, by 18.1.4 also $S = G_1 \cup G_2 \cup (Q - R)$ is connected. We have $(a) \cup (b) \subset S \subset (P - Q) \cup (Q - R) = P - R$. If $P - R = A \cup B$ with separated summands, $a \in A$, we have, by 18.1.2, $S \subset A$ and hence $b \in A$ so that R does not separate a from b in P.

II. Let Q be an irreducible cut of P between the points a, b. Then Q is a closed set by 18.5.4. Q separates a from b in P, so that $(a) \cup (b) \subset P - Q$ and the space $P - Q$ is not connected between a and b. Thus, by 18.3.3, $G_1 \neq G_2$, if G_1, G_2 are components of $P - Q$ such that $a \in G_1$, $b \in G_2$. By 22.1.9, $B(G_1) \subset Q$ and hence $a \in G_1 - B(G_1)$. By 8.7.1 and 18.2.2 $\overline{G}_1 - Q = G_1$ so that $b \in P - \overline{G}_1$. Thus, by 18.5.2, the set $B(G_1) \subset Q$ separates a from b in P, so that $B(G_1) = Q$. Similarly, $B(G_2) = Q$.

22.1.11. *Let P be a locally connected space. Let $G \subset P$ be an open connected set. Let $a \in G$, $b \in P - \overline{G} \subset P - B(G)$. Let Γ be the component of $P - B(G)$ containing b. Then $B(\Gamma)$ is an irreducible cut of P between the points a, b.*

Proof: $B(G)$ and $B(\Gamma)$ are closed by 10.3.1. As Γ is a component of $P - B(G)$, we have, by 22.1.9, $B(\Gamma) \subset B(G)$. Γ is open by 22.1.4, so that $B(\Gamma) = \overline{\Gamma} - \Gamma$ by 10.3.2. Similarly, $B(G) = \overline{G} - G$. Thus, $a \in G \subset P - B(G) \subset P - B(\Gamma)$. Let Δ be the component of $P - B(\Gamma)$ containing a. As $a \in G \subset P - B(\Gamma)$, we have $G \subset \Delta$ by 18.2.5. If $\Delta = \Gamma$, we have $G \subset \Gamma$, so that the connected set Γ contains the point $a \in G$ and the point $b \in P - \overline{G} \subset P - G$, so that, by 18.1.8, $\emptyset \neq \Gamma \cap B(G)$, which is a contradiction. Thus, $\Gamma \neq \Delta$. As $G \subset \Delta$, we have $B(\Gamma) \subset B(G) \subset \overline{G} \subset \overline{\Delta}$. On the other hand, Δ is open by 22.1.4, so that $B(\Delta) = \overline{\Delta} - \Delta$ by 10.3.2. As $\Delta \subset \subset P - B(\Gamma)$, $B(\Gamma) \subset \overline{\Delta}$, we have $B(\Gamma) \subset \overline{\Delta} - \Delta = B(\Delta)$. On the other hand, by 22.1.9, we have $B(\Delta) \subset B(\Gamma)$. Thus, Γ and Δ are distinct connected sets such that $a \in \Delta$, $b \in \Gamma$, $\Gamma \cup \Delta \subset P - B(\Gamma)$, $B(\Gamma) = B(\Delta)$. Thus, by 22.1.10, $B(\Gamma)$ is an irreducible cut of P between the points a and b.

22.1.12. *Let P be a locally connected space. Let $Q \subset P$ separate a point a from a point b in P. Then there is an irreducible cut M of P between the points a, b such that $M \subset Q$.*

Proof: By 18.5.1 there is a closed set $F \subset Q$ which separates a from b in P. Let G be the component of $P - F$ containing a. The set G is open by 22.1.4. By 18.3.3 G does not contain b, so that $b \in (P - F) - G$. By 8.7.1 and 18.2.2 $\bar{G} - F = G$ so that $b \in P - \bar{G} \subset P - B(G)$. Let Γ be the component of $P - B(G)$ containing b. By 22.1.11 the set $M = B(\Gamma)$ is an irreducible cut of P between the points a, b. In proving theorem 22.1.11 we noted that $B(\Gamma) \subset \bar{G} - G$. Since $\bar{G} - F = G$ we obtain $M \subset F \subset Q$.

22.1.13. *Let P be a connected and locally connected space. Let $C \subset P$ be a closed connected set. Let K be a component of $P - C$. Then $P - K$ is connected.*

Proof: K is open by 22.1.4. $C \cup \bar{K}$ and $P - K$ are closed. The set

$$(C \cup \bar{K}) \cup (P - K) = P$$

is connected. By 8.7.1 and 18.2.2 we have $K = \bar{K} - C$, so that the set

$$(C \cup \bar{K}) \cap (P - K) = C \cup (\bar{K} - K) = C$$

is also connected. Thus, $P - K$ is connected by 18.1.12.

22.1.14. *The spherical space $\mathbf{S}_m$ $(m = 1, 2, 3, \ldots)$ is locally connected.*

This follows easily by 17.10.4 and 22.1.8.

22.1.15. *Let P be a locally connected space. Let K be a component of a set $M \subset P$ Then $B(K) \subset B(M)$.*

Proof: Let there be, on the contrary, a point $a \in B(K) - B(M)$. Then $P - B(M)$ is a neighborhood of a. Let C be the component of $P - B(M)$ containing a. Since P is locally connected, a is an interior point of C. On the other hand, $a \in B(K) \subset \bar{K}$ so that $C \cap K \neq \emptyset$ and hence $C \cap M \neq \emptyset$. Since C is connected and $C \cap B(M) \neq \emptyset$, we have $C \subset M$ by 18.1.8. Thus, $C \subset K$ by 18.2.5. This is, however, evidently impossible, since $a \in B(K)$ and a is an interior point of C.

22.2. Let P be a metric space, $Q \subset P$. Define a set $L(Q) \subset \bar{Q}$ as follows: If $a \in \bar{Q}$, then $a \in L(Q)$ if and only if for every neighborhood U of a there is a component K of $Q \cap U$ such that a is an interior point of $K \cap (P - Q)$.

22.2.1. *Let $a \in Q$. We have $a \in L(Q)$ if and only if Q is locally connected at the point a.*

Proof: I. Let Q be locally connected at a. Let U be a neighborhood of a in P. Then (see 8.7.5) $Q \cap U$ is a neighborhood of a in Q. Thus, if K is the component of $Q \cap U$ containing a, then a is an interior point of K *in the space* Q; i.e. there is an $\varepsilon > 0$ such that $x \in Q \cap \Omega(a, \varepsilon)$ implies $x \in K$. Thus, in the space P, $x \in \Omega(a, \varepsilon)$

implies $x \in K$. Thus, in the space P, $x \in \Omega(a, \varepsilon)$ implies $x \in K \cup (P - Q)$, i.e., a is an interior point of $K \cup (P - Q)$.

II. Let $a \in Q \cap L(Q)$. Let V be a neighborhood of a in Q and let K be the component of V containing a. We have to prove that, for suitable $\varepsilon > 0$, $x \in Q \cap \cap \Omega(a, \varepsilon)$ implies $x \in K$. By 8.7.5 there is a neighborhood U of the point a in P such that $V = Q \cap U$. As $a \in L(Q)$, there is a component H of V such that, for suitable $\varepsilon > 0$, $x \in \Omega(a, \varepsilon)$ implies $x \in H \cup (P - Q)$. In particular, $a \in H \cup (P - Q)$. As $a \in Q$, we have $a \in H$ and hence $H = K$ by 18.2.1. Thus, $x \in \Omega(a, \varepsilon)$ implies $x \in K \cup (P - Q)$, i.e. $x \in Q \cap \Omega(a, \varepsilon)$ implies $x \in K$.

22.2.2. $Q \subset L(Q)$ *if and only if* Q *is locally connected.*

This follows by 22.2.1.

22.2.3. $L(Q)$ *is a* $\mathbf{G}_\delta$*-set for every* $Q \subset P$.

Proof: I. If $\varepsilon > 0$, denote by $A(\varepsilon)$ the set of all $a \in P$ such that there exists a connected $S \subset Q$ such that: [1] $d(S) < \varepsilon$, [2] a is an interior point of $S \cup (P - Q)$.

II. For every $\varepsilon > 0$ we have $L(Q) \subset A(\varepsilon)$. If $a \in L(Q)$, then $\Omega(a, \frac{1}{3}\varepsilon)$ is a neighborhood of the point a, so that $Q \cap \Omega(a, \frac{1}{3}\varepsilon)$ has a component K such that a is an interior point of $K \cup (P - Q)$. K is connected and $K \subset Q$. Moreover, $K \subset \subset \Omega(a, \frac{1}{3}\varepsilon)$ implies $d(K) < \varepsilon$. Thus, $a \in A(\varepsilon)$.

III. The sets $A(\varepsilon)$ are open. If $a \in A(\varepsilon)$, there is a connected $S \subset Q$ such that: [1] $d(S) < \varepsilon$, [2] there is a $\delta > 0$ with $\Omega(a, \delta) \subset S \cup (P - Q)$. Evidently, every $x \in \Omega(a, \delta)$ is an interior point of $S \cup (P - Q)$, so that $\Omega(a, \delta) \subset A(\varepsilon)$. Thus, $A(\varepsilon)$ is open by 8.6.1.

IV. $Q \cap \bigcap_{n=1}^{\infty} A(1/n)$ is a $\mathbf{G}_\delta$-set by 13.1.2, since $\bar{Q}$ is $\mathbf{G}_\delta$ by 13.2 and $A(1/n)$ are $\mathbf{G}_\delta$ by III and 13.1.1.

V. It remains to be proved that $L(Q) = \bar{Q} \cap \bigcap_{n=1}^{\infty} A(1/n)$. As $L(Q) \subset \bar{Q}$, we have, by II, $L(Q) \subset \bar{Q} \cap \bigcap_{n=1}^{\infty} A(1/n)$. On the other hand, choose an $a \in \bar{Q} \cap \bigcap_{n=1}^{\infty} A(1/n)$. We shall prove that $a \in L(Q)$. Let U be a neighborhood of a. There exists an index n with $\Omega(a, 2/n) \subset U$. We have $a \in A(1/n)$, so that there is a connected $S \subset Q$ such that $d(S) < 1/n$ and that a is an interior point of $S \cup (P - Q)$. There exists a $\delta > 0$ with $\Omega(a, \delta) \subset S \cup (P - Q)$; we may suppose that $\delta < 1/n$. As $a \in \bar{Q}$, there is a point $b \in Q \cap \Omega(a, \delta) \subset Q \cap (S \cup (P - Q)) = Q \cap S = S$. As $b \in \Omega(a, \delta)$, $b \in S$, $d(S) < 1/n$, we have $S \subset \Omega(a, \delta + 1/n) \subset \Omega(a, 2/n) \subset U$. Since $S \subset Q$, we have $S \subset Q \cap U$, so that (see 18.2.5) there exists a component K of $Q \cap U$ such that $S \subset K$, so that a is an interior point of $K \cup (P - Q)$. Since U was an arbitrary neighborhood of a, we have $a \in L(Q)$.

22.2.4. *Let $Q \subset M \subset L(Q)$ (so that Q is locally connected by 22.2.2). Then M is locally connected.*

Proof: Choose a point $a \in M$ and a number $\varepsilon > 0$. Then $\Omega(a, \frac{1}{3}\varepsilon)$ is a neighborhood of the point $a \in L(Q)$. Hence, there is a component K of $Q \cap \Omega(, \frac{1}{3}\varepsilon)$ and a number $\delta > 0$ such that $\varrho(a, x) < \delta$ implies $x \in K \cup (P - Q)$. We may suppose that $\delta < \frac{1}{3}\varepsilon$. Put $S = M \cap \bar{K} \cap \Omega(a, \frac{1}{3}\varepsilon)$. By 8.7.1 and 18.2.2, $K = \bar{K} \cap Q \cap \Omega(a, \frac{1}{3}\varepsilon)$. Thus, $K \subset S$ and, moreover, $S \subset \bar{K}$, so that S is connected by 18.1.7. As $S \subset \subset \Omega(a, \frac{1}{3}\varepsilon)$ we have $d(S) < \varepsilon$. Moreover, $S \subset M$, so that, by 22.1.1 it suffices to prove that $x \in S$ whenever $x \in M$, $\varrho(a, x) < \delta$ (in particular, for $x = a$). As $\delta < \frac{1}{3}\varepsilon$, it suffices to prove that $x \in M$, $\varrho(a, x) < \delta$ imply $x \in \bar{K}$. Thus, let $x \in M$, $\varrho(a, x) < \delta$. Choose an $\eta > 0$ with $\varrho(a, x) + \eta < \delta$. As $M \subset \bar{Q}$, we have $\varrho(x, Q) = 0$. Thus, there exists a point $z \in Q$ such that $\varrho(x, z) < \eta$. We have then $\varrho(a, z) \leqq \varrho(a, x) + \varrho(x, z) < \delta$, hence $z \in K \cup (P - Q)$, i.e. $z \in K$, since $z \in Q$. Thus, $\varrho(x, K) \leqq \varrho(x, z) < \eta$ for every sufficiently small $\eta > 0$. Thus, $\varrho(x, K) = 0$, i.e. $x \in \bar{K}$.

22.2.5. *Let P be a continuum. Let $a \in P - L(P)$. Then, there is a continuum K such that $a \in K$, $K \subset P - L(P)$. Moreover, there is a point $b \neq a$ and a disjoint sequence of continua $\{K_n\}_1^\infty$ such that* $\lim \varrho(a, K_n) = \lim \varrho(b, K_n) = 0$.

Proof: I. By the definition of $L(P)$ there is a neighborhood U of a such that a is not an interior point of C, if C is the component of U containing a. Choose a neighborhood V of a with $\bar{V} \subset U$.

II. For $n = 1, 2, 3, \ldots$ we may, by I, determine recursively the components A_n of U such that $a \in U - A_n$, $V \cap A_n \neq \emptyset$, $\varrho(a, A_n) < n^{-1}$. By 18.2.2, $\varrho(a, A_n) > 0$, so that we may evidently determine the A_n to be distinct and hence (see 18.2.1) disjoint. For $n = 1, 2, 3, \ldots$ choose an $a_n \in V \cap A_n$ such that $\varrho(a, a_n) < n^{-1}$.

III. The set $A_n \subset U$ is evidently connected, so that $\bar{A}_n$ is either a one-point set, or a continuum. On the other hand evidently $U \neq P$, so that, by 19.3.2, (see also 10.3.2), $\bar{A}_n - U \neq \emptyset$. Thus, $\bar{A}_n$ is a continuum and $\bar{A}_n - \bar{V} \neq \emptyset$.

Denote by B_n the component of $\bar{A}_n \cap \bar{V}$ containing a_n. By 19.3.1 we obtain easily that $B_n \cap (\bar{V} - V) \neq \emptyset$, so that B_n is not a one-point set. B_n is a closed (see 18.2.2) connected subset of $\bar{A}_n \cap \bar{V}$. Thus, B_n is a continuum. Choose a $b_n \in B_n \cap (\bar{V} - V)$.

IV. As P is compact, there are indices $i_1 < i_2 < i_3 < \ldots$ such that there exists $\lim b_{i_n} = b$. Evidently $b \in \bar{V} - V$, and hence $a \neq b$. Since $B_n \subset \bar{A}_n \cap \bar{V} \subset \bar{A}_n \cap \cap U = A_n$ (see 18.2.2) and since the sets A_n are disjoint, B_n are disjoint continua. We have $\varrho(a, B_n) \leqq \varrho(a, a_n)$, $\varrho(b, B_n) \leqq \varrho(b, b_n)$, so that, for $K_n = B_{i_n}$, $\lim \varrho(a, K_n) = \lim \varrho(b, K_n) = 0$.

V. Put $K = \overline{\operatorname{Lim}}\, B_n$. Evidently $a \in K$, $b \in K$. We even have $a \in \underline{\operatorname{Lim}}\, B_n$, so that K is a continuum (see 19.1.7). We have $K \subset \bar{V} \subset U$.

VI. It remains to be proved that $K \subset P - L(P)$. Let there be, on the contrary, a point c in $K \cap L(P)$. Since $a \in K$, $c \in K$, and since K is a connected subset of U, both points belong to the same component of U i.e. (see I), $c \in C$. As $c \in L(P)$, c is an interior point of C. Thus, there is a $\delta > 0$ such that $\varrho(c, x) < \delta$ implies $x \in C$. Since $c \in K = \overline{\text{Lim}}\, B_n$, there exists an index p and a point $u \in B_p$ such that $\varrho(c, u) < \delta$, and hence $u \in C$. Thus, $C \cap B_p \neq \emptyset$. On the other hand, $B_p \subset A_p$ (see IV) and C, A_p are components of U. Thus (see 18.2.1), $A_p = C$, which is a contradiction.

22.3. 22.3.1. *Let P be a topologically complete connected and locally connected space. Let $a \in P$, $b \in P$, $a \neq b$. Then there is a simple arc $C \subset P$ with the end points a, b.*

Proof: I. By 15.6.3 we may suppose that P is complete.

II. For every $x \in P$ denote by $V(x)$ the component of $\Omega(x, \frac{1}{2})$ containing the point x. Thus, $V(x)$ is connected and $x \in V(x)$. Moreover, $d(V(x)) \leqq 1$ and $V(x)$ is open by 22.1.4. As all the $V(x)$ are open and since $\bigcup_{x \in P} V(x) = P$, there is, by 18.4.2, a finite point sequence $\{x_i\}_{i=0}^{k_1}$ such that $x_0 = a$, $x_{k_1} = b$ and that $V(x_{i-1}) \cap V(x_i) \neq \emptyset$ for $1 \leqq i \leqq k_1$. Evidently $\{x_i\}$ contains a finite subsequence $\{y_i\}_{i=0}^{h_1}$ such that $y_0 = a$, $y_{h_1} = b$, $V(y_{i-1}) \cap V(y_i) \neq \emptyset$ for $1 \leqq i \leqq h_1$ and $V(y_i) \cap V(y_j) = \emptyset$ for $0 \leqq i \leqq h_1$, $0 \leqq j \leqq h_1$, $|i - j| \geqq 2$. Put $U_i^{(1)} = V(y_i)$ for $0 \leqq i \leqq h_1$.

III. Suppose that for a given n there is a finite sequence $\{U_i^{(n)}\}_{i=0}^{h_n}$ of point sets (as it was just done for $n = 1$) such that

$[1]_n$ $a \in U_0^{(n)}$, $b \in U_{h_n}^{(n)}$,

$[2]_n$ $U_{i-1}^{(n)} \cap U_i^{(n)} \neq \emptyset$ for $1 \leqq i \leqq h_n$,

$[3]_n$ $U_i^{(n)} \cap U_j^{(n)} = \emptyset$ for $0 \leqq i \leqq h_n$, $0 \leqq j \leqq h_n$, $|i - j| \geqq 2$,

$[4]_n$ $U_i^{(n)}$ $(0 \leqq i \leqq h_n)$ are open,

$[5]_n$ $U_i^{(n)}$ $(0 \leqq i \leqq h_n)$ are connected,

$[6]_n$ $d(U_i^{(n)}) \leqq n^{-1}$ $(0 \leqq i \leqq h_n)$.

Put $c_0 = a$, $c_{h_n+1} = b$ and, for $1 \leqq i \leqq h_n$ choose a $c_i \in U_{i-1}^{(n)} \cap U_i^{(n)}$, which may be done by $[2]_n$. By $[4]_n$ we may choose for every $x \in U_i^{(n)}$ $(0 \leqq i \leqq h_n)$ an open set $H_i(x)$ such that $x \in H_i(x) \subset \overline{H_i(x)} \subset U_i^{(n)}$, $d(H_i(x)) \leqq (n + 1)^{-1}$. Let $W_i(x)$ be the component of $H_i(x)$ containing x. Thus, $W_i(x)$ is connected and $x \in W_i(x) \subset \overline{W_i(x)} \subset U_i^{(n)}$. Moreover, $d[W_i(x)] \leqq (n + 1)^{-1}$ and the set $W_i(x)$ is open by 22.1.4.

We have $c_i \in U_i^{(n)}$, $c_{i+1} \in U_i^{(n)}$, the sets $W_i(x)$ are open in $U_i^{(n)}$ and $\bigcup_{x \in U_i^{(n)}} W_i(x) = U_i^{(n)}$. Thus, by 18.4.2 (see also 18.4.1) and $[5]_n$ there is a finite point sequence $\{z_{ir}\}_{r=0}^{p_i}$ such that $z_{i0} = c_i$, $z_{ip_i} = c_{i+1}$ and that $W_i(z_{i,r-1}) \cap W_i(z_{ir}) \neq \emptyset$ for $1 \leqq r \leqq p_i$. Combine all the sequences $\{z_{ir}\}_{r=0}^{p_i}$ $(0 \leqq i \leqq h_n)$ into a new finite point sequence $\{v_j\}_{j=0}^{k}$ where $k = \sum_{i=0}^{h_n} (p_i + 1) - 1$ in the following manner: the first elements of $\{v_j\}_{j=0}^{k}$ are the points z_{0r} $(r = 0, 1, \ldots, p_0)$, they are followed by the points z_{1r} $(r = 0, 1, \ldots, p_1)$ etc., and the sequence is finished by the points $z_{h_n,r}$ $(r = 0, 1, \ldots, p_{h_n})$. Put $W_i(z_{ir}) = W(v_j)$ for $z_{ir} = v_j$. Then we have $v_0 = a$, $v_k = b$ and $W(v_{j-1}) \cap W(v_j) \neq \emptyset$ for $1 \leqq j \leqq k$. Evidently, $\{v_j\}_{j=0}^{k}$ contains a finite subsequence $\{u_j\}_{j=0}^{h_{n+1}}$ such that $u_0 = a$, $u_{h_{n+1}} = b$, $W(u_{j-1}) \cap W(u_j) \neq \emptyset$ for $1 \leqq j \leqq h_{n+1}$ and $W(u_i) \cap W(u_j) = \emptyset$ for $0 \leqq i \leqq h_{n+1}$, $0 \leqq j \leqq h_{n+1}$, $|i - j| \geqq 2$. Put $U_j^{(n+1)} = W(u_j)$ for $0 \leqq j \leqq h_{n+1}$. Then the conditions $[1]_{n+1} - [6]_{n+1}$ are satisfied. Moreover, we have

$[7]_n$ for every i $(0 \leqq i \leqq h_{n+1})$ there is an index $\lambda(i)$ $(0 \leqq \lambda(i) \leqq h_n)$ such that $\overline{U_i^{(n+1)}} \subset U_{\lambda(i)}^{(n)}$; the indices $\lambda(i)$ may be chosen in such a way that

$[8]_n$ $0 \leqq i \leqq j \leqq h_{n+1}$ implies $\lambda(i) \leqq \lambda(j)$.

IV. Thus, we may construct recursively, for $n = 1, 2, 3, \ldots$, finite sequences $\{U_i^{(n)}\}_{i=0}^{h_n}$ of point sets such that, for every n, $[1]_n - [8]_n$ hold. Put

$$G_n = \bigcup_{i=0}^{h_n} U_i^{(n)} \quad (n = 1, 2, 3, \ldots).$$

G_n are open by $[4]_n$ and connected by $[2]_n$, $[5]_n$ and 18.1.4. Moreover, by $[7]_n$,

$$\overline{G}_{n+1} \subset G_n \quad (n = 1, 2, 3, \ldots)$$

and hence

$$C = \bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \overline{G}_n .$$

V. We have $a \in C$, $b \in C$ since, by $[1]_n$, $a \in G_n$, $b \in G_n$ for every n.

VI. C is compact by $[6]_n$ and 17.5.2.

VII. C is a continuum. Let us assume the contrary. By V and VI $C = A \cup B$ with non-void separated summands. By 10.2.7 there are open sets Γ, Δ such that $\overline{\Gamma} \cap \overline{\Delta} = \emptyset$, $\Gamma \supset A$, $\Delta \supset B$. As Γ, Δ are open and $\overline{G}_{n+1} \subset G_n$, $\overline{G_{n+1} - (\Gamma \cup \Delta)} \subset G_n - (\Gamma \cup \Delta)$. Since $G_n \supset C$ are connected and $G_n \cap (\Gamma \cup \Delta) = (G_n \cap \Gamma) \cup (G_n \cap \Delta)$ with separated non-void summands, we have $G_n - (\Gamma \cup \Delta) \neq \emptyset$. Thus, by $[6_n]$ and 15.7.2 we have $\emptyset \neq \bigcap_{n=1}^{\infty} [G_n - (\Gamma \cup \Delta)] = C - (\Gamma \cup \Delta)$, which is a contradiction.

VIII. Let us choose a $c \in C$, $a \neq c \neq b$, and prove that $C - (c) = C' \cup C''$ with separated summands, $a \in C'$, $b \in C''$.

Since $C \subset G_n$ for every n, there are indices s_n such that $0 \leqq s_n \leqq h_n$, $c \in U_{s_n}^{(n)}$. Choose an index p with

$$3 \, . \, p^{-1} < \min [\varrho(a, c), \varrho(b, c)] .$$

$[1]_n$, $[2]_n$ and $[6]_n$ yield:

$$\varrho(a, c) \leqq d(\bigcap_{i=0}^{s_n} U_i^{(n)}) \leqq \sum_{i=0}^{s_n} d(U_i^{(n)}) \leqq (s_n + 1) \cdot n^{-1},$$

$$\varrho(b, c) \leqq d(\bigcap_{i=s_n}^{h_n} U_i^{(n)}) \leqq \sum_{i=s_n}^{h_n} d(U_i^{(n)}) \leqq (h_n - s_n + 1) \cdot n^{-1},$$

so that

$$3 \leqq s_n \leqq h_n - 3 \quad \text{for} \quad n \geqq p .$$

Let

$$G'_n = \bigcup_{i=0}^{s_n-3} U_i^{(n)}, \qquad G''_n = \bigcup_{i=s_n+3}^{h_n} U_i^{(n)} \qquad (n = p, p+1, \ldots),$$

$$G' = \bigcup_{n=p}^{\infty} G'_n, \qquad G'' = \bigcup_{n=p}^{\infty} G''_n .$$

G' and G'' are open by $[4]_n$ and we have $a \in G'$, $b \in G''$ by $[1]_n$. Thus, it suffices to prove first that $G' \cap G'' = \emptyset$ and secondly that $C - (c) \subset G' \cup G''$ since then we may put $C' = C \cap G'$, $C'' = C \cap G''$.

We prove easily that $C - (c) \subset G' \cup G''$. Let $d \in C - (c)$. Choose an n such that $n \geqq p$ and that $5 \cdot n^{-1} < \varrho(c, d)$. There exists an index i such that $0 \leqq i \leqq h_n$ and $d \in U_i^{(n)}$. If $|s_n - i| \leqq 2$, we obtain, by $[2]_n$ and $[6]_n$,

$$\varrho(c, d) \leqq d(\bigcup_{j=s_n-2}^{s_n+2} U_j^{(n)}) \leqq \sum_{j=s_n-2}^{s_n+2} d(U_j^{(n)}) \leqq 5 \cdot n^{-1},$$

which is a contradiction. Thus, $|s_n - i| \geqq 3$, so that $d \in G'_n \cup G''_n \subset G' \cup G''$.

It remains to be proved that $G' \cap G'' = \emptyset$.

Let $p \leqq l \leqq m$. By $[7]_n$ and $[8]_n$ $(n = l, l+1, \ldots, m-1)$ we may associate, with every i $(0 \leqq i \leqq h_n)$, a $\mu_n(i)$ $[0 \geqq \mu(i) \leqq h_l]$ such that $U_i^{(m)} \subset U_{\mu(i)}^{l}$ and that

$$0 \leqq i < j \leqq h_m \quad \text{implies} \quad \mu(i) \leqq \mu(j) .$$

Assume that there is a point $d \in G'_m \cap G''_l$. As $d \in G'_m$, there is an index i with $0 \leqq i \leqq s_m - 3$, $d \in U_i^{(m)}$ and hence $d \in U_{\mu(i)}^{(l)}$. As $d \in G''_l$, there is an index j with $s_l + 3 \leqq j \leqq h_l$, $d \in U_j^{(l)}$. Thus $U_{\mu(i)}^{(l)} \cap U_j^{(l)} \neq \emptyset$, so that, by $[3]_l$, $\mu(i) \geqq j - 1 \geqq$ $\geqq s_l + 2$. Since $i < s_m$, we have $\mu(i) \leqq \mu(s_m)$, so that $\mu(s_m) \geqq s_l + 2$, so that, by $[3]_l$, $U_{\mu(s_m)}^{(l)} \cap U_{s_l}^{(l)} = \emptyset$; this is a contradiction, since obviously $c \in U_{\mu(s_m)}^{(l)} \cap U_{s_l}^{(l)}$. Thus,

$$p \leqq l \leqq m \quad \text{implies} \quad G'_m \cap G''_l = \emptyset . \tag{1}$$

Suppose that there is a point $d \in G'_l \cap G''_m$. As $d \in G''_m$, there is an index i such that $s_m + 3 \leqq i \leqq h_m$, $d \in U_i^{(m)}$, so that $d \in U_{\mu(i)}^{(l)}$. As $d \in G'_l$, there is an index j such that $0 \leqq j \leqq s_l - 3$, $d \in U_j^{(l)}$. Thus, $U_{\mu(i)}^{(l)} \cap U_j^{(l)} \neq \emptyset$, so that, by $[3]_l$, $\mu(i) \leqq$ $\leqq j + 1 \leqq s_l - 2$. As $i > s_m$, we have $\mu(i) \geqq \mu(s_m)$, so that $\mu(s_m) \leqq s_l - 2$, and that, by $[3]_l$, $U_{\mu(s_m)}^{(l)} \cap U_{s_l}^{(l)} = \emptyset$, which is a contradiction, since ibviously $c \in U_{\mu(s_m)}^{(l)} \cap U_{s_l}^{(l)}$.

Thus

$$p \leqq l \leqq m \quad \text{implies} \quad G'_l \cap G''_m = \emptyset. \tag{2}$$

(1) and (2) yield $G' \cap G'' = \emptyset$.

IX. By V, VII, VIII and 20.3, C is a simple arc with end points a, b.

22.3.2. *In a locally connected topologically complete space P the constituants are identical with the components.*

Proof: Let K be a component of P. By 12.8.1, 19.5.5 and 19.5.8 it suffices to prove that K is a semicontinuum. K is open in P by 22.1.4. Thus, K is locally connected by 22.1.3 and K is a topologically complete space by 15.5.3 (see 13.1.1). Thus, K is a semicontinuum by 20.1.1 and 22.3.1.

22.3.3. *Let P be a locally connected topologically complete space. Let a closed $Q \subset P$ cut P between points a, b. Then Q separates a from b in P.*

Proof: By 19.5.10 the points a, b belong to distinct constituants of $P - Q$. The set $P - Q$ is open in P and hence it is locally connected by 22.1.3 and $P - Q$ is topologically complete by 15.5.3. Thus, the constituants of $P - Q$ coincide with its quasicomponents by 22.1.5 and 22.3.2. Thus, by 18.3.5, $P - Q = A \cup B$ with separated summands, $a \in A$, $b \in B$, i.e. Q separates a from b in P.

Exercises

22.1. Every connected subset of $\mathbf{E}_1$ is locally connected. This is not true in $\mathbf{E}_n$ ($n \geqq 2$).

22.2. If $P \times Q \neq \emptyset$ is locally connected, then $P \times Q$ are locally connected (see ex. 18.10).

22.3. $P \times Q$ is locally connected if and only if for every M open in P and for every N open in Q every component of $M \times N$ is open in $P \times Q$.

22.4. Let P, Q be locally connected spaces. Let $M \subset P \times Q$. Let f be a continuous function on $P \times Q$. Let, for every $a \in M$,

$$x \in P \times Q, \qquad f(x) = f(a) \Rightarrow x = a.$$

Then M has no cluster points in $P \times Q$ (see ex. 19.14).

22.5. Let $M_1 \subset P$, $M_2 \subset P$, $a \in M_1 \cap M_2$. Let both the sets M_1, M_2 be locally connected at the point a. Then $M_1 \cup M_2$ is also locally connected at the point a.

22.6. If the sets $M_1 \subset P$, $M_2 \subset P$ are locally connected, then $M_1 \cup M_2$ need not be locally connected. This may be shown by means of an example with $M_1 \cup M_2 = P_5$ (see ex. to § 19).

22.7. Let the sets $M_1 \subset P$, $M_2 \subset P$ be locally connected and closed in $M_1 \cup M_2$. Then $M_1 \cup M_2$ is locally connected.

22.8. We may replace the word "closed" in ex. 22.7. by the word "open".

22.9. Let G be an open set in a separable locally connected space P. Then the system of all components of G is countable.

22.10. A space P is locally connected if and only if it has the following property: If a point $a \in P$ and a positive number ε are given, then there is a connected open set G such that $a \in G$, $d(G) < \varepsilon$.

22.11. Let $a \in P$. For every $\varepsilon > 0$ let there be a connected open G with $a \in G$, $d(G) < \varepsilon$. Then P is locally connected at a.

22.12. There exists a space P and a point $a \in P$ such that P is locally connected at a and that, for suitable $\varepsilon > 0$ there is no connected open G with $a \in G$, $d(G) < \varepsilon$. This may be shown by means of the example with $a = (0, 0)$, $P = P_7$ (see exercises to § 19).

22.13. Let $a \in M \subset P$. Let M be locally connected at the point a. Let $M \subset N \subset \overline{M}$. Then N is locally connected at the point a.

22.14. Let $M \subset P$. If M is locally connected, $\overline{M}$ need not be locally connected. This may be shown by an example with $P = \mathbf{E}_1$ and also by an example with $P = \mathbf{E}_2$ and with open connected M.

22.15. Let P be a locally connected space. Let $Q \subset P$ be compact. Let G be a neighborhood of Q. Then there exists an open M such that $Q \subset M \subset G$ and that M has a finite number of components.

22.16. We may replace the word "open" in ex. 22.15 by the word "closed".

22.17. Let P be a connected and locally connected space. Let G_1, G_2 be connected open sets. Let the sets $B(G_1)$, $B(G_2)$ be connected and disjoint. Let

$$G_1 \cap G_2 \neq \emptyset \neq P - (\overline{G}_1 \cup \overline{G}_2).$$

Then either $G_1 \subset G_2$ or $G_2 \subset G_1$.

Remark: V. Knichal noticed that in ex. 22.17 we may: [1] omit the assumption of local connectedness, [2] replace the assumption of G_1, G_2 open by a weaker assumption of $G_1 \cup G_2$ open, [3] replace the assumption $P - (\overline{G}_1 \cup \overline{G}_2) \neq \emptyset$ by a weaker asssumption $P - (G_1 \cup G_2) \neq \emptyset$.

22.18. Let there exist a one-to-one continuous mapping of a connected and locally connected space P onto a simple arc. Then P is a simple arc.

22.19. It is not possible to omit the assumption of local connectedness in ex. 22.18. This may be shown by means of an example with $P \subset P_5$ (see exercises to § 19).

22.20. A one-to-one continuous image of a locally connected space need not be locally connected.

22.21. There exists a connected space P such that $P - L(P)$ is an n-point set ($n = 1, 2, 3, \ldots$) or an infinite countable set. This may be shown by means of an example with $P \subset P_1$ (see exercises to § 19).

22.22. A space P satisfying the condition from ex. 22.21 cannot be compact; this follows by 22.2.5. Prove that P cannot be locally compact. P may be topologically complete.

22.23. Let P be the set of all couples $(x, y) \in \mathbf{E}_2$ such that at least one of x, y is irrational. Then P is connected, locally connected and topologically complete.

22.24. Let P be a connected, locally connected and topologically complete space. Let $a \in P$. Let a be an end point of every simple arc $C \subset P$ such that $a \in C$. Then the set $P - (a)$ is either void or connected.

22.25. Let P be a locally connected and topologically complete space. Let $C \subset P$ be a simple arc with end points a, b. Let $c \in C$, $a \neq c \neq b$. Let $P - (c)$ be connected. Then there exists a simple loop $D \subset P$ such that $c \in D$.

22.26. Let P be a locally connected and topologically complete space. Let $A \subset P$ be a closed and locally connected set. Let B be a union of some components of $P - A$. Then $A \cup B$ is closed and locally connected

Remark: V. Jarník noticed that the assumption of topological completeness in ex. 22.26 is superfluous.

22.27. Let P be a locally connected and topologically complete space. Let $G \subset P$ be an open set. Let M be the set of all $x \in B(G)$ such that there is a continuum K with $x \in K$, $K - (x) \subset G$. Then M is dense in $B(G)$.

22.28. Let P be a locally connected space. Let $\mathfrak{S}$ be a system of points sets. Let M be the union of all $X \in \mathfrak{S}$. Let N be the union of all sets $B(X)$ ($X \in \mathfrak{S}$). Then $\overline{M} \subset M \cup \overline{N}$.

§ 23. Locally connected continua

23.1. 23.1.1. *Let P be a metric space. For every $\varepsilon > 0$ let there be a $\delta > 0$ such that for every $a \in P$, $b \in P$ with $\varrho(a, b) < \delta$ there is a connected $S \subset P$ with $a \in S$, $b \in S$, $d(S) < \varepsilon$. Then P is locally connected.*

This follows by 22.1.1.

23.1.2. *Let P be a compact locally connected space. Then, for every $\varepsilon > 0$, there is a $\delta > 0$ such that for any $a \in P$, $b \in P$, $\varrho(a, b) < \delta$ there is a connected $S \subset P$ with $a \in S$, $b \in S$, $d(S) < \varepsilon$.*

Proof: On the contrary, let there be an $\varepsilon > 0$ such that no $\delta = n^{-1}$ ($n = 1, 2, 3, \ldots$) has the required property. Then there are point sequences $\{x_n\}$, $\{y_n\}$ such that [1] $\varrho(x_n, y_n) < n^{-1}$ [2] if $S \subset P$ is connected and $x_n \in S$, $y_n \in S$, then $d(S) \geqq \varepsilon$. Since P is compact, there are indices $i_1 < i_2 < i_3 < \ldots$ such that $\lim x_{i_n} = a$ exists. As P is locally connected at the point a, there is a $\delta > 0$ such that there is a connected $S \subset P$ with $a \in S$, $x \in S$, $d(S) < \frac{1}{2}\varepsilon$ whenever $\varrho(a, x) < \delta$. There is an index n such that $\varrho(a, x_{i_n}) < \frac{1}{2}\delta$ and $i_n^{-1} < \frac{1}{2}\delta$ and hence $\varrho(a, y_{i_n}) \leqq \varrho(a, x_{i_n}) + \varrho(x_{i_n}, y_{i_n}) < \delta$. There exist connected $S_1 \subset P$, $S_2 \subset P$ such that $a \in S_1 \cap S_2$, $x_{i_n} \in S_1$, $y_{i_n} \in S_2$, $d(S_1) < \frac{1}{2}\delta$, $d(S_2) < \frac{1}{2}\delta$. We have $x_{i_n} \in S_1 \cup S_2$, $y_{i_n} \in S_1 \cup S_2$, $d(S_1 \cup S_2) \leqq d(S_1) + d(S_2) < \delta$ and $S_1 \cup S_2$ is connected by 18.1.4. This is a contradiction.

23.1.3. *A metric space P is locally connected if and only if every its component is open and locally connected.*

Proof: I. Let P be locally connected and let K be its component. K is open by 22.1.4, so that K is locally connected by 22.1.3.

II. Let every component of P be open and locally connected. Choose a point $a \in P$ and a number $\varepsilon > 0$. Let K be the component containing the point a. Then there is a $\delta_1 > 0$ such that $\Omega(a, \delta_1) \subset K$. Since K is locally connected, by 22.1.1 there is a $\delta_2 > 0$ such that for every $x \in K$ with $\varrho(a, x) < \delta_2$ there is a connected $S \subset K$ with $a \in S$, $x \in S$, $d(S) < \varepsilon$. Put $\delta = \min(\delta_1, \delta_2)$. If $x \in P$, $\varrho(a, x) < \delta$, we have $\varrho(a, x) < \delta_1$ and hence $x \in K$. Moreover, $\varrho(a, x) < \delta_2$, so that there exists a connected $S \subset K \subset P$ such that $a \in S$, $x \in S$, $d(S) < \varepsilon$. Thus, P is locally connected at the point a by 22.1.1.

23.1.4. *A compact space P is locally connected if and only if:* [1] *P has a finite number of components,* [2] *every component is locally connected.*

Proof: I. If P has a finite number of components and if K is one of them, then $P - K$ is the union of the remaining ones, so that, by 8.3.4 and 18.2.2, $P - K$ is closed and hence K is open. If, moreover, every K is locally connected, P is locally connected by 23.13.

In this part of the proof the compactness of P was not used.

II. Let P be locally connected, so that the components are open by 23.1.3. By 18.2.1 and 17.5.4 the number of components is finite.

23.1.5. *Let P be a metric space. Let, for every $\varepsilon > 0$, $P = \bigcup_{i=1}^{m} K_i$ with a finite number of closed connected summands of diameters less than ε. Then P is locally connected.*

Proof: Choose an $a \in P$. Let F be the union of all K_i ($1 \leqq i \leqq m$) which do not contain the point a (if $a \in K_i$ for every i, $F = \emptyset$). Denote by S the union of the remaining K_i, so that $a \in S$ and S is connected by 18.1.4 or by 18.1.5. F is obviously closed, so that there is a $\delta > 0$ such that $\Omega(a, \delta) \subset P - F \subset S$. Evidently, $d(S) < 2\varepsilon$. Thus, P is locally connected at the point a by 22.1.1.

23.1.6. *Let $P \neq \emptyset$ be a compact locally connected space. Let $\varepsilon > 0$. Then $P = \bigcup_{i=1}^{m} K_i$ with a finite number of closed and connected summands of diameters less than ε.*

Proof: For every $x \in P$ denote by $U(x)$ the component of $\Omega(x, \frac{1}{3}\varepsilon)$ containing the point x. The sets $U(x)$ are open by 22.1.4 and $\bigcup_{x \in P} U(x) = P$ so that, by 17.5.4 there is a finite sequence $\{x_i\}_1^m$ such that $\bigcup_{i=1}^{m} U(x_i) = P$ and consequently $P = \bigcup_{i=1}^{m} K_i$ where $K_i = \overline{(Ux_i)}$. As $U(x_i) \subset \Omega(x_i, \frac{1}{3}\varepsilon)$, we have evidently $d(K_i) \leqq \frac{2}{3}\varepsilon < \varepsilon$. Moreover, the sets K_i are closed and, by 18.1.6, also connected.

23.1.7. *Let P be a continuum. P is locally connected if and only if for every $\varepsilon > 0$, P is a union of a finite number of continua of diameters less than ε.*

Proof: I. The condition is sufficient by 23.1.5, since every continuum is closed by 17.2.2.

II. Let P be a locally connected continuum and let $\varepsilon > 0$. By 23.1.6, $P = \bigcup_{i=1}^{m} K_i$, where K_i are connected and closed (and hence compact by 17.2.2) and $d(K_i) < \varepsilon$. Thus, every K_i is either a continuum, or a one-point set. We may suppose that there is an index $n \leqq m$ such that K_i is a one-point set if and only if $i > n$. By 18.1.9 $n \geqq 1$. We have $P = A \cup B$, where $A = \bigcup_{i=1}^{n} K_i$, $B = P - A$. A is closed and non-void. B is finite and hence also closed. Moreover $A \cap B = \emptyset$, so that A, B are separated. Since P is connected and $A \neq \emptyset$, we have $B = \emptyset$, i.e. $P = \bigcup_{i=1}^{n} K_i$.

23.1.8. *Simple arcs are locally connected continua.*

This follows, e.g., from 20.1.1, 20.1.12 and 23.1.7.

23.1.9. *Simple loops are locally connected continua.*
This follows, e.g., from 21.1.1, 23.1.7 and 23.1.8.

23.1.10. *Let a continuum P not be locally connected at a point $a \in P$. Then there exists a continuum $K \subset P$ such that $a \in K$ and that P is locally connected at no point $x \in K$.*

Proof: By 22.2.1 $a \in P - L(P)$. Thus, by 22.2.5, there exists a continuum K such that $a \in K \subset P - L(P)$. By 22.2.1 P is locally connected at no $x \in K$.

23.1.11. *Let P be a metric space. Let there be a finite number of locally connected compact sets A_i* ($1 \leqq i \leqq m$) such that $P = \bigcup_{i=1}^{m} A_i$. *Then P is a locally connected compact space.*

Proof: P is compact by ex. 17.4. P is locally connected by 23.15 and 23.1.6 (see also 17.2.2).

23.2. 23.2.1. *Let P be a locally connected continuum. Let Q be a metric space containing more than one point. Let there exist a continuous mapping f of P onto Q. Then Q is a locally connected continuum.*

Proof: By 17.4.2 and 18.1.10 Q is a continuum. Choose an $\varepsilon > 0$. By 17.4.4 there is a $\delta > 0$ such that

$$M \subset P,\ d(M) < \delta \quad \text{imply} \quad d[f(M)] < \varepsilon\,. \tag{1}$$

By 23.1.7 $P = \bigcup_{i=1}^{m} K_i$ where K_i are continua and $d(K_i) < \delta$ ($1 \leqq i \leqq m$). We have $Q = \bigcup_{i=1}^{m} f(K_i)$. By (1), $d[f(K_i)] < \varepsilon$. The sets $f(K_i)$ are compact by 17.4.2 and hence closed in Q by 17.2.2 and connected by 18.1.10. Thus, Q is locally connected by 23.1.5.

23.2.2. *Let P be a metric space containing more than one point. Put* $J = \mathop{\mathrm{E}}_{t}[0 \leqq t \leqq 1]$. *Let there exist a continuous mapping f of the interval J onto P. Then P is a locally connected continuum.*

This is a particular case of theorem 23.2.1, since J is a locally connected continuum (e.g. by 23.1.8).

23.2.3. *Let P be a locally connected continuum. Then there exists a continuous mapping f of the interval* $J = \mathop{\mathrm{E}}_{t}[0 \leqq t \leqq 1]$ *onto P.*

First proof: I. Let D be the Cantor discontinuum (see 17.8.3). Let $\mathop{\mathrm{E}}_{t}[u_n < t < v_n]$ ($n = 1, 2, 3, \ldots$) be the contiguous intervals of D, so that $v_n - u_n \to 0$.

II. By 17.8.4 there is a continuous mapping φ of D onto P. Put $\eta_n = \varrho[\varphi(u_n), \varphi(v_n)]$, hence $\eta_n \geqq 0$. As D is a compact space, we have, by 17.4.4, $\eta_n \to 0$.

III. By 23.1.2 we may associate with every m $(= 1, 2, 3, \ldots)$ a number $\delta_m > 0$ such that for every $a \in P$, $b \in P$ with $\varrho(a, b) < \delta_m$ there is a connected $S \subset P$ with $a \in S$, $b \in S$, $d(S) < m^{-1}$.

IV. As $\eta_n \geqq 0$, $\eta_n \to 0$, $\delta_m > 0$, we may associate with every m $(= 1, 2, 3, \ldots)$ an index i_m such that

$$n \geqq i_m \quad \text{implies} \quad \eta_n < \delta_m.$$

We may assume that $1 < i_1 < i_2 < i_3 < \ldots$.

V. We shall define, for every n $(= 1, 2, 3, \ldots)$ a continuous mapping ψ_n of the interval $\mathop{\mathrm{E}}_t[u_n \leqq t \leqq v_n]$ into P such that $\psi_n(u_n) = \varphi(u_n)$, $\psi_n(v_n) = \varphi(v_n)$. We shall distinguish the following three cases: [1] $\varphi(u_n) = \varphi(v_n)$, [2] $1 \leqq n < i_1$, $\varphi(u_n) \neq \varphi(v_n)$, [3] $i_m \leqq n < i_{m+1}$ $(m = 1, 2, 3, \ldots)$, $\varphi(u_n) \neq \varphi(v_n)$.

VI. First, if $\varphi(u_n) = \varphi(v_n)$, we put $\psi_n(t) = \varphi(u_n)$ for every $t \in \mathop{\mathrm{E}}_t[u_n \leqq t \leqq v_n]$.

VII. Secondly, let $1 \leqq n < i_1$, $\varphi(u_n) \neq \varphi(v_n)$. By 17.2.1 and 22.3 there is a simple arc $C_n \subset P$ with the end points $\varphi(u_n)$, $\varphi(v_n)$. Let ψ_n be a homeomorphic mapping of the interval $\mathop{\mathrm{E}}_t[u_n \leqq t \leqq v_n]$ onto C_n such that $\psi_n(u_n) = \varphi(u_n)$, $\psi_n(v_n) = \varphi(v_n)$.

VIII. Thirdly, let $i_m \leqq n < i_{m+1}$, $\varphi(u_n) \neq \varphi(v_n)$. By IV, we have $\eta_n = \varrho[\varphi(u_n), \varphi(v_n)] < \delta_m$, so that, by III, there exists a connected $S_n \subset P$ such that $\varphi(u_n) \in S_n$, $\varphi(v_n) \in S_n$, $d(S) < m^{-1}$ and hence $S_n \subset \Omega(\varphi(u_n), m^{-1})$. Let G_n be the component of $\Omega[\varphi(u_n), m^{-1}]$ containing the point $\varphi(u_n) \in S_n$. By 18.2.5 $S_n \subset G_n$ and hence $\varphi(v_n) \in G_n$. The set G_n is connected. By 22.1.4 G_n is open, so that, by 22.1.3, G_n is locally connected. By 17.2.1 and 15.5.3 G_n is a topologically complete space. Thus, by 22.3, there exists a simple arc $C_n \subset G_n$ with the end points $\varphi(u_n)$, $\varphi(v_n)$. Let ψ_n be a homeomorphic mapping of the interval $\mathop{\mathrm{E}}_t[u_n \leqq t \leqq v_n]$ onto C_n such that $\psi_n(u_n) = \varphi(u_n)$, $\psi_n(v_n) = \varphi(v_n)$.

IX. Define a mapping f of the interval $J = \mathop{\mathrm{E}}_t[0 \leqq t \leqq 1]$ into P as follows: Evidently $J = D \cup \bigcup_{n=1}^{\infty} \mathop{\mathrm{E}}_t[u_n < t < v_n]$ with disjoint summands. If $t \in D$, put $f(t) = \varphi(t)$; if $u_n < t < v_n$, put $f(t) = \psi_n(t)$. As $\varphi(D) = P$ we have $f(J) = P$, i.e. f is a mapping of J onto P. It remains to prove that f is continuous. Assume the contrary. Then there is a number $a \in J$ and a sequence of numbers $\{t_k\}_1^\infty$ such that $t_k \to a$, while $f(t_k)$ does not converge to $f(a)$. Then there exists a positive number α and a subsequence $\{x_k\}_1^\infty$ of $\{t_k\}_1^\infty$ such that $\varrho[f(x_k), f(a)] > 2\alpha$ for every k. We see easily that some of the following three cases occur: [1] there is a subsequence $\{y_k\}_1^\infty$ of $\{x_k\}_1^\infty$ such that $y_k \in D$ for each k, [2] there is an index n such that a subsequence $\{y_k\}_1^\infty$ of $\{x_k\}_1^\infty$ may be chosen with $y_k \in \mathop{\mathrm{E}}_t[u_n < t < v_n]$ for each k, [3] there are indices n_k ($k = 1, 2, 3, \ldots$) such that $n_1 < n_2 < n_3 < \ldots$ and that a subsequence $\{y_k\}_1^\infty$ of $\{x_k\}_1^\infty$ may be chosen with $y_k \in \underset{t}{\mathrm{E}}[u_{n_k} < t < v_{n_k}]$ for each k.

In the first case $y_k \to a$, $y_k \in D$, hence $a \in D$, so that $f(y_k) = \varphi(y_k)$, $f(a) = \varphi(a)$, and hence $f(y_k) \to f(a)$ which is a contradiction, since $\varrho[f(y_k), f(a)] > 2\alpha > 0$ for all k.

In the second case $y_k \to a$, $u_n < y_k < v_n$, hence $u_n \leqq a \leqq v_n$, so that $f(y_k) = \psi_n(y_k)$, $f(a) = \psi_n(a)$, and hence $f(y_k) \to f(a)$, which is a contradiction.

In the third case $y_k \to a$, $u_{n_k} < y_k < v_{n_k}$, hence $| y_k - u_{n_k} | < v_{n_k} - u_{n_k} \to 0$, hence $u_{n_k} \to a$, hence $a \in D$, hence $f(u_{n_k}) = \varphi(u_{n_k})$, $f(a) = \varphi(a)$ and hence $f(u_{n_k}) \to f(a)$. As $\alpha > 0$, there is an index p such that $p^{-1} < \alpha$. Since $n_1 < n_2 < n_3 < \ldots$ and since $f(u_{n_k}) \to f(a)$, there is an index k such that $i_p \leqq n_k$ and that $\varrho[f(u_{n_k}), f(a)] < \alpha$. As $\varrho[f(y_k), f(a)] < 2\alpha$, we have evidently $\varrho[f(y_k), f(u_{n_k})] > \alpha$. Thus, $f(y_k) \neq f(u_{n_k})$. Since $u_{n_k} < y_k < v_{n_k}$, we have, by VI, $\varphi(u_{n_k}) \neq \varphi(v_{n_k})$. Since $i_p \leqq n_k$, there is an index $m \geqq p$ such that $i_m \leqq n_k < i_{m+1}$. Thus, by VIII, we have $f(y_k) = \psi_{n_k}(y_k) \in C_{n_k} \subset G_{n_k} \subset \Omega[\varphi(u_{n_k}), m^{-1}]$, i.e. $\varrho[f(y_k), \varphi(u_{n_k})] < m^{-1}$. As $\varphi(u_{n_k}) = f(u_{n_k})$, $\varrho[f(y_k), f(u_{n_k})] > \alpha$, we have $m^{-1} > \alpha$. This is a contradiction, since $m \geqq p$, $p^{-1} < \alpha$.

The proof just finished is simple; however, it is based not only on theorem 17.8.4, but also on theorem 22.3.

Second proof of theorem 23.2.3: I. For every $x \in P$ denote by $V(x)$ the component of $\Omega(x, \frac{1}{4})$ containing x. Thus, $V(x)$ is connected and $x \in V(x)$. Moreover, $d(V(x) \leqq \frac{1}{2}$ and $V(x)$ is open by 22.1.4. As $\bigcup_{x \in P} V(x) = P$, by 17.5.4 there is a finite point sequence $\{x_\lambda\}_{\lambda=1}^p$ such that $\bigcup_{\lambda=1}^{p} V(x_\lambda) = P$. By 18.4.2 there is a finite point sequence $\{y_i\}_{i=1}^h$ such that $\{y_i\}$ is a subsequence of $\{x_\lambda\}$, every term of $\{y_i\}$ is equal to some member of $\{x_\lambda\}$ and $V(y_i) \cap V(y_{+1}) \neq \emptyset$ for $1 \leqq i \leqq h - 1$. The sequence $\{y_i\}$ may be modified by repeating the last term several times, so that we may assume $h = 2^{m_1} = 2^{N_1}$. Put $U_i^{(1)} = V(y_i)$ $(1 \leqq i \leqq h)$.

II. Assume that we have determined for some n ($= 1, 2, 3, \ldots$) a finite sequence $\{U_i^{(n)}\}_{i=1}^{h_n}$ $(h = 2^{N_n})$ of point sets (as just done for $n = 1$) such that

$$[1]_n \quad \bigcup_{i=1}^{h_n} U_i^{(n)} = P,$$

$[2]_n$ $U_i^{(n)} \cap U_i^{(n+1)} \neq \emptyset$ for $1 \leqq i \leqq h_n - 1$,

$[3]_n$ the sets $U_i^{(n)}$ $(1 \leqq i \leqq h_n)$ are open,

$[4]_n$ the sets $U_i^{(n)}$ $(1 \leqq i \leqq h_n)$ are connected,

$[5]_n$ $d(U_i^{(n)}) \leqq 2^{-n}$ $(1 \leqq i \leqq h_n)$.

For a given $i (1 \leq i \leq h_n)$ denote by $W_i(x)$, for every $x \in \overline{U_i^{(n)}}$, the component of $\Omega(x, 2^{-n-2})$ containing x. Thus, $W_i(x)$ is connected and $x \in W_i(x)$. Moreover, $d[W_i(x)] \leqq 2^{-n-1}$ and $W_i(x)$ is open by 22.1.4. As $\bigcup_{x \in \overline{U_i^{(n)}}} [\overline{U_i^{(n)}} \cap W_i(x)] = \overline{U_i^{(n)}}$

and as $\overline{U_i^{(n)}}$ is compact by 17.2.2, by 17.5.4 there exists a finite point sequence $\{z_\mu^i\}_{\mu=1}^q$ such that $\bigcup_{\mu=1}^q W_i(z_\mu^{(i)}) \supset \overline{U_i^{(n)}}$.

By $[2]_n$ we may assume that, for every $i \geqq 2$, we have $z_1^{(i)} \in U_{i-1}^{(n)} \cap U_i^{(n)}$ and that, for $i \leqq h_n - 1$, we have $z_q^{(i)} \in U_i^{(n)} \cap U_{i+1}^{(n)}$. Therefore, we see easily that we may assume $z_1^{(i+1)} = z_q^{(i)}$ for $1 \leqq i \leqq h_n - 1$. Since $\overline{U_i^{(n)}}$ is connected by $[4]_n$ and 18.1.6, by 18.4.2 there is a finite point sequence $\{u_r^{(i)}\}_{r=1}^{k_i}$ such that the sequence $\{z_\mu^{(i)}\}$ is a subsequence of $\{u_r^{(i)}\}$, every term of $\{u_r^{(i)}\}$ is equal to some term of $\{z_\mu^{(i)}\}$ (in particular $u_1^{(i)} = z_1^{(i)}$, $u_{k_i}^{(i)} = z_{q_i}^{(i)}$), and $W_i(u_r^{(i)}) \cap W_i(u_{r+1}^{(i)}) \neq \emptyset$ for $1 \leqq r \leqq k_i - 1$. The sequence $\{u_r^{(i)}\}$ may be modified by repeating the last term several times, so that we may assume $k_i = 2^{m_{n+1}}$, where the number m_{n+1} is the same for all i $(1 \leqq i \leqq h_n)$. Let us combine the sequences $\{W_i(u_r^{(i)})\}_{r=1}^{2^{m_n}}$ into a new sequence $\{U_j^{(n+1)}\}_{j=1}^{h_{n+1}}$ where $h_{n+1} = 2^{N_{n+1}}$, $N_{n+1} = N_n + m_{n+1}$. We take first the sets $W_1(z_r^{(1)})$ $(1 \leqq r \leqq 2^{m_{n+1}})$; they are followed by $W_2(u_r^{(2)})$ $(1 \leqq r \leqq 2^{m_{n+1}})$ etc. and, finally, by the sets $W_{h_n}(u_r^{(h_n)})$ $(1 \leqq r \leqq 2^{m_{n+1}})$. Then all the properties $[1]_{n+1} - [5]_{n+1}$ are satisfied. Moreover, we have (see 10.2.6).

$$[6]_n \quad 1 \leqq i \leqq h_n\,,\ 1 \leqq r \leqq 2^{m_{n+1}} \Rightarrow U_{(i-1)h_n+r}^{(n+1)} \cap U_i^{(n)} \neq \emptyset\,.$$

III. Thus, we may construct recursively, for $n = 1, 2, 3, \ldots$, finite sequences $\{U_i^{(n)}\}_{i=1}^{h_n}$ of point sets such that, for every n, $[1]_n - [6]_n$ hold. We have $h_n = 2^{N_n}$, $N_1 = m_1$, $N_{n+1} = N_n + m_{n+1}$ and hence $N_n = \sum_{s=1}^n m_s$. By $[2]_n$ or $[4]_n$, $U_i^{(n)} \neq \emptyset$. Choose a $z_i^{(n)} \in U_i^{(n)}$ $(n = 1, 2, 3, \ldots, 1 \leqq i \leqq h_n)$.

IV. For $n = 1, 2, 3, \ldots$ define a mapping f_n of the interval $J = \mathop{E}_t[0 \leqq t \leqq 1]$ as follows: Put $I_i = \mathop{E}_t[(i-1)\,.\,2^{-N_n} \leqq t < i\,.\,2^{-N_n}]$ $(i = 1, 2, \ldots, h_n - 1)$, $I_{h_n} = \mathop{E}_t[(h_n - 1)\,.\,2^{-N_n} \leqq t \leqq 1]$.

Then, put $f_n(t) = z_i^{(n)}$, where i is uniquely determined by the relation $t \in I_i$. (If $t \in J$, then there is a unique index i $(1 \leqq i \leqq h_n = 2^{N_n})$ such that $(i-1)\,.\,2^{-N_n} \leqq t < i\,.\,2^{-N_n}$ provided $i < h_n$ and $(i-1)\,.\,2^{-N_n} \leqq t \leqq i\,.\,2^{-N_n}$ provided $i = h_n$. Put $f_n(t) = z_i^{(n)}$.)

V. Let $t \in J$, $f_n(t) = z_i^{(n)}$, $f_{n+1}(t) = z_j^{(n+1)}$. We see easily that there is an index r such that $1 \leqq r \leqq 2^{m_{n+1}}$, $j = (i-1)\,2^{m_{n+1}} + r$. As $z_i^{(n)} \in U_i^{(n)}$, $z_j^{(n+1)} \in U_j^{(n+1)}$, we have, by $[5]_n$ and $[6]_n$

$$\varrho[f_n(t), f_{n+1}(t)] \leqq d(U_i^{(n)}) + d(U_j^{(n+1)}) < 2^{-n+1}\,.$$

Thus, for $n = 1, 2, 3, \ldots$; $m = n+1, n+2, n+3, \ldots$ we have

$$\varrho[f_n(t), f_m(t)] < \sum_{s=n}^{\infty} 2^{-s+1} = 2^{-n+2}.$$

Thus, $\{f_n(t)\}$ is a Cauchy sequence, so that, by 17.2.1, there exists

$$f(t) = \lim_{n \to \infty} f_n(t) \in P.$$

VI. If $t_1 \in J$, $t_2 \in J$, $|t_1 - t_2| < 2^{-N_n}$, $f_n(t_1) = z_{i_1}^{(n)}$, $f_n(t_2) = z_{i_2}^{(n)}$, we have $(i_1 - 1) . 2^{-N_n} \leqq t_1 . 2^{-N_n}$, $(i_2 - 1) . 2^{-N_n} \leqq t_2 \leqq i_2 . 2^{-N_n}$, so that evidently $|i_1 - i_2| \leqq 1$. Thus, by $[2]_n$ and $[5]_n$,

$$\varrho[f_n(t_1), f_n(t_2)] \leqq d(U_{i_1}^{(n)}) + d(U_{i_2}^{(n)}) \leqq 2^{-n+1}.$$

This yields easily that f is a *continuous* mapping of the interval J into P. In fact, let $t_\nu \in J$, $\tau \in J$, $t_\nu \to \tau$ and let $\varepsilon > 0$. There is an index n with $2^{-n+4} < \varepsilon$. As $t_\nu \to \tau$, there is an index p such that $|t_\nu - \tau| < 2^{-N_n}$ for $\nu > p$ and hence $\varrho[f_n(t_\nu), f_n(\tau)] \leqq \leqq 2^{-n+1}$. On the other hand, by V,

$$\varrho[f_n(t_\nu), f(t_\nu)] \leqq 2^{-n+2}, \quad \varrho[f_n(\tau), f(\tau)] \leqq 2^{-n+2},$$

so that, for $\nu > p$, $\varrho[f(t_\nu), f(\tau)] \leqq 2^{-n+1} + 2^{-n+2} + 2^{-n+2} < 2^{-n+4} < \varepsilon$. Thus, $f(t_\nu) \to f(\tau)$.

VII. It remains to prove that $f(J) = P$. Let, there be on the contrary, an $a \in P - f(J)$.

By VI and 17.4.2, $f(J)$ is compact, so that (see 17.2.2) $f(J)$ is closed, and hence $P - f(J)$ is open. As $a \in P - f(J)$, there is a $\delta > 0$ such that $\Omega(a, \delta) \subset P - f(J)$. There is an index n with $2^{-n+3} < \delta$. As $a \in P$, by $[1]_n$ there is an index i ($1 \leqq i \leqq h_n$) such that $a \in U_i^{(n)}$, so that, by $[5]_n$, we have $\varrho(a, z_i^{(n)}) \leqq 2^{-n}$. If $t = (i - 1) . 2^{-N_n} \in J$, then $f_n(t) = z_i^{(n)}$, so that, by V, $\varrho(z_i^{(n)}, f(t)) \leqq 2^{-n+2}$. Thus, $\varrho[a, f(t)] \leqq 2^{-n} + 2^{-n+2} < 2^{-n+3} < \delta$, so that $f(t) \in \Omega(a, \delta) \subset P - f(J)$, which is a contradiction.

23.2.4. *Let P be a locally connected continuum. Let $\varepsilon > 0$. Then there exists a finite number of locally connected continua P_i ($1 \leqq i \leqq m$) such that $P = \bigcup_{i=1}^{m} P_i$ and $d(P_i) \leqq \varepsilon$ ($1 \leqq i \leqq m$).*

Proof: By 23.2.3 there exists a continuous mapping f of the interval $J = \underset{t}{\mathrm{E}}[0 \leqq t \leqq 1]$ onto P. By 17.4.4 (see also 9.6.1) there is a $\delta > 0$ such that $0 \leqq t_1 < t_2 \leqq 1$, $t_2 - t_1 < \delta$ imply $\varrho[f(t_1), f(t_2)] < \varepsilon$. Choose a natural number $n > \delta^{-1}$ and denote by A_k ($1 \leqq k \leqq n$) the set of all $t \in J$ with $(k - 1) n^{-1} \leqq t \leqq kn^{-1}$. Then $P = \bigcup_{k=1}^{n} f(A_k)$ and the sets $f(A_k)$ are less than or equal to ε in diameter. We see easily by 23.2.2 that every $f(A_k)$ which is not a one-point set is a locally connected continuum. On the other hand, the equation $P = \bigcup_{i=1}^{m} f(A_k)$ remains valid after omitting the one-point summands on the right-hand side (see the proof of theorem 23.1.7).

23.2.5. *Let P be a locally connected continuum. Let $\emptyset \neq F \subset G \subset P$. Let F be closed. Let G be open and connected. Then there exists a locally connected continuum K with $F \subset K \subset G$.*

Proof: F is compact by 17.2.2. Thus (see 17.3.4) there is an $\varepsilon > 0$ such that $x \in G$ whenever $\varrho(x, F) \leqq \varepsilon$. By 23.2.4, $P = \bigcup_{i=1}^{m} P_i$, where P_i are locally connected continua of diameter less than or equal to ε.

G is a topologically complete space by 15.5.2 (see also 17.2.1). Moreover, G is connected and locally connected (see 22.1.3).

Denote by N the system of all couples (i, k) with $1 \leqq i \leqq k \leqq m$, $F \cap P_i \neq \emptyset \neq F \cap P_k$. If $(i, k) \in N$, choose points $a \in P_i$, $b \in P_k$, $a \neq b$. We have $a \in G$, $b \in G$, so that, by 22.3.1 there exists a simple arc $C_{ik} \subset G$ with end points a, b.

Denote by K_1 the union of all P_i $(1 \leqq i \leqq m)$ with $F \cap P_i \neq \emptyset$. As $P = \bigcup_{i=1}^{m} P_i$, we have $F \subset K_1$. Since $d(P_i) \leqq \varepsilon$, $\varrho(x, F) < \varepsilon$ imply $x \in G$, we have $K_1 \subset G$. Denote by K_2 the union of all C_{ik} with $(i, k) \in N$. Put $K = K_1 \cup K_2$. Evidently $F \subset K \subset G$. By 23.1.11 K is a locally connected compact set. Evidently K is not a one-point set, and we obtain easily by 18.1.5 that K is connected. Thus, K is a locally connected continuum.

Exercises

The spaces $P_1, P_2, \ldots, P_{12}$ were defined in the exercises to § 19.

23.1. P_4 and P_6 are locally connected continua. Moreover, every continuum embedded into P_4 is locally connected (this is not true for P_6).

23.2. At which points are P_1, P_2, P_5, P_7 locally connected?

23.3. The continuum P_3 is locally connected at a unique point; P_{12} is locally connected at no point.

23.4. Let $C \subset P_5$ or $C \subset P_{12}$. Let C be a locally connected continuum. Then C is a simple arc.

23.5. Let P be a locally connected continuum. Let $\varepsilon > 0$. Then there is a number $\delta > 0$ such that for every $a \in P$, $b \in P$ with $0 < \varrho(a, b) < \delta$, there is a simple arc $C \subset P$ with end points a, b and with diameter less than ε.

23.6. Let $K \subset \mathbf{E}_m$ be a continuum. There exist locally connected continua $K_n \subset \mathbf{E}_m$ $(n = 1, 2, 3, \ldots)$ such that $K_n \supset K_{n+1}$, $\bigcap_1^{\infty} K_n = K$.

23.7. Let $K \subset \mathbf{E}_m$ be a continuum. There are simple arcs $C_n \subset \mathbf{E}_m$ $(n = 1, 2, 3, \ldots)$ such that $K \cup \bigcup_1^{\infty} C_n$ is a locally connected continuum.

23.8. Pet P be a continuum. P is locally connected, if and only if for any two disjoint closed F_1, F_2 there are separated A_1, A_2 and a closed Φ such that Φ has a finite number of components and $P - \Phi = A_1 \cup A_2$, $A_1 \supset F_1$, $A_2 \supset F_2$.

23.9. There exists a continuous mapping of $\mathbf{E}_1$ onto P if and only if there exist locally connected continua $K_n \subset P$ $(n = 1, 2, 3, \ldots)$ such that $K_n \subset K_{n+1}$, $P = \bigcup_1^{\infty} K_n$.

23.10. Let P be a locally connected continuum. Let $Q \subset \mathbf{E}_m$ be a locally connected continuum. There exists a continuous mapping of Q onto P.