CHAPTER V

SEPARATION

(Sections 26 – 30)

This chapter is devoted to the separation properties of closure spaces. It is rather difficult to say what is meant by a separation property. Given a single-valued relation α for closure spaces and proximities (in some cases generalized proximities, usually such that $\alpha\mathscr{P}$ is a proximity for $\mathscr{P}$, not necessarily continuous), and a property $\mathfrak{P}$ of pairs of closure spaces and proximities, we ask whether $\langle\mathscr{P}, \alpha\mathscr{P}\rangle$ has the property $\mathfrak{P}$. E.g., if $\mathfrak{P}$ means that any two distinct points of $\mathscr{P}$ are distant under $\alpha\mathscr{P}$, or that $\alpha\mathscr{P}$ induces the closure structure of $\mathscr{P}$, then the relation $\alpha = \{\mathscr{P} \to \{X \to Y \mid X \text{ and } Y \text{ are not semi-separated in } \mathscr{P}\}\}$ leads to the so-called semi-separated spaces (each singleton is closed) or semi-uniformizable spaces, respectively, and the relation $\alpha = \{\mathscr{P} \to \to \{X \to Y \mid X \text{ and } Y \text{ are not separated in } \mathscr{P}\}\}$ leads to the separated spaces or regular spaces, respectively. If $\mathfrak{P}$ means that $\alpha\mathscr{P}$ is uniformizable and induces the closure structure of $\mathscr{P}$, and if $\alpha\mathscr{P}$ is the proximity $\{X \to Y \mid \overline{X} \cap \overline{Y} \neq \emptyset\}$, then we obtain normal spaces. It should be noted that we do not intend to give a theory of separation; we are interested in those separation properties which often enter into important theorems; e.g., the following conditions on a space $\mathscr{P}$ are equivalent: $\mathscr{P}$ is separated, each net in $\mathscr{P}$ has at most one limit point, if f is a continuous mapping of a space into $\mathscr{P}$ then $\operatorname{gr} f$ is closed in $\mathbf{D}^* f \times \mathscr{P}$.

Section 26 is concerned with an investigation of those properties which depend on closures of finite sets. Section 27 is devoted to an examination of separated and regular spaces; these appear in essential assumptions of many results. We mention the uniqueness theorem for mappings into separated spaces, the theorem on simultaneous continuous extension of a mapping into a regular space, and the theorem on removable discontinuities of a mapping into a regular space. Section 28 is devoted to uniformizable spaces, i.e. spaces whose closure structures are induced by uniformities. It is to be noted that these spaces are often termed completely regular. The exposition is based on the results of sections 23 – 25 on uniform and proximity spaces. In addition some purely topological proofs are sketched. Section 29 is devoted to normal spaces. Particular attention is given to counter-examples and uniformizable covers. Here we shall prove that every open cover of a pseudometrizable space has a locally finite σ-discrete open refinement. The development is again based upon sections 23 – 25. The last section is devoted to some special kinds of normal spaces, namely hereditarily and perfectly normal spaces, paracompact spaces and hereditarily paracompact spaces. Here we shall prove the Bing-Nagata-Smirnov metrization theorem. Special attention is given to localization of properties in paracompact and hereditarily paracompact spaces. It will be shown in the exercises to 41 that paracompactness is a separation property in the sense explained above.

26. QUASI-DISCRETE SPACES

Let $\langle P, u\rangle$ be a closure space; for each $X \subset P$ let vX be the set of all $y \in P$ such that $y \in u(x)$ for some x in X. It is easily seen that $\{X \to vX \mid X \subset P\}$ is a closure operation. This closure is called the quasi-discrete modification of u, and a closure is said to be quasi-discrete if it coincides with its quasi-discrete modification. In subsection A quasi-discrete spaces are investigated. In this connection the important notions of a discrete family and a quasi-discrete family of subsets of a closure space are introduced. In subsection B we shall be concerned with the examination of some of those properties of a closure operation which depend only on the quasi-discrete modification; namely, we shall introduce feebly semi-separated spaces (sometimes called T_0-spaces) and semi-separated spaces (sometimes called T_1-spaces). It has already been shown (17 C.18) that every closure space admits an embedding into a product $P^{\aleph}$ where P is a certain three-point space and $\aleph$ is a suitable cardinal. Now we shall prove three interesting theorems of this type, namely that every topological space can be embedded into a product of two-point topological spaces, every feebly semi-separated topological space can be embedded into a product of two-point topological feebly semi-separated spaces and finally, every semi-separated space $\langle P, u\rangle$ can be embedded into the product $\langle P, v\rangle^{\exp P}$ where v is the coarsest semi-separated closure for P.

A. QUASI-DISCRETE MODIFICATION

26 A.1. Definition. The *quasi-discrete modification of a closure operation* u for a set P is defined to be the closure operation $v = \{X \to \bigcup\{u(x) \mid x \in X\} \mid X \subset P\}$ for P. A closure operation u will be called *quasi-discrete* if u coincides with its quasi-discrete modification, that is, if $uX = \bigcup\{u(x) \mid x \in X\}$ for each $X \subset P$. *A space* is termed *quasi-discrete* if its closure structure is quasi-discrete.

For example discrete, accrete and finite spaces are quasi-discrete.

26 A.2. Theorem. *Suppose that* $\langle P, u\rangle$ *is a closure space and* v *is the quasi-discrete modification of* u, *and consider the relation*

$$\varrho = \mathbf{E}\{\langle x, y\rangle \mid x \in P, y \in u(x)\} = \Sigma\{u(x) \mid x \in P\}\,.$$

Then the relation ϱ *is a reflexive relation on* P *(i.e.* $\mathrm{J}_P \subset \varrho$*) and the closure* v *is the* *expansion of ϱ (i.e. $vX = \varrho[X]$ for each $X \subset P$). Next, ϱ is symmetric if and only if the closure operation u is semi-uniformizable, and ϱ is transitive (i.e. $\varrho \circ \varrho = \varrho$) if and only if the closure v is topological.*

Proof. The relation ϱ is a reflexive relation on P because $x \in u(x)$ for each $x \in P$. Clearly $\varrho[X] = \bigcup\{\varrho[(x)] \mid x \in X\} = \bigcup\{u(x) \mid x \in X\}$ and the latter set is, by definition, vX. The relation ϱ is symmetric if and only if $y \in u(x)$ implies $x \in u(y)$, and (by 23 B.3) this is a necessary and sufficient condition for the closure u to be induced by a semi-uniformity. The equivalence of the transitivity of ϱ to the fact that v is topological is evident.

According to this result every quasi-discrete closure operation for a set P is the expansion of a reflexive relation on P, i.e. of a vicinity of the diagonal of $P \times P$. Conversely, if ϱ is a vicinity of the diagonal of $P \times P$, then obviously the expansion u of ϱ, i.e. the relation $\{X \to \varrho[X] \mid X \subset P\}$, is a quasi-discrete closure for P. It is to be noted that this result was proved in Example 14 A.5 (f). This result and a related simple result are stated in the proposition which follows.

26 A.3. *Suppose that P is a set, R is the set of all vicinities of the diagonal of $P \times P$ (thus R is the uniformly discrete uniformity for P) and u_ϱ is the expansion of ϱ for each ϱ in R. Then the relation $\{\varrho \to u_\varrho \mid \varrho \in R\}$ is one-to-one and ranges on the set of all quasi-discrete closure operations for P; next, $\varrho \supset \sigma$ if and only if the closure u_ϱ is coarser than u_σ.*

Thus the study of quasi-discrete closure operations is reduced to the study of reflexive relations.

26 A.4. *The quasi-discrete modification of a closure operation v is the coarsest quasi-discrete closure finer than u.* — Evident.

Now we proceed to various characterizations of quasi-discrete spaces. The required concepts are introduced in the following definition.

26 A.5. Definition. Let $\mathscr{P}$ be a closure space. The *star of a subset* X of $\mathscr{P}$ (in the space $\mathscr{P}$) is the intersection of all neighborhoods of X. A *family* $\{X_a\}$ of subsets of $\mathscr{P}$ will be called *discrete* if $\{X_a\}$ is closure-preserving and $\{\overline{X}_a\}$ is disjoint. A *family* $\{X_a\}$ will be called *quasi-discrete* if $\{X_a\}$ is disjoint and closure-preserving.

26 A.6. *The star of a subset X of a space $\mathscr{P}$ is the set of all points x of $\mathscr{P}$ the closure of which intersects X, that is, the star of X is the set $\mathsf{E}\{x \mid \overline{(x)} \cap X \neq \emptyset, x \in \mathscr{P}\}$. In particular, if U is a neighborhood of a set X, and a point x does not belong to the star of X, then $U - (x)$ is a neighborhood of x.*

Proof. If $\overline{(x)} \cap X \neq \emptyset$, then every neighborhood U of X contains x, because U is a neighborhood of some $y \in X \cap \overline{(x)}$. Conversely, if $\overline{(x)} \cap X = \emptyset$, then for each y in X there exists a neighborhood U_y of y such that $x \notin U_y$. The union U of $\{U_y \mid y \in X\}$ is a neighborhood of each point of X, and hence of X, and does not contain the point x.

It is to be noted that the star of a point x in a space $\mathscr{P}$ need not be identical with the star of x in the cover $\{\overline{(y)} \mid y \in \mathscr{P}\}$ of $\mathscr{P}$ (as may be verified by examples).

26 A.7. Remark. For each closure u for a set P let u^* be the single-valued relation on $\exp P$ which assigns to each $X \subset P$ the star of X in $\langle P, u \rangle$. It is easily seen that each u^* is a quasi-discrete closure for P and $(u^*)^*$ is the quasi-discrete modification of u. In particular, if u is a quasi-discrete closure, then $(u^*)^* = u$. Finally, the inverse of the relation $\{\langle x, y \rangle \mid y \in u(x)\}$ is identical with the relation $\{\langle x, y \rangle \mid y \in u^*(x)\}$. As a consequence, $u = u^*$ if and only if u is a semi-uniformizable quasi-discrete closure operation.

A comment on quasi-discrete and discrete families may be in place. First, a quasi-discrete family need not be discrete; e.g. if $\mathscr{P}$ is an accrete space, then the family $\{(x) \mid x \in \mathscr{P}\}$ is evidently quasi-discrete (because $\overline{(x)} = |\mathscr{P}|$ for each x) but not discrete whenever the underlying set of $\mathscr{P}$ contains at least two points (because $\overline{(x)} \cap \overline{(y)} = |\mathscr{P}|$ for each x and y in $\mathscr{P}$). Moreover, this example shows that a quasi-discrete family need not be locally finite. By 14 B.18 a family $\{X_a\}$ is locally finite if and only if the family $\{\overline{X}_a\}$ is point-finite and the family $\{X_a\}$ is closure-preserving. Thus everv discrete family is locally finite. Now we shall prove essentially more.

26 A.8. Theorem. *A family $\{X_a \mid a \in A\}$ of subsets of a space $\mathscr{P}$ is discrete in $\mathscr{P}$ if and only if each point $x \in \mathscr{P}$ has a neighborhood intersecting at most one member of $\{X_a\}$.*

Proof. If the condition is fulfilled, then clearly the family $\{X_a\}$ is locally finite, and hence closure-preserving (by 14 B.17), and evidently the family $\{\overline{X}_a\}$ is disjoint. Conversely, suppose that the family $\{X_a\}$ is discrete. If $x \in \mathscr{P}$, then $x \in \overline{X}_\alpha$ for at most one index $\alpha \in A$ (because $\{\overline{X}_a\}$ is disjoint) and hence $U = |\mathscr{P}| - \bigcup\{X_a \mid a \in A,\ a \neq \alpha\}$ is a neighborhood of x because $\{X_a\}$ is closure-preserving and hence the closure of $|\mathscr{P}| - U$ is $\bigcup\{\overline{X}_a \mid a \in A,\ a \neq \alpha\}$.

A space $\mathscr{P}$ is discrete if and only if the family $\{(x) \mid x \in \mathscr{P}\}$ is discrete. Similar characterizations for quasi-discrete spaces are listed in the following theorem.

26 A.9. Theorem. *Each of the following conditions is necessary and sufficient for a closure space $\mathscr{P}$ to be quasi-discrete:*

(a) *The family $\{(x) \mid x \in \mathscr{P}\}$ is quasi-discrete.*

(b) *Every family of subsets of $\mathscr{P}$ is closure-preserving.*

(c) *For each subset X of $\mathscr{P}$ the star of X is a neighborhood of X.*

(d) *For each $x \in \mathscr{P}$ the star of x is a neighborhood of x.*

Proof. It will be shown that (a) is necessary, (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) and (d) is sufficient. The necessity of (a) is obvious, and clearly (a) implies (b). Suppose (b) and let $\mathscr{U}$ be the neighborhood system of a subset X of $\mathscr{P}$. We must show that $V = \bigcap \mathscr{U}$ is a neighborhood of X. Since each $U \in \mathscr{U}$ is a neighborhood of X we have $X \cap \overline{|\mathscr{P}| - U} = \emptyset$ for each U in $\mathscr{U}$. According to (b), $\overline{|\mathscr{P}| - V} = \bigcup\{\overline{|\mathscr{P}| - U} \mid U \in \mathscr{U}\}$ and consequently $\overline{|\mathscr{P}| - V} \cap X = \emptyset$, which means that V is a neighborhood of X.

The implication (c) $\Rightarrow$ (d) being self-evident, it remains to prove the sufficiency of (d). If $x \in \overline{X}$, then every neighborhood of x intersects X, in particular the star V of x intersects X. By definition of stars, $x \in \overline{(y)}$ for each $y \in V \cap X$.

Corollary. *A closure space $\mathscr{P}$ is quasi-discrete if and only if $\mathscr{P}$ is of a finite local character.*

26 A.10. Theorem. *Subspaces and sums of quasi-discrete spaces are quasi-discrete spaces. The product of a family $\{\mathscr{P}_a\}$ of non-void spaces is quasi-discrete if and only if all $\mathscr{P}_a$ are quasi-discrete spaces and all, excepting a finite number of $\mathscr{P}_a$, are accrete spaces.*

Proof. The invariance of the class of all quasi-discrete spaces under formation of sums and subspaces is evident. Suppose that $\mathscr{P}$ is the product of a family $\{\mathscr{P}_a \mid a \in A\}$ of quasi-discrete spaces such that for some finite subset A_0 of A all $\mathscr{P}_a$, $a \in A - A_0$, are accrete spaces. If $A_0 = \emptyset$, then $\mathscr{P}$ is obviously an accrete space and hence a quasi-discrete space. If $A_0 \neq \emptyset$ and $x \in \mathscr{P}$, then the collection of all the canonical neighborhoods of x of the form $\mathsf{E}\{x \mid a \in A_0 \Rightarrow \mathrm{pr}_a x \in U_a\}$ is a local base at x. Now apply 26 A.9 (d). Conversely, if $\mathscr{P}$ is quasi-discrete, then all coordinate spaces are quasi-discrete because each coordinate space can be embedded in the product. The fact that almost all $\mathscr{P}_a$ must be accrete spaces follows from statement 17 ex. 3c asserting that an infinite product is of an infinite local character whenever none of the coordinate spaces is an accrete space.

Remark. According to 17 C.18 every closure space can be embedded into the product space $\mathscr{P}^{\aleph}$ where $\mathscr{P}$ is certain three-point space and $\aleph$ is a suitable cardinal. Consequently, every closure space can be embedded into a product of quasi-discrete spaces.

B. SEMI-SEPARATED SPACES

Let $\langle P, u\rangle$ be a closure space and let us consider the relation $\varrho = \mathsf{E}\{\langle x, y\rangle \mid y \in u(x)\}$. We know that ϱ is symmetric if and only if the space $\langle P, u\rangle$ is semi-uniformizable, and ϱ is transitive if and only if the quasi-discrete modification of u is topological. Here we shall investigate spaces $\langle P, u\rangle$ such that either $\varrho \cap \varrho^{-1}$ or ϱ is the diagonal of P. For the sake of completeness we shall first state the basic properties of the class of all semi-uniformizable spaces and of the class of all spaces whose quasi-discrete modification is topological; only then we shall turn to the proper subject of this subsection.

26 B.1. Theorem. *The class of all semi-uniformizable spaces as well as the class of all spaces whose quasi-discrete modification is topological is hereditary and closed under sums and products.*

Proof. A simple proof consists in showing that the symmetry and the transitivity of the relation ϱ, mentioned above, is preserved under the operations in question.

Moreover, an alternate proof of the statements concerning semi-uniformizable spaces was given in 23 D.11.

Later we shall need the following simple proposition:

26 B.2. *The following condition is sufficient for a closure space $\mathscr{P}$, to be semi-uniformizable: if U is a neighborhood of x in $\langle P, u\rangle$, then U is a neighborhood of any $y \in \overline{(x)}$ (i.e. if U is a neighborhood of a point x, then U is a neighborhood of the closure of the singleton (x)). If $\mathscr{P}$ is topological, then this condition is also necessary.* — Obvious.

26 B.3. Definition. A *closure space* $\mathscr{P}$ will be termed *feebly semi-separated* if $x \in \overline{(y)}$, $y \in \overline{(x)}$ imply $x = y$. A *closure space* $\mathscr{P}$ will be termed *semi-separated* if each two distinct points of $\mathscr{P}$ are semi-separated, i.e. $((x) \cap \overline{(y)} \cup (\overline{(x)} \cap (y)) = \emptyset$ whenever $x \neq y$. It is to be noted that feebly semi-separated spaces are often called T_0-spaces, and semi-separated spaces are called T_1-spaces.

In the series of theorems which follows we shall give various characterizations of feebly semi-separated and semi-separated spaces, and we shall derive basic properties of the class of all feebly semi-separated spaces and of the class of all semi-separated spaces. The proofs are simple and therefore often omitted.

26 B.4. Theorem. *Each of the following three conditions is necessary and sufficient for a closure space $\mathscr{P}$ to be feebly semi-separated:*

(a) *For any two distinct points x and y of $\mathscr{P}$ either the star of x is disjoint with (y) or the star of y is disjoint with (x).*

(b) *At least one of any two distinct points of $\mathscr{P}$ possesses a neighborhood which does not contain the other point.*

(c) *If $x, y \in \mathscr{P}$, $x \neq y$, then either $|\mathscr{P}| - (x)$ is a neighborhood of y or $|\mathscr{P}| - (y)$ is a neighborhood of x.*

26 B.5. Theorem. *Each of the following conditions is necessary and sufficient for a closure space $\mathscr{P}$ to be semi-separated:*

(a) *Every one-point subset of $\mathscr{P}$ is closed in $\mathscr{P}$.*

(b) *$\mathscr{P}$ is simultaneously semi-uniformizable and feebly semi-separated.*

Proof. Obviously (a) is necessary and (a) implies (b). Assuming (b), if $x \in \overline{(y)}$, then $y \in \overline{(x)}$ because $\mathscr{P}$ is semi-uniformizable, and finally $x = y$ because $\mathscr{P}$ is feebly semi-separated. Thus (b) is sufficient.

26 B.6. Theorem. *The class of all feebly semi-separated spaces as well as the class of all semi-separated spaces is hereditary and closed under products and sums.*

Proof. The assertions concerning the invariance under sums and the operation of forming subspaces are obvious and the assertion concerning products follows from the fact that the product closure of the product of sets coincides with the product of closures (by 17 C.2).

26 B.7. *Suppose that a closure u is finer than a closure v. If v is feebly semi-separated or semi-separated, then u possesses the same property.*

If u is a closure for a set P and v is the quasi-discrete closure for P such that $x \in v(y)$ if and only if $y \in u(x)$, then v is feebly semi-separated if and only if u is feebly semi-separated, and no closure coarser than both u and v is feebly semi-separated provided that u is not semi-separated. It follows that the least upper bound of two feebly semi-separated closures need not be feebly semi-separated. On the other hand, we shall prove that the least upper bound of two semi-separated closures is a semi-separated closure, and moreover there exists a coarsest semi-separated closure for a given set.

26 B.8. The coarsest semi-separated closure operation for a given set P. *Let P be a set. By* 14 A.5 (a) *there exists exactly one topological closure operation u for P such that $X \subset P$ is open if and only if either $X = \emptyset$ or the complement $P - X$ of X is finite. A subset X of $\langle P, u\rangle$ is closed if and only if X is finite or $X = P$. In particular, all one-point sets are closed, and the space is semi-separated according to* 26 B.5. *It will ve shown that u is the coarsest semi-separated closure for P.* Let v be any semi-separated closure for P. We must show that v is finer than u. It is sufficient to prove that each closed subset of $\langle P, u\rangle$ is closed in $\langle P, v\rangle$. This is evident, however, because each one-point subset and hence each finite subset of $\langle P, v\rangle$ is closed.

Now we proceed to the embedding theorems. Let us recall that each closure space $\mathscr{P}$ can be embedded in the product $\mathscr{Q}^{\aleph}$ where $\mathscr{Q}$ is a certain three-point space and $\aleph$ is a suitable cardinal depending on $\mathscr{P}$ (17 C.18). The section concludes with three similar embedding theorems.

26 B.9. Theorem. *Let $\mathscr{P}_1$ be the two-point set $(0, 1)$ endowed with the accrete closure operation, and let $\mathscr{P}_2$ be the two-point set $(0, 1)$ endowed with the closure operation defined as follows:* $\overline{(0)} = (0)$, $\overline{(1)} = (0, 1)$. *Then*

(a) *In order that a closure space $\mathscr{P}$ be a feebly semi-separated topological space it is necessary and sufficient that $\mathscr{P}$ can be embedded into the product space $\mathscr{P}_2^{\aleph}$ where $\aleph$ is a cardinal depending on $\mathscr{P}$ (for $\aleph$ one may take* card exp $|\mathscr{P}|$).

(b) *In order that a closure space $\mathscr{P}$ be topological it is necessary and sufficient that $\mathscr{P}$ admit an embedding into the product space $\mathscr{P}_1^{\aleph'} \times \mathscr{P}_2^{\aleph''}$ where $\aleph'$ and $\aleph''$ are suitable cardinals depending on $\mathscr{P}$ (one may take $\aleph''$* = card exp $|\mathscr{P}|$, $\aleph'$ = card $|\mathscr{P}|$).

Corollary. *Every topological space admits an embedding into the product of topological quasi-discrete spaces.*

Proof. I. The conditions in (a) and (b) are sufficient because both spaces $\mathscr{P}_1$ and $\mathscr{P}_2$ are topological, $\mathscr{P}_2$ is feebly semi-separated and the class of all topological spaces as well as the class of all feebly semi-separated topological spaces is hereditary and closed under products.

II. Let $\mathscr{P}$ be a topological space and let $\mathscr{B}$ be an open base of $\mathscr{P}$. For each B in $\mathscr{B}$ let f_B be the mapping of $\mathscr{P}$ into $\mathscr{P}_2$ which is 1 on B and 0 on $|\mathscr{P}| - B$, i.e. $f_B x = 1$ if $x \in B$ and $f_B x = 0$ if $x \in |\mathscr{P}| - B$. Obviously the mapping $f_B : \mathscr{P} \to \mathscr{P}_2$ is continuous for each B in $\mathscr{B}$ (inverse images of open sets are open). Let us consider the reduced product f of the family $\{f_B \mid B \in \mathscr{B}\}$, that is $f = \{x \to \{f_B x \mid B \in \mathscr{B}\}\} : : \mathscr{P} \to \mathscr{P}_2^{\mathscr{B}}$. According to **17 C.13** the mapping f is continuous. Let $\mathscr{Q}$ be the subspace $f[|\mathscr{P}|]$ of $\mathscr{P}_2^{\mathscr{B}}$. We shall prove that

$$(*) \qquad \text{if} \quad x \in |\mathscr{P}|, X \subset |\mathscr{P}|, x \notin \overline{X}, \text{ then } fx \notin \overline{f[X]}.$$

Indeed, if $x \notin \overline{X}$, then $x \in B \subset |\mathscr{P}| - \overline{X}$ for some B in $\mathscr{B}$; we have $f_B x = 1$ and $f_B[X] \subset (0)$, and consequently the set $\mathsf{E}\{x \mid x \in \mathscr{P}_2^{\mathscr{B}}, \operatorname{pr}_B x = 1\}$ is a neighborhood of fx in $\mathscr{P}_2^{\mathscr{B}}$ which does not intersect $f[X] \subset \mathsf{E}\{x \mid x \in \mathscr{P}_2^{\mathscr{B}}, \operatorname{pr}_B x = 0\}$; it follows that $fx \notin \overline{f[X]} \supset \overline{f[X]}^{\mathscr{Q}}$.

III. If f is a one-to-one mapping, then f is necessarily an embedding because f is continuous and $f^{-1} : \mathscr{Q} \to \mathscr{P}$ is continuous by $(*)$. If $\mathscr{P}$ is feebly semi-separated and $x, y \in |\mathscr{P}|$, $x \neq y$, then there exists a B in $\mathscr{B}$ ($\mathscr{B}$ is an open base) which contains only one of the points x or y. It follows that $f_B x \neq f_B y$ and consequently $fx \neq fy$. Thus f is an embedding whenever $\mathscr{P}$ is feebly semi-separated, which establishes the necessity of the condition in statement (a).

IV. It remains to prove the necessity of the condition in the statement (b). For each x in $\mathscr{P}$ let g_x be the mapping of $\mathscr{P}$ into $\mathscr{P}_1$ such that $g_x x = 1$ and $g_x y = 0$ for $y \neq x$. For each $x \in \mathscr{P}$ the mapping $g_x : \mathscr{P} \to \mathscr{P}_1$ is continuous because $\mathscr{P}_1$ is an accrete space and any mapping into an accrete space is continuous. Let g be the reduced product of the family $\{g_x \mid x \in \mathscr{P}\}$, that is, $gx = \{g_y x \mid y \in \mathscr{P}\} \in \mathscr{P}_1^{\mathscr{P}}$ for each $x \in \mathscr{P}$. By **17 C.13** the mapping g is continuous and again by **17 C.13** the reduced product $h = \{x \to \langle fx, gx \rangle\} : \mathscr{P} \to (\mathscr{P}_2^{\mathscr{B}} \times \mathscr{P}_1^{\mathscr{P}})$ is continuous. The mapping h is one-to-one because g is obviously one-to-one. To prove that h is an embedding it remains to show that

$$(**) \qquad \text{if } x \in |\mathscr{P}|, X \subset |\mathscr{P}|, x \notin \overline{X}, \text{ then } hx \notin \overline{h[X]}.$$

But this follows from $(*)$. Indeed, if π is the projection of $\mathscr{P}_2^{\mathscr{B}} \times \mathscr{P}_1^{\mathscr{P}}$ onto $\mathscr{P}_2^{\mathscr{B}}$, then $f = \pi \circ h$ and hence $\pi hx = fx$, $\pi[h[X]] = f[X]$; since π is continuous, $hx \in \overline{h[X]}$ implies $fx = \pi hx \in \overline{\pi[h[X]]} = \overline{f[X]}$ but this contradicts $(*)$. This contradiction establishes $(**)$ and concludes the proof.

26 B.10. Theorem. *A necessary and sufficient condition for a closure space $\mathscr{P}$ to be a semi-separated topological space is that $\mathscr{P}$ be homeomorphic to a subspace of a product space $\mathscr{Q}^{\aleph}$, where $\mathscr{Q}$ is a set of an appropriate cardinal endowed with the coarsest semi-separated closure operation and $\aleph$ is an appropriate cardinal.*

Proof. The sufficiency is an immediate consequence of **26 B.6** and **26 B.8**. To prove necessity, let us suppose that $\mathscr{P}$ is a topological semi-separated space. Let $\mathscr{Q}$ be a set with cardinal at least that of $\mathscr{P}$ endowed with the coarsest semi-separated closure operation and let $\mathscr{C}$ be a closed base of $\mathscr{P}$. Clearly for each C in $\mathscr{C}$ there exists a mapping f_C of $\mathscr{P}$ into $\mathscr{Q}$ such that $f_C[C]$ is a point of $\mathscr{Q}$, $f_C[C] \cap f_C[|\mathscr{P}| - C] = \emptyset$ and the restriction of f_C to $|\mathscr{P}| - C$ is a one-to-one mapping. Since the inverse images under f_C of points are closed and $\mathscr{Q}$ is endowed with the coarsest semi-separated closure operation, the mapping f_C is continuous (see 26 B.8). By 17 C.13 the reduced product f of the family $\{f_C \mid C \in \mathscr{C}\}$ is also continuous. Since clearly f is also one-to-one, to show that f is a homeomorphism it remains to prove the following

$$(*) \qquad \text{if } x \in |\mathscr{P}|,\ X \subset |\mathscr{P}|,\ x \notin \overline{X}, \text{ then } fx \notin \overline{f[X]}.$$

$\mathscr{C}$ being a closed base of $\mathscr{P}$, we can choose a C in $\mathscr{C}$ with $\overline{X} \subset C \subset |\mathscr{P}| - (x)$. We have $f_C[X] \subset f_C[C]$, $(f_C x) \neq f_C[C]$. If π_C is the C-th projection, i.e. $\pi_C \circ f = f_C$, then $\pi_C^{-1}[f_C[C]]$ and $\pi_C^{-1}[f_C x]$ are disjoint closed sets, the first one containing $f[X]$ and the second one fx. Thus $fx \notin \overline{f[X]}$, which establishes $(*)$ and completes the proof.

Remark. It is easy to show that in 26 B.10 the cardinal of $\mathscr{Q}$ essentially depends on P (see ex. 8).

26 B.11. Definition. A *coarse semi-separated closure* operation is defined to be the coarsest semi-separated closure for some set. A *coarse semi-separated space* is a closure space whose closure structure is a coarse semi-separated closure.

27. SEPARATED AND REGULAR SPACES

In the preceding section we studied those properties of closure spaces which depend only on the quasi-discrete modification, i.e. on closures of finite sets. Here we begin the study of separation properties of spaces which cannot be described by means of the quasi-discrete modification.

A closure space is said to be separated if any two distinct points are separated, and a space is said to be regular if $x \notin \overline{X}$ implies that the sets (x) and X are separated. It turns out that each separated space is semi-separated, a separated space need not be regular and a regular feebly semi-separated space is separated. The class of all separated spaces as well as the class of all regular spaces is hereditary and closed under sums and products.

It will appear that both separatedness and regularity often enter into definitions and theorems. The significance of separated spaces is seen from the fact that each of the following two conditions is necessary and sufficient for a closure space $\mathscr{P}$ to be separated: (a) each net in $\mathscr{P}$ has at most one limit point (Theorem 27 A.6); (b) if f and g are continuous mappings of a space $\mathscr{Q}$ into $\mathscr{P}$ and if f and g coincide on a dense subset of $\mathscr{Q}$, then $f = g$ (Theorem 27 A.8).

The importance of regularity will be shown by theorem 27 B.10 on the continuous extension of mappings (if g is a mapping of a topological closure space $\mathscr{Q}$ into a regular space $\mathscr{P}$ such that the domain restriction of g to each subspace $R \cup (x)$ of $\mathscr{Q}$ is continuous, where R is a dense subspace of $\mathscr{Q}$, then g is continuous) and by theorem 27 B.17 on removable discontinuities of a mapping into a regular space. The purely topological theorem 27 B.10 mentioned above will be applied to uniformly continuous extensions of uniformly continuous mappings for uniform spaces (27 B.16).

A. SEPARATED SPACES

27 A.1. Definition. A closure *space* $\mathscr{P}$ is said to be *separated* if any two distinct points of $\mathscr{P}$ are separated. It is to be noted that separated topological spaces are often called *Hausdorff spaces* or *T_2-spaces.*

Since any two separated sets are semi-separated, every separated space is semi-separated. On the other hand, a semi-separated space need not be separated. For example, an infinite set endowed with the coarsest semi-separated closure operation is not separated. By 20 A.5 two sets X and Y are separated if and only if the closure of a neighborhood of X does not intersect Y. This fact enables us to restate the definition of separated spaces as follows:

27 A.2. *In order that a closure space $\mathscr{P}$ be separated it is necessary and sufficient that, for each point x of $\mathscr{P}$, the intersection of closures of all neighborhoods of x is (x).*

27 A.3. Theorem. *The class of all separated closure spaces is hereditary and closed under sums and products.*

Proof. If $\mathscr{Q}$ is a subspace of $\mathscr{P}$ and U and V are disjoint neighborhoods of $x \in \mathscr{Q}$ and $y \in \mathscr{Q}$ in $\mathscr{P}$, then $U \cap |\mathscr{Q}|$ and $V \cap |\mathscr{Q}|$ are disjoint neighborhoods of x and y in $\mathscr{Q}$. It follows that subspaces of separated spaces are separated. That the class is closed under products and sums follows from the following more general result.

27 A.4. *Each of the following two conditions is necessary and sufficient for a closure space $\mathscr{P}$ to be separated:*

(a) *For any distinct points x and y of $\mathscr{P}$ there exists a continuous mapping f of $\mathscr{P}$ into a separated space $\mathscr{Q}$ such that $fx \neq fy$.*

(b) *For each point x of $\mathscr{P}$ there exists a neighborhood U of x such that the subspace $\overline{U}$ of $\mathscr{P}$ is separated.*

Indeed, if $\mathscr{P}$ is the product of a family $\{\mathscr{P}_a\}$ of separated spaces and x and y are distinct points of $\mathscr{P}$, then there exists an index a such that $\pi_a x \neq \pi_a y$, where π_a is the projection of $\mathscr{P}$ onto $\mathscr{P}_a$. If $\mathscr{P}$ is the sum of a family $\{\mathscr{P}_a\}$ of separated spaces, then the subspace $\operatorname{inj}_a [\mathscr{P}_a] = \mathsf{E}\{\langle a, x\rangle \mid x \in \mathscr{P}_a\}$ of $\mathscr{P}$ is simultaneously open and closed (17 B.2) and homeomorphic with $\mathscr{P}_a$ (17 B.2).

Proof of 27 A.4. If $\mathscr{P}$ is separated, $\mathscr{P} = \mathscr{Q}$, f is the identity mapping of $\mathscr{P}$ onto $\mathscr{P}$ and $U = |\mathscr{P}|$, then conditions (a) and (b) are fulfilled. The sufficiency of (a) is a consequence of the fact that inverse images under a continuous mapping of separated sets are separated (see 20 A.8). To prove that (b) is sufficient, assume $x, y \in \mathscr{P}$, $x \neq y$ and U is a neighborhood of x such that $\overline{U}$ is a separated subspace of $\mathscr{P}$. If $y \notin \overline{U}$, then x and y are obviously separated. If $y \in \overline{U}$, then there exist disjoint neighborhoods U_1 of x and V_1 of y in $\overline{U}$. Obviously, U_1 and $V_1 \cup (|\mathscr{P}| - \overline{U})$ are disjoint neighborhoods of x and y in $\mathscr{P}$.

27 A.5. Examples. (a) Every generalized ordered space is separated. To prove this, let u be a generalized order closure for a monotone ordered set $\langle P, \leqq\rangle$ and let x and y be any two distinct points of P. We have $x < y$ or $y < x$. We may and shall assume that $x < y$; if the interval $]\!]\, x, y \,[\![$ is empty then the intervals $[\![\leftarrow, y \,[\![$ and $]\!]\, x, \rightarrow]\!]$ are disjoint neighborhoods of x and y in $\langle P, u\rangle$, and if $]\!]\, x, y \,[\![\neq \emptyset$, then we can choose a z such that $x < z < y$, and then clearly $[\![\leftarrow, z \,[\![$ and $]\!]\, z, \rightarrow]\!]$ are disjoint neighborhoods of x and y in $\langle P, u\rangle$. In particular, the space of reals is separated. By 27 A.3 subspaces of products of generalized ordered spaces are separated.

(b) Every metrizable space is separated. Indeed, if a closure u for a set P is induced by a metric d and if x and y are distinct points of $\langle P, u\rangle$ then $f = \{z \to d\langle x, z\rangle\} : \langle P, u\rangle \to \mathsf{R}$ is a continuous mapping of $\langle P, u\rangle$ into a separated space such that $fx \neq fy$.

(c) Every feebly semi-separated pseudometrizable space is metrizable and hence separated. Indeed, if a feebly semi-separated closure u for a set P is induced by a pseudometric d, then clearly d is a metric.

(d) According to Conventions 23 C.6 a semi-uniform space $\langle P, \mathscr{U}\rangle$ is said to be separated, semi-separated or feebly semi-separated if the induced closure space has this property. It is easily seen that a semi-separated semi-uniform space need not be separated, and the following conditions on a semi-uniform space $\langle P, \mathscr{U}\rangle$ are equivalent: $\langle P, \mathscr{U}\rangle$ is semi-separated, $\langle P, \mathscr{U}\rangle$ is feebly semi-separated, the intersection of $\mathscr{U}$ is the diagonal of $P \times P$. On the other hand a semi-separated uniform space $\langle P, \mathscr{U}\rangle$ is necessarily separated. Indeed, if x and y are distinct points of P and $\langle x, y\rangle \notin U \in \mathscr{U}$, then we can choose a symmetric element V of $\mathscr{U}$ so that $V \circ V \subset U$; clearly $V[x]$ and $V[y]$ are disjoint neighborhoods of x and y in $\langle P, \mathscr{U}\rangle$.

(e) It is worth noticing that each of the following two conditions is necessary and sufficient for a uniform space $\langle P, \mathscr{U}\rangle$ to be separated: if x and y are distinct points of P then there exists a uniformly continuous or continuous pseudometric d for $\langle P, \mathscr{U}\rangle$ such that $d\langle x, y\rangle \neq 0$.

(f) Condition (b) of 27 A.4 cannot be weakened by requiring U to be a separated subspace instead of $\overline{U}$ being separated. For example, let x be a cluster point of a separated space $\langle P, v\rangle$ and let Q be a set consisting of all elements of $P - (x)$ as well as two distinct elements x_1 and x_2. Let us consider the closure u for Q such that $P - (x)$ is a subspace of both $\langle P, v\rangle$ and $\langle Q, u\rangle$ and, if $X \subset Q$, then $x_i \in uX$ if and only if $x_i \in X$ or $x \in v(X \cap (P - (x))$. Clearly $\langle Q, u\rangle$ is a semi-separated space. On the other hand, $\langle Q, u\rangle$ is not separated because the points x_1 and x_2 are not separated. Moreover, both $Q - (x_i)$, $i = 1, 2$, are open subspaces of Q homeomorphic to the separated space $\langle P, v\rangle$.

We have already shown that a net in a closure space may be convergent to many points, and moreover, in an accrete space, every point is a limit point of every net. Now it will be shown that in a separated space a net possesses at most one limit point and that this property characterizes the separated spaces.

27 A.6. Theorem. *In order that a closure space $\mathscr{P}$ be separated it is necessary and sufficient that every net in $\mathscr{P}$ possess at most one limit point.*

Proof. If x is a limit point of a net N, and if U is any neighborhood of x, then clearly all accumulation points of N are elements of $\overline{U}$. If $\mathscr{P}$ is separated, then the intersection of closures of neighborhoods of x is a one-point set (x). The necessity follows. Conversely, suppose that $\mathscr{P}$ is not separated. There exist two distinct points x and y of $\mathscr{P}$ such that $W = U \cap V \neq \emptyset$ for any neighborhoods U of x and V of y. Let $\mathscr{W}$ be the collection of all such W and $\{N_W \mid W \in \mathscr{W}\}$ a family such that $N_W \in W$ for each

W in $\mathscr{W}$. Obviously $\mathscr{W}$ is directed by the inverse inclusion $\supset$ and the net $\{N_W\}$ is convergent to both x and y.

27 A.7. *Each of the following three conditions is necessary and sufficient for a closure space $\mathscr{P}$ to be separated:*

(a) *If $\mathscr{Q}$ is a space and $\{f_a \mid a \in A\}$ is a family of continuous mappings of $\mathscr{Q}$ into $\mathscr{P}$, then the set $D = \mathsf{E}\{x \mid x \in \mathscr{Q}, f_{a_1}x = f_{a_2}x$ for each a_1 and a_2 in $A\}$ is closed in $\mathscr{Q}$.*

(b) *If $\mathscr{Q}$ is the product of a family $\{\mathscr{Q}_a \mid a \in A\}$ of subspaces of $\mathscr{P}$ then the set $\mathsf{E}\{x \mid x = \{x_a\} \in \mathscr{Q}, x_{a_1} = x_{a_2} \in \bigcap\{\mathscr{Q}_a \mid a \in A\}$ for each a_1 and a_2 in $A\}$ is closed in $\mathscr{Q}$.*

(c) *The diagonal $\Delta (= \mathsf{E}\{\langle x, x\rangle \mid x \in \mathscr{P}\})$ of $\mathscr{P} \times \mathscr{P}$ is closed in the product space $\mathscr{P} \times \mathscr{P}$.*

Proof. It will be shown that (a) implies (b), (b) implies (c), condition (c) is sufficient and (a) is necessary.

I. The implication (a) $\Rightarrow$ (b) is almost self-evident. For each a in A let f_a be the mapping $\mathrm{pr}_a : \mathscr{Q} \to \mathscr{P}$. Since $x \in D$ if and only if $\mathrm{pr}_{a_1}x = \mathrm{pr}_{a_2}x$ for each a_1 and a_2 in A, the set D from (a) coincides with the set described in (b).

II. Applying (b) to $A = (1, 2)$, $\mathscr{Q}_1 = \mathscr{Q}_2 = \mathscr{P}$, f_i the identity mapping of $\mathscr{Q}_i$ onto $\mathscr{P}$, we obtain (c), because $f_1 \times f_2$ is a homeomorphism of $\mathscr{Q}$ onto $\mathscr{P} \times \mathscr{P}$ which carries the set from (b) onto the diagonal of $\mathscr{P} \times \mathscr{P}$.

III. The sufficiency of (c) is proved as follows. Let us suppose that Δ is closed in $\mathscr{P} \times \mathscr{P}$. If x and y are two distinct points of $\mathscr{P}$, then $\langle x, y\rangle \notin \Delta$, and consequently there exists a canonical neighborhood $U \times V$ of $\langle x, y\rangle$ which does not intersect Δ, that is, $U \cap V = \emptyset$. Now the sets U and V are disjoint neighborhoods of x and y respectively.

IV. It remains to prove the most delicate part, namely, the necessity of condition (a). Suppose that $\mathscr{P}$ is separated. To prove that D is closed, that is, $\bar{D} \subset D$, it is sufficient to show that if N is a net in D which converges to a point x of $\mathscr{Q}$, then $x \in D$. Let $N = \{N_b\}$ be a net in D which is convergent to a point x of $\mathscr{Q}$. Since all mappings f_a are continuous, for each a in A the net $f_a \circ N$ is convergent to the point $f_a x$. However, N is in D and hence $f_{a_1}N_b = f_{a_2}N_b$ for each a_1 and a_2 in A. It follows that all nets $f_a \circ N$ coincide, and consequently, all points $f_a x$ are limit points of a net in $\mathscr{P}$. According to the preceding theorem, all points $f_a x$, $a \in A$, coincide. By the definition of D, $x \in D$; this establishes the necessity of (a) and concludes the proof of the theorem.

The preceding theorem has two corollaries which will be stated as theorems because of their importance.

27 A.8. Uniqueness Theorem. *Let f and g be two continuous mappings of a closure space $\mathscr{Q}$ into a separated closure space $\mathscr{P}$. If $fx = gx$ for each x from a dense subset X of $\mathscr{Q}$, then $f = g$.*

Corollary. *If $\mathscr{P}$ is a separated space and the density character of a space $\mathscr{Q}$ is at most $\aleph$, then the cardinal of $\mathsf{C}(\mathscr{Q}, \mathscr{P})$ is at most* (card $|\mathscr{P}|)^{\aleph}$. *In particular, if $\mathscr{Q}$ is of a countable density character, then the cardinal of $\mathsf{C}(\mathscr{Q}, \mathsf{R})$ is* $\exp \aleph_0$.

Proof. Let X be a dense subset of $\mathcal{Q}$ with cardinal at most $\aleph$. Let π be the mapping of $\mathbf{C}(\mathcal{Q}, \mathcal{P})$ into $\mathbf{C}(X, \mathcal{P})$ which carries each f into its restriction $f \mid X$ to X. According to the preceding uniqueness theorem, the mapping is one-to-one, and consequently the cardinal of $\mathbf{C}(\mathcal{Q}, \mathcal{P})$ is less than or equal to the cardinal of $\mathbf{C}(X, \mathcal{P})$. But the cardinal of $\mathbf{C}(X, \mathcal{P})$ is less than or equal to the cardinal of all mappings of X into $\mathcal{P}$, that is, $(\text{card } |\mathcal{P}|)^{\text{card } X} \leqq (\text{card}|\mathcal{P}|)^{\aleph}$. The second part is an immediate consequence of the facts that R is separated (example 27 A.5), the cardinal of R is $\exp \aleph_0$ and $(\exp \aleph_0)^{\aleph_0} = \exp (\aleph_0 \,.\, \aleph_0) = \exp \aleph_0$.

27 A.9. Theorem. *The graph of a continuous mapping f of a closure space $\mathcal{P}$ into a separated closure space $\mathcal{Q}$ is a closed subset of the product space $\mathcal{P} \times \mathcal{Q}$.*

Proof. The graph of f is the set of all points $\langle x, fx \rangle \in \mathcal{P} \times \mathcal{Q}, x \in \mathcal{P}$. The mappings $f_1 = \{\langle x, y \rangle \to y\}$ and $f_2 = \{\langle x, y \rangle \to fx\}$ of $\mathcal{P} \times \mathcal{Q}$ into $\mathcal{Q}$ are continuous because f_1 is the projection of the product onto a coordinate space and f_2 is the composite of the projection of $\mathcal{P} \times \mathcal{Q}$ onto $\mathcal{P}$, which is continuous, and of the mapping f which is continuous by our assumption. But the graph of f is identical with the set D of all $z \in \mathcal{P} \times \mathcal{Q}$ such that $f_1 z = f_2 z$, and the set D is closed in $\mathcal{P} \times \mathcal{Q}$ by 27 A.7.

Remark. If the graph of a mapping f of a space $\mathcal{P}$ into a space $\mathcal{Q}$ is closed in the product space $\mathcal{P} \times \mathcal{Q}$, then f need not be continuous (see ex. 5), but it is continuous if $\mathcal{P}$ is compact (41 C.6).

27 A.10. Definition. A *subset* X of a closure space $\mathcal{P}$ will be called *relatively discrete* if X is a discrete subspace of $\mathcal{P}$, that is, if the family $\{(x) \mid x \in X\}$ is discrete in the subspace X of $\mathcal{P}$. A *subset* X of $\mathcal{P}$ is called *discrete* if the family $\{(x) \mid x \in X\}$ is discrete in $\mathcal{P}$. It is to be noted that in the literature a relatively discrete subset of $\mathcal{P}$ is sometimes called an isolated subset of $\mathcal{P}$ or a discrete subset of $\mathcal{P}$.

27 A.11. *Let $\mathcal{P}$ be a closure space. Every discrete subset of $\mathcal{P}$ is relatively discrete. A closed relatively discrete subset of $\mathcal{P}$ is discrete. A discrete subset of a semi-separated space is closed. In particular, in a semi-separated space a set is discrete if and only if it is closed and relatively discrete.*

Proof. If X is a closed relatively discrete subset of a space $\mathcal{P}$, then $|\mathcal{P}| - X$ is a neighborhood of each of its points which intersects no member of $\{(x) \mid x \in X\}$, and $(y) \cup (|\mathcal{P}| - X)$, $y \in X$, is a neighborhood of y which intersects exactly one member of the family, namely (y). If X is a discrete subset of $\mathcal{P}$, that is, the family $\{(x) \mid x \in X\}$ is discrete, then the family is closure-preserving, and consequently $\overline{X} = \bigcup\{\overline{(x)} \mid x \in X\}$. If $\mathcal{P}$ is semi-separated, then the one-point sets are closed and hence $\overline{X} = X$ which means that X is closed. The remaining statements are obvious.

It should be noted that a relatively discrete set need not be discrete. For example, the set X of all $1/n$, $n = 1, 2, \ldots$, is relatively discrete but not discrete in R (each neighborhood of zero meets X in an infinite set).

A semi-separated infinite space may contain no infinite relatively discrete subset. For example, no space whose closure structure is the coarsest semi-separated closure contains an infinite relatively discrete subset. This is not the case, however, if the space is separated, as stated in the following proposition.

27 A.12. *Every infinite separated space P contains an infinite relatively discrete subspace X, that is, an infinite discrete space.*

Proof. If P is discrete, then we can put $X = P$. I. If P is not discrete, then there exists a point x of P such that $x \in \overline{P - (x)}$. By induction it is easy to prove that there exists a sequence $\{x_n\}$ of points and a decreasing sequence $\{U_n\}$ of neighborhoods of x such that

(a) $x_n \in \operatorname{int}(P - U_n) = P - \overline{U}_n$ for each n; and

(b) $x_{n+m} \in U_n$ for each $n \in \mathsf{N}$ and $m \in \mathsf{N}$, $m \geqq 1$.

Indeed, if x_0 is any point of $P - (x)$, then there exists a neighborhood U_0 of x such that $x_0 \notin \overline{U}_0$ because of 27 A.2. If finite sequences $\{U_n \mid n \leqq k\}$ and $\{x_n \mid n \leqq k\}$ possess the required properties, (condition (b) under the additional hypothesis $n + m \leqq k$), then we can choose an x_{k+1} in $U_k - (x)$ for $x \in \overline{U_k - (x)}$ because U_k is a neighborhood of x and $x \in \overline{P - (x)}$. By 27 A.2 we can choose a neighborhood V of x such that $x_{k+1} \notin \overline{V}$. Setting $U_{k+1} = U_k \cap V$ we obtain finite sequences $\{U_n \mid n \leqq k + 1\}$ and $\{x_n \mid n \leqq k + 1\}$ with the required properties, condition (b) being fulfilled under the additional hypothesis that $n + m \leqq k + 1$.

II. Now let $\{U_n\}$ be a decreasing sequence of sets and $\{x_n\}$ a sequence of points such that conditions (a) and (b) are fulfilled. If $n > m$, then $x_m \in (P - U_m)$ and $x_n \in U_{n-1} \subset U_m$ and hence $x_n \neq x_m$. It follows that the sequence $\{x_n\}$ is one-to-one and hence the set X of all x_n, $n \in \mathsf{N}$, is infinite. If y is any point of X, say $y = x_n$, then $P - U_n$ is a neighborhood of x_n which contains x_k for $k \leqq n$ only. Since P is separated and hence semi-separated, the set $(x_0, x_1, \ldots, x_{n-1})$ is closed and the set $(P - U_n) - (x_0, x_1, \ldots, x_{n-1})$ is a neighborhood of $x_n = y$ which intersects X in only one point, namely $y = x_n$.

Corollary. *The cardinal of the collection of all closed subsets of an infinite separated topological space is at least* $\exp \aleph_0$.

Proof. If $\mathscr{P}$ is an infinite separated space, then there exists an infinite relatively discrete subset X of $\mathscr{P}$. The cardinal of $\exp X$ is at least $\exp \aleph_0$. If $X_1, X_2 \in \exp X$, $X_1 \neq X_2$, then $\overline{X}_1 \neq \overline{X}_2$ because $\overline{X}_1 \cap X = X_1 \neq X_2 = \overline{X}_2 \cap X$. It follows that $\{Y \to \overline{Y}\}$ is a one-to-one mapping of $\exp X$ into $\exp \mathscr{P}$. If $\mathscr{P}$ is topological, then each $\overline{Y}$ is closed, and consequently $\mathscr{Y} = \mathsf{E}\{\overline{Y} \mid Y \in \exp X\}$ is a collection of closed subsets of $\mathscr{P}$ and the cardinal of $\mathscr{Y}$ is that of $\exp X$.

B. REGULAR SPACES

We begin with a definition and various characterizations of regular spaces.

27 B.1. Definition. A closure *space* $\mathscr{P}$ is said to be *regular* if for each point x of $\mathscr{P}$ and each subset X of $\mathscr{P}$ such that $x \notin \overline{X}$ there exist neighborhoods U of x and V of X

such that $U \cap V = \emptyset$; stated in other words, if $x \in |\mathscr{P}| - \overline{X}$, then the sets (x) and X are separated.

From the definition we obtain immediately the following characterizations of regular spaces (compare with 27 A.2):

27 B.2. *Each of the following two conditions is necessary and sufficient for a closure space $\mathscr{P}$ to be regular:*

(a) *For each x in $\mathscr{P}$ and each neighborhood U of x in $\mathscr{P}$ there exists a neighborhood V of x such that $\overline{V} \subset U$.*

(b) *The closure of any subset X of $\mathscr{P}$ is the intersection of closures of neighborhoods of X.*

Corollary. *In a regular space any closed set is the intersection of closures of its neighborhoods.*

The non-trivial part of the next theorem, which corresponds to a similar result in 27 A.4 for separated spaces, asserts that a space is regular whenever there exist sufficiently many continuous mapping into regular spaces as well as whenever the space satisfies a somewhat stronger condition than "feeble local regularity".

27 B.3. *Each of the following two conditions is necessary and sufficient for a closure space $\mathscr{P}$ to be regular:*

(a) *If $x \in |\mathscr{P}| - \overline{X}$, then there exists a continuous mapping f of $\mathscr{P}$ into a regular space such that fx does not belong to the closure of $f[X]$.*

(b) *Each point of $\mathscr{P}$ has a neighborhood W in $\mathscr{P}$ such that the subspace $\overline{W}$ of $\mathscr{P}$ is regular.*

Proof. Clearly both conditions are necessary. I. Assuming (a), let $x \in (|\mathscr{P}| - \overline{X})$ and f be the corresponding continuous mapping into a regular space. Since the sets (fx) and $f[X]$ are separated in $\mathbf{E}^*f$, by 20 A.8 the sets $f^{-1}[fx]$ and $f^{-1}[f[X]]$ are separated in $\mathscr{P}$ and hence the sets (x) and X are separated.

II. Assuming (b) we shall prove that condition (a) of 27 B.2 is fulfilled. Let U be a neighborhood of x in $\mathscr{P}$ and let W be a neighborhood of x in $\mathscr{P}$ such that the subspace $\overline{W}$ of $\mathscr{P}$ is regular. The set $W \cap U$ is a neighborhood of x in $\mathscr{P}$ and hence in $\overline{W}$, and therefore, by 27 B.2, there exists a neighborhood V of x in $\overline{W}$ such that the relative closure $\overline{V} \cap \overline{W}$ of V in $\overline{W}$ is contained in $W \cap U$. Clearly V is a neighborhood of x in $\mathscr{P}$ and $\overline{V} \subset W \cap U \subset U$ (observe that $\overline{V} \subset \overline{W}$ and hence $\overline{V} = \overline{V} \cap \overline{W}$).

In 25 A.18 we introduced various continuous proximities for a closure space $\langle P, u\rangle$. If p is the proximity $\mathbf{E}\{\langle X, Y\rangle \mid X$ and Y are not semi-separated$\}$, then p induces the closure u if and only if $\langle P, u\rangle$ is semi-uniformizable, and each two distinct points x and y are distant in $\langle P, p\rangle$ if and only if the space $\langle P, u\rangle$ is semi-separated. Now let p be the proximity $\mathbf{E}\{\langle X, Y\rangle \mid X$ and Y are not separated in $\langle P, u\rangle\}$; we have seen that every two distinct points of $\mathscr{P}$ are distant in $\langle P, p\rangle$ if and only if the space $\langle P, u\rangle$ is separated. From the definition of regularity we obtain immediately the following

27 B.4. Theorem. *A closure space $\mathscr{P}$ is regular if and only if the closure structure of $\mathscr{P}$ is induced by the proximity* $\mathbf{E}\{\langle X, Y\rangle \mid X \subset |\mathscr{P}|,\ Y \subset |\mathscr{P}|$, *$X$ and Y are not separated in $\mathscr{P}$*$\}$. *In particular, every regular space is semi-uniformizable.*

According to Definition 21 A.9 a space is said to be locally closed if each neighborhood of any point x contains a closed neighborhood of x. Evidently every locally closed space is regular, but example 27 B.9 (f) will show that a regular space need not be locally closed. Nevertheless, for topological spaces, the property of being locally closed is equivalent to regularity as stated in the next theorem.

27 B.5. Theorem. *Each of the following two conditions is necessary and sufficient for a topological space $\mathscr{P}$ to be regular:*

(a) *$\mathscr{P}$ is locally closed.*

(b) *Every closed subset of $\mathscr{P}$ is the intersection of closures of its neighborhoods.*

Proof. If $\mathscr{P}$ is a topological space, then conditions (a) of 27 B.2 and (a) of 27B.5 are equivalent, and similarly for conditions (b) of 27 B.2 and (b) of 27 B.5.

Sometimes the following two characterizations are convenient:

27 B.6. *Each of the following two conditions is necessary and sufficient for a topological space $\mathscr{P}$ to be regular:*

(a) *Every open subset U of $\mathscr{P}$ is the union of open sets whose closures are contained in U.*

(b) *Every closed subset of $\mathscr{P}$ is the intersection of its neighborhoods.*

Now we turn to an examination of the class of all regular spaces.

27 B.7. *Every uniformizable space, in particular, every pseudometrizable space, is regular and every regular space is semi-uniformizable. A feebly semi-separated regular space is separated.*

Proof. The first statement follows from 24 A.2 (b), the second has already been proved (27 B.4) and the last one can be proved as follows: if x and y are two distinct points of a feebly semi-separated space $\mathscr{P}$, then one of them does not belong to the closure of the other one, and hence, by regularity, the sets (x) and (y) are separated.

On the other hand a regular space need not be uniformizable (the corresponding example is not trivial, see example 33 D.6) nor separated (every accrete space is regular), and a semi-uniformizable space need not be regular (27 B.9).

27 B.8. Theorem. *The class of all regular spaces is hereditary and closed under products and sums.*

Proof. I. If $\mathscr{Q}$ is a subspace of a regular space $\mathscr{P}$ and U is a neighborhood of x in $\mathscr{Q}$, then there is a neighborhood U_1 of x in $\mathscr{P}$ such that $U_1 \cap |\mathscr{Q}| = U$; by 27 B.2 we can choose a neighborhood V_1 of x in $\mathscr{P}$ so that $\overline{V}_1^{\mathscr{P}} \subset U_1$. Clearly the closure in $\mathscr{Q}$ of the neighborhood $V_1 \cap |\mathscr{Q}|$ of x is contained in U. By 27 B.2 $\mathscr{Q}$ is regular.

II. Invariance under sums follows from 27 B.3 because, if $\mathscr{P}$ is the sum of a family

$\{\mathscr{P}_a\}$ of closure spaces, then the subspace $\mathrm{inj}_a[\mathscr{P}_a]$ of $\mathscr{P}$ is a homeomorph of $\mathscr{P}_a$ and the set $\mathrm{inj}_a[\mathscr{P}_a]$ is simultaneously closed and open in $\mathscr{P}$.

III. Now let $\mathscr{P}$ be the product of a family $\{\mathscr{P}_a \mid a \in A\}$ of regular spaces and let W be a neighborhood of a point x of $\mathscr{P}$. Choose a canonical neighborhood $U = \mathbf{E}\{x \mid a \in A_0 \Rightarrow \mathrm{pr}_a x \in U_a\}$ of x contained in W, where A_0 is a finite subset of A and each U_a, $a \in A_0$, is necessarily a neighborhood of $\mathrm{pr}_a x$ in $\mathscr{P}_a$. For each a in A_0 let V_a be a neighborhood of $\mathrm{pr}_a x$ such that the closure of V_a (in $\mathscr{P}_a$) is contained in U_a. Clearly the closure of the neighborhood $V = \mathbf{E}\{y \mid a \in A_0 \Rightarrow \mathrm{pr}_a y \in V_a\}$ of x is contained in U.

27 B.9. Examples. (a) *Every generalized ordered space is regular* because, for each point x, the order-closed intervals which are neighborhoods of x form a local base at x, and order-closed intervals are closed.

(b) *Every regular quasi-discrete space is topological.* Assuming $x \in \overline{(y)}$, $y \in \overline{(z)}$, we shall prove $x \in \overline{(z)}$. The space is regular and therefore it is sufficient to show that the closure of each neighborhood of x contains z. Let U be a neighborhood of x. We have $y \in U$ becaeus $x \in \overline{(y)}$. Next, the space is regular and so semi-uniformizable, and hence $y \in \overline{(z)}$ implies $z \in \overline{(y)}$. Thus $x \in \overline{U}$.

(c) A regular separated space need not be topological; e.g. take a separated regular space $\mathscr{P}$ such that the set X of all cluster points of $\mathscr{P}$ is infinite and discrete in $\mathscr{P}$ (e.g. take an infinite separated regular space $\mathscr{R}$ with exactly one cluster point and put $\mathscr{P} = \Sigma\{\mathscr{R} \mid a \in A\}$ where A is an infinite set), choose a free filter $\mathscr{U}$ on X (i.e. a filter $\mathscr{U}$ on X such that $\bigcap\mathscr{U} = \emptyset$) and add a point, say x, to the set $|\mathscr{P}|$; now consider the closure space $\mathscr{Q}$ such that $|\mathscr{Q}| = |\mathscr{P}| \cup (x)$, $\mathscr{P}$ is an open subspace of $\mathscr{Q}$ and $(x) \cup [\mathscr{U}]$ is a local base at x. It is easily seen that $\mathscr{Q}$ is a regular space (the set $X \cup (x)$ is closed in $\mathscr{Q}$, and hence every set $(x) \cup U$, $U \in \mathscr{U}$, is closed in $\mathscr{Q}$). On the other hand, $\mathscr{Q}$ is not topological because the closure of $|\mathscr{P}| - X$ is $|\mathscr{P}|$ and the closure of $|\mathscr{P}|$ is $|\mathscr{Q}|$ $(\neq |\mathscr{P}|)$.

(d) Let $\langle P, u\rangle$ be a closure space and let X be a subset of P. Consider the closure v for P such that $vY = X \cap uY$ if $Y \subset X$ and $vY = uY$ if $Y \subset (P - X)$. Notice that the relativizations of u and v to X coincide and X is closed in $\langle P, u\rangle$ if and only if $u = v$. Always v is finer than u and X is closed in $\langle P, v\rangle$. Now if $\langle P, u\rangle$ is a topological space, if $P - X$ is dense in $\langle P, u\rangle$ and there exists an $x \in uX - X$, then the space $\langle P, v\rangle$ is not regular because the closure (in $\langle P, v\rangle$) of each neighborhood of x in $\langle P, v\rangle$ intersects X but $P - X$ is an open neighborhood of x in $\langle P, v\rangle$.

(e) Let $\langle P, u\rangle$ be the space R of reals and let X be the set of all rational numbers. The space $\langle P, u\rangle$ is regular and separated but the space $\langle P, v\rangle$, where v is the closure constructed in (d), is not regular by (d).

(f) If $\langle P, u\rangle$ is the ordered space $[\![0, 1]\!]$ and X is the set consisting of all n^{-1}, $n = 1, 2, \ldots$, then space $\langle P, v\rangle$ is not regular (by (d)) but $\langle P, v\rangle$ is clearly locally regular, i.e., every point has arbitrarily small neighborhoods U such that the subspace U of $\langle P, v\rangle$ is regular. Moreover, $\langle P, v\rangle$ is separated because $\langle P, u\rangle$ is separated and v is finer than u. Thus a separated locally regular space need not be regular.

Now we proceed to the domain-extension of continuous mappings. Let f be a continuous mapping of a subspace $\mathscr{R}$ of a space $\mathscr{Q}$ into a space $\mathscr{P}$. It is natural to ask under what conditions there exists a continuous mapping g of $\mathscr{Q}$ into $\mathscr{P}$ such that f is the restriction of g, i.e., under what conditions f has a continuous domain-extension to $\mathscr{Q}$. Here we will be concerned with the case when $\mathscr{R}$ is dense in $\mathscr{Q}$. The answer is simple if $|\mathscr{Q}| - |\mathscr{R}|$ is a singleton, say (x), and either the set (x) is closed or both spaces $\mathscr{P}$ and $\mathscr{Q}$ are semi-uniformizable. Then, by 17 ex. 4, there exists a continuous domain-extension of f to $\mathscr{Q}$ if and only if there exists a point y of $\mathscr{P}$ such that each neighborhood V of y contains a set $f[U \cap |\mathscr{R}|]$ where U is an appropriate neighborhood of x (depending on V). The following important theorem (with the essential assumptions that $\mathscr{Q}$ is topological and $\mathscr{P}$ is regular) reduces the general problem to the existence of continuous domain-extensions to each subspace $|\mathscr{R}| \cup (x)$ of $\mathscr{Q}$.

27 B.10. Extension Theorem. *A continuous mapping f of a dense subspace $\mathscr{R}$ of a topological space $\mathscr{Q}$ into a regular space $\mathscr{P}$ has a continuous domain-extension to $\mathscr{Q}$ if and only if f has a continuous domain-extension on each subspace $\mathscr{R} \cup (x)$, $x \in |\mathscr{Q}| - |\mathscr{R}|$.*

Proof. Every restriction of a continuous mapping is a continuous mapping and therefore "only if" is true without any assumption on $\mathscr{P}$, $\mathscr{R}$ and $\mathscr{Q}$. Suppose that for each x in $|\mathscr{Q}| - |\mathscr{R}|$ there exists a continuous domain-extension g_x of f to the subspace $\mathscr{R} \cup (x)$ of $\mathscr{Q}$, and let us consider the mapping g of $\mathscr{Q}$ into $\mathscr{P}$ such that $gx = fx$ if $x \in \mathscr{R}$ and $gx = g_x x$ if $x \in |\mathscr{Q}| - |\mathscr{R}|$. By definition, g is a domain-extension of f to $\mathscr{Q}$; we shall prove that g is continuous. I. First we shall prove that

$$(*) \qquad g[U] \subset \overline{f[U \cap |\mathscr{R}|]}^{\mathscr{P}}$$

for each open subset U of $\mathscr{Q}$. Assuming $x \in U$, it is required to show that gx belongs to the closure in $\mathscr{P}$ of the set $f[U \cap |\mathscr{R}|]$. If $x \in \mathscr{R}$, then $gx = fx$ and hence gx belongs to the set $f[U \cap |\mathscr{R}|]$ and therefore to its closure. Now suppose $x \in$ $\in |\mathscr{Q}| - |\mathscr{R}|$. The set U is a neighborhood of x and therefore $x \in \overline{U \cap |\mathscr{R}|}^{\mathscr{Q}}$ because $\mathscr{R}$ is dense in $\mathscr{Q}$. It follows that x belongs to the closure of $U \cap |\mathscr{R}|$ in $\mathscr{R} \cup (x)$. Now the mapping g_x is continuous at x and hence the point $g_x x$ belongs to the closure of $g[U \cap |\mathscr{R}|]$ in $\mathscr{P}$; but $f[U \cap |\mathscr{R}|] = g_x[U \cap |\mathscr{R}|]$ and $gx = g_x x$ and therefore $gx \in \overline{f[U \cap |\mathscr{R}|]}^{\mathscr{P}}$. — II. Now let $x \in \mathscr{Q}$ and let W be a neighborhood of gx in $\mathscr{P}$. We must find a neighborhood U of x so that $g[U] \subset W$. First, $\mathscr{P}$ being regular, we can choose a neighborhood V of gx the closure of which is contained in W. Now if $x \in \mathscr{R}$, then we can choose a neighborhood U_1 of x in $\mathscr{R}$ so that $f[U_1] \subset V$ (because f is continuous at x), and then an open neighborhood U of x in $\mathscr{Q}$ such that $U \cap |\mathscr{R}| \subset$ $\subset U_1$; by $(*)$ $g[U]$ is contained in $\overline{f[U \cap |\mathscr{R}|]}^{\mathscr{P}} \subset \overline{f[U_1]}^{\mathscr{P}} \subset \overline{V}^{\mathscr{P}}$ and hence in W. If $x \in |\mathscr{Q}| - |\mathscr{R}|$, then we can choose a neighborhood U_1 of x in $|\mathscr{R}| \cup (x)$ such that $g_x[U_1] \subset V$ (since g_x is continuous at x and $gx = g_x x$), and then an open neighborhood U of x in $\mathscr{Q}$ such that $U \cap (|\mathscr{R}| \cup (x)) \subset U_1$ and hence $(U \cap |\mathscr{R}|) \subset$

$\subset (U_1 \cap |\mathscr{R}|)$. Now by virtue of $(*)$ the set $g[U]$ is contained in the closure of $f[U \cap |\mathscr{R}|] \subset f[U_1 \cap |\mathscr{R}|] = g_x[U_1 \cap |\mathscr{R}|] \subset g_x[U_1] \subset V$ and hence in W.

27 B.11. Theorem. *Suppose that f is a continuous mapping of a dense subspace $\mathscr{R}$ of a topological space $\mathscr{Q}$ into a regular space $\mathscr{P}$. There exists a unique subspace $\mathscr{S}$ of $\mathscr{Q}$ with the following property: if g is any continuous domain-extension of f to a subspace $\mathscr{S}_1$ of $\mathscr{Q}$, then g is the domain-restriction of some continuous domain-extension of f to the subspace $\mathscr{S}$ of $\mathscr{Q}$, and in particular $|\mathscr{S}_1| \subset |\mathscr{S}|$.*

Supplement. $\mathscr{S}$ consists of all the points x of $\mathscr{Q}$ such that f has a continuous domain-extension to the subspace $\mathscr{R} \cup (x)$ of $\mathscr{Q}$, i.e., at least if $\mathscr{Q}$ is semi-uniformizable then $\mathscr{S}$ consists of all $x \in \mathscr{Q}$ for which there exists a $y \in \mathscr{P}$ such that each neighborhood V of y contains a set $f[|\mathscr{R}| \cap U]$ where U is a neighborhood of x in $\mathscr{Q}$.

Proof. Let $\mathscr{S}$ be the subspace of $\mathscr{Q}$ consisting of all points x such that f has a continuous domain-extension to the subspace $\mathscr{R} \cup (x)$ of $\mathscr{Q}$. For each x in $\mathscr{S}$ let Y_x be the set of all $y \in \mathscr{P}$ such that $gx = y$ for some continuous domain-extension g of f. By the preceding theorem g is a continuous domain-extension of f to a subspace of $\mathscr{Q}$ if and only if $\mathbf{D}g \subset \mathscr{S}$ and $gx \in Y_x$ for each $x \in \mathbf{D}g$; the theorem follows.

Combining Theorem 27 B.11 with the uniqueness theorem (27 A.8) we obtain the following important theorem which, roughly speaking, asserts that every mapping of a dense subspace of a topological space $\mathscr{Q}$ into a separated regular space possesses a largest continuous domain-extension to a subspace of $\mathscr{Q}$.

27 B.12. Theorem. *Suppose that f is a continuous mapping of a dense subspace $\mathscr{R}$ of a topological space $\mathscr{Q}$ into a separated regular space $\mathscr{P}$. There exists a unique continuous domain-extension g of f to a subspace of $\mathscr{Q}$ such that every continuous domain-extension of f to a subspace of $\mathscr{Q}$ is the restriction of g.*

It is to be noted that the above three theorems apply to continuous functions because the space R of reals is separated and regular. Now we shall prove that each of these three theorems is true for uniformly continuous mappings of uniform spaces (but not semi-uniform spaces). The crucial fact which is needed is the following theorem which is important by itself. Keep in mind that every uniformly continuous mapping is continuous but a continuous mapping for uniform spaces need not be uniformly continuous.

27 B.13. Theorem. *If g is a continuous mapping of a uniform space $\langle Q, \mathscr{V} \rangle$ into a uniform space $\langle P, \mathscr{W} \rangle$ and if there exists a dense subspace $\langle R, \mathscr{U} \rangle$ of $\langle Q, \mathscr{V} \rangle$ such that the domain-restriction f of g to $\langle R, \mathscr{U} \rangle$ is uniformly continuous, then g is uniformly continuous.*

Proof. Remember that, by 24 A.2, the open elements of a uniformity as well as the closed elements of a uniformity form a base for this uniformity. I. Let v be the closure induced by $\mathscr{V}$. We shall show that the closure in the product space $\langle Q, v \rangle \times \times \langle Q, v \rangle$ of each element of $\mathscr{U}$ is an element of $\mathscr{V}$. Let $U \in \mathscr{U}$. Since $\mathscr{U}$ is a relativization of $\mathscr{V}$ we can choose a V_1 in $\mathscr{V}$ so that $(V_1 \cap (R \times R)) \subset U$. Since open elements (in $\langle Q, v\rangle \times \langle Q, v\rangle$) of $\mathscr{V}$ form a base for $\mathscr{V}$ we can choose an open element V of $\mathscr{V}$ contained in V_1. The set R is dense in $\langle Q, v\rangle$ and therefore the set $R \times R$ is dense in $\langle Q, v\rangle \times \langle Q, v\rangle$. However, V is open and consequently the set $(R \times R) \cap V$ is dense in V, i.e. the closure of $(R \times R) \cap V$ contains V. Since $(V \cap (R \times R)) \subset U$, the closure of U also contains V, and hence the closure U belongs to $\mathscr{V}$. — II. Now we shall prove that g is uniformly continuous. It is sufficient to find a base $\mathscr{W}'$ for $\mathscr{W}$ such that each set $(g \times g)^{-1}[W]$, $W \in \mathscr{W}'$, contains an element of $\mathscr{V}$ (depending on W). Let $\mathscr{W}'$ be the collection of all closed elements of $\mathscr{W}$ (i.e., those $W \in \mathscr{W}$ which are closed in the product space $\langle P, w\rangle \times \langle P, w\rangle$ where w is induced by $\mathscr{W}$). We know that $\mathscr{W}'$ is actually a base for $\mathscr{W}$. The mapping g is continuous and therefore the product mapping $g \times g$ is also continuous. It follows that $(g \times g)^{-1}[W]$ is a closed set in the product space $\langle Q, v\rangle \times \langle Q, v\rangle$ for each W in $\mathscr{W}'$. Next, the domain-restriction f of g to $\langle R, \mathscr{U}\rangle$ is uniformly continuous and $(f \times f)^{-1}[W]$ belongs therefore to $\mathscr{U}$ for each W in $\mathscr{W}$ and hence each W in $\mathscr{W}'$. By I the closure of each set $(f \times f)^{-1}[W]$, $W \in \mathscr{W}$ belongs to $\mathscr{V}$. However, $(f \times f)^{-1}[W] = (R \times R) \cap (g \times g)^{-1}[W]$ for each W in $\mathscr{W}$ and therefore the closure of each $(g \times g)^{-1}[W]$, $W \in \mathscr{W}$ belongs to $\mathscr{V}$. Now, if $W \in \mathscr{W}'$, then $(g \times g)^{-1}[W]$ is closed and hence belongs to $\mathscr{V}$.

Remark. The preceding theorem remains true if $\langle P, \mathscr{W}\rangle$ and $\langle Q, \mathscr{V}\rangle$ are semi-uniform spaces such that the closed elements of $\mathscr{W}$ form a base for $\mathscr{W}$ and the open elements of $\mathscr{V}$ form a base for $\mathscr{V}$. Actually, only these properties of $\mathscr{W}$ and $\mathscr{V}$ are required in the proof.

27 B.14. Corollary. *Let f be a uniformly continuous mapping of a dense subspace $\mathscr{R}$ of a uniform space $\mathscr{Q}$ into a uniform space $\mathscr{P}$. A domain-extension of f to a subspace of $\mathscr{Q}$ is uniformly continuous if and only if it is continuous.*

Since every uniform space is regular we obtain from Corollary 27 B.14 and from Theorems 27 B.11 and 27 B.12 the following important theorem.

27 B.15. Theorem. *Suppose that f is a uniformly continuous mapping of a dense subspace $\mathscr{R}$ of a uniform space $\mathscr{Q}$ into a uniform space $\mathscr{P}$. There exists a unique subspace $\mathscr{S}$ of $\mathscr{Q}$ with the following property: if g_1 is any uniformly continuous domain-extension of f to a subspace $\mathscr{S}_1$ of $\mathscr{Q}$, then $\mathscr{S}_1$ is a subspace of $\mathscr{S}$ and there exists a uniformly continuous domain-extension g of f to $\mathscr{S}$ such that g_1 is the restriction of g. The set $\mathscr{S}$ consists of all points x of $\mathscr{Q}$ such that f has a continuous domain-extension to the subspace $|\mathscr{R}| \cup (x)$ of $\mathscr{Q}$. In addition, if $\mathscr{P}$ is separated, then there exists a uniformly continuous domain-extension g of f to a subspace of $\mathscr{Q}$ such that every uniformly continuous extension of f to a subspace of $\mathscr{Q}$ is a restriction of g.*

The assumption of regularity of the range carrier of a mapping enters into the situation which is going to be described. Let f be a mapping of a closure space $\mathscr{Q}$ into another one $\mathscr{P}$ such that, for each x in $\mathscr{Q}$, either f is continuous at x or f has a removable discontinuity at x, i.e. by changing the value of f at x we obtain a mapping continuous at x. More precisely, let f be a mapping of $\mathcal{Q}$ into $\mathcal{P}$ such that there exists a family $\{g_x \mid x \in \mathcal{Q}\}$ of mappings of $\mathcal{Q}$ into $\mathcal{P}$ such that each g_x is continuous at x and agrees with f on $\mathcal{Q} - (x)$. Let g be the mapping of $\mathcal{Q}$ into $\mathcal{P}$ such that $gx = g_x x$ for each x in $\mathcal{Q}$. The mapping g need not be continuous even if $\mathcal{P} = \mathsf{R}$ and $\mathcal{Q} \subset \mathsf{R}$. E.g. if $\mathcal{Q}$ is a space with exactly one cluster point, say x, and f is the function on $\mathcal{Q}$ which is 0 at x and 1 otherwise, $g_x = \{z \to 1\}$ and g_y, $y \neq x$, are appropriate functions such that $g_y y = 0$, then g is not continuous.

On the other hand, if $g_x x = fx$ whenever x is an isolated point (in particular, if $g_x x = fx$ whenever f is continuous at x), $\mathcal{Q}$ is topological and semi-separated and $\mathcal{P}$ is regular, then g is necessarily continuous as is stated in the following theorem.

27 B.16. Theorem. *Let f be a mapping of a topological semi-separated space $\mathcal{Q}$ into a regular space $\mathcal{P}$ and let $\{g_x \mid x \in \mathcal{Q}\}$ be a family of mappings of $\mathcal{Q}$ into $\mathcal{P}$ such that each g_x is continuous at x and agrees with f on $|\mathcal{Q}| - (x)$ and moreover, if x is an isolated point of $\mathcal{Q}$, then $g_x x = fx$ (the latter condition is fulfilled if $fx = g_x x$ whenever f is continuous at x). Then the mapping $g = \{x \to g_x x\} : \mathcal{Q} \to \mathcal{P}$ is continuous.*

Proof. I. First we shall prove that the set $g[U]$ is contained in the closure (in $\mathcal{P}$) of the set $f[U]$ for each open subset U of $\mathcal{Q}$. Let $x \in U$. If x is an isolated point of $\mathcal{Q}$, then $fx = g_x x = gx$ and hence gx belongs to $f[U]$. If x is not isolated then x belongs to the closure (in $\mathcal{Q}$) of $U - (x)$, and g_x being continuous, $g_x x$ belongs to the closure of $g_x[U - (x)]$; however $gx = g_x x$ and $f[U - (x)] = g_x[U - (x)]$. (Notice that no assumptions on the spaces $\mathcal{P}$ and $\mathcal{Q}$ were needed.) – II. Now let $x \in \mathcal{Q}$ and let W be a neighborhood of gx in $\mathcal{P}$. Choose a neighborhood V of gx so that $\overline{V} \subset W$. The mapping g_x is continuous at x, $g_x x = gx$ and $\mathcal{Q}$ is topological, and therefore there exists an open neighborhood U of x such that $g_x[U] \subset V$. The set $U_1 = U - (x)$ is open in $\mathcal{Q}$ (because (x) is closed since $\mathcal{Q}$ is semi-separated). By I the set $g[U_1]$ is contained in the closure of the set $f[U_1]$ which coincides with $g_x[U_1]$, and consequently $g[U_1] \subset \overline{V} \subset W$.

In conclusion we shall state results obtained by combining the preceding theorems on continuous or uniformly continuous extensions with the result mentioned above that, under certain assumptions, $f : \mathcal{R} \to \mathcal{P}$ has a continuous domain-extension on a space $\mathcal{Q}$ such that $|\mathcal{Q}| - |\mathcal{R}| = (x)$ if and only if there exists a y in $|\mathcal{P}|$ such that each neighborhood of y contains $f[U \cap |\mathcal{R}|]$ for some neighborhood U of x in $\mathcal{Q}$, i.e. such that the neighborhood system of y in $\mathcal{P}$ is contained in the f-transform of the filter $[\mathcal{U}] \cap |\mathcal{R}|$, where $\mathcal{U}$ is the neighborhood system of x in $\mathcal{Q}$.

27 B.17. Theorem. *Let f be a continuous mapping of a dense subspace $\mathcal{R}$ of a topological semi-uniformizable space $\mathcal{Q}$ into a regular closure space $\mathcal{P}$. Then*

(a) *f has a continuous domain-extension to $\mathcal{Q}$ if and only if for each x in $|\mathcal{Q}| - |\mathcal{R}|$ there exists a y in $\mathcal{P}$ such that each neighborhood of y contains a set $f[U \cap |\mathcal{R}|]$ where U is a neighborhood of x in $\mathcal{Q}$.*

(b) *A domain-extension g of f to $\mathcal{Q}$ is continuous if and only if the following condition is fulfilled: for each x in $|\mathcal{Q}| - |\mathcal{R}|$, each neighborhood of gx contains the set $f[U \cap |\mathcal{R}|]$ for some neighborhood U of x in $\mathcal{Q}$.*

Remark. In 15 ex. 7 we defined what is meant by a filter converging to a point in a space. Clearly statements (a) and (b) can be formulated as follows:

(a′) f has a continuous domain-extension on $\mathcal{Q}$ if and only if the f-transform of the filter $[\mathcal{U}_x] \cap |\mathcal{R}|$ converges in $\mathcal{P}$ for each x in $|\mathcal{Q}| - |\mathcal{R}|$, where $\mathcal{U}_x$ is the neighborhood system at x.

(b′) A domain-extension g of f to $\mathcal{Q}$ is continuous if and only if the f-transform of the filter $[\mathcal{U}_x] \cap |\mathcal{R}|$ converges to gx for each x in $|\mathcal{Q}| - |\mathcal{R}|$, where $\mathcal{U}_x$ is the neighborhood system of x in $\mathcal{Q}$.

27 B.18. *Let f be a uniformly continuous mapping of a dense subspace $\mathcal{R}$ of a uniform space $\mathcal{Q}$ into a uniform space $\mathcal{P}$. Then*

(a) *f has a uniformly continuous domain-extension to $\mathcal{Q}$ if and only if for each x in $|\mathcal{Q}| - |\mathcal{R}|$ there exists a y in $|\mathcal{P}|$ such that each neighborhood of y contains the set $f[|\mathcal{R}| \cap U]$ for some neighborhood U of x in $\mathcal{Q}$.*

(b) *A domain-extension g of f to $\mathcal{Q}$ is uniformly continuous if and only if the following condition is fulfilled: for each x in $|\mathcal{Q}| - |\mathcal{R}|$, each neighborhood of gx contains $f[|\mathcal{R}| \cap U]$ for some neighborhood U of x in $\mathcal{Q}$.*

We leave to the reader the task of formulating (a) and (b) in a manner similar to (a′) and (b′) in the remark following 27 B.17.

28. UNIFORMIZABLE SPACES

By Definition 24 A.1 a closure space $\langle P, u\rangle$ is said to be uniformizable if the closure u is induced by a uniformity $\mathscr{U}$ for P; this means (by definition 23 A.3) that the set of all $U[x]$, $U \in \mathscr{U}$, is the neighborhood system at x in $\langle P, u\rangle$ for each x in P, or equivalently (by 23 B.5), that $uX = \bigcap\{U[X] \mid U \in \mathscr{U}\}$ for each subset $X \subset P$. For example every pseudometrizable space is uniformizable; in particular every discrete space, every accrete space and the space R of reals are uniformizable. Indeed if d is a pseudometric inducing the closure structure of a space $\langle P, u\rangle$, then u is induced by the uniformity induced by d. It is to be noted that uniformizable spaces are often termed completely regular spaces (see Remark 28 A.6). It seems that the class of all uniformizable closure spaces is the most important class of spaces in general topology; probably because it is precisely the uniformizable spaces which can be described in terms of continuous functions; more precisely, a space $\langle P, u\rangle$ is uniformizable if and only if u is the coarsest closure for P such that all functions continuous in u are continuous, i.e. if v is a closure for P such that $f : \langle P, v\rangle \to \mathsf{R}$ is continuous whenever $f \in \mathbf{C}(\langle P, u\rangle, \mathsf{R})$ then v is finer than u (see Remark 28 A.6). In terms of Section 32 we can formulate this condition as follows: a space $\mathscr{P}$ is uniformizable if and only if $\mathscr{P}$ is projectively generated by functions. Roughly speaking, a space $\mathscr{P}$ is uniformizable if and only if there are "enough" continuous functions on $\mathscr{P}$ (see 28 A.5 (e), (f)). A great deal of the results of this section (mainly subsections A and C) is an immediate consequence of the results of sections 23, 24 and 25. Nevertheless, the theory of uniformizable spaces can be built up independently of the theory of uniform spaces and, in fact, in this section we shall prove most of the topological results without any reference to sections 23 – 25.

In the first subsection we shall prove, besides various characterizations of uniformizable spaces, that the class of all uniformizable spaces is hereditary, closed under products and sums, and contained in the class of all regular spaces. One of the most important results is Theorem 28 A.9 which states, in the terminology of Section 32, that a space $\mathscr{P}$ is uniformizable if and only if it is projectively generated by a mapping into a product $\mathsf{R}^{\aleph}$, and its Corollary 28 A.10 which states that a separated space $\mathscr{P}$ is uniformizable if and only if $\mathscr{P}$ is the homeomorph of a subspace of a cube $\mathsf{R}^{\aleph}$ (i.e. $\mathscr{P}$ admits an embedding into a cube $\mathsf{R}^{\aleph}$). In subsection B we shall introduce exact open sets (as the sets of the form $f^{-1}[G]$ where f is a continuous function and G is open in R) and exact closed sets (as the sets of the form $f^{-1}[F]$ where f is continuous and F is closed in R), in another terminology N-sets and Z-sets, and we shall prove that a space is uniformizable if and only if it is locally exact open (compare with similar characterizations of topological spaces), i.e. the exact open sets form an open base. We shall show that two sets X_1 and X_2 are distant with respect to the Čech proximity of a space $\mathscr{P}$ if and only if there exist exact closed sets Y_i, $i = 1, 2$, such that $X_i \subset Y_i$ and $Y_1 \cap Y_2 \neq \emptyset$.

In the last subsection the uniformizable modification of a closure operation u will be introduced (the finest uniformizable closure coarser than u) and studied. It is easily seen that the uniformizable modification of a space $\mathscr{P}$ is the unique uniformizable space $\mathscr{Q}$ such that $|\mathscr{P}| = |\mathscr{Q}|$, and $f : \mathscr{P} \to \mathsf{R}$ is continuous if and only if $f : \mathscr{Q} \to \mathsf{R}$ is continuous. Thus the uniformizable modification of a space $\mathscr{P}$ is uniquely determined by the graphs of continuous functions. Moreover we shall show that the uniformizable modification $\mathscr{Q}$ of $\mathscr{P}$ is uniquely determined by the exact open sets; the exact open sets of $\mathscr{P}$ form an open base for $\mathscr{Q}$.

It is to be noted that a great deal of the results of this section are also corollaries of the results of Chapter VI.

A. PROPERTIES OF UNIFORMIZABLE SPACES

28 A.1. Definition. A uniformity $\mathscr{U}$ for a closure space $\mathscr{P}$ will be termed the *fine uniformity of* $\mathscr{P}$ if $\mathscr{U}$ is the uniformly finest continuous uniformity for $\mathscr{P}$, i.e. $\mathscr{U}$ is a continuous uniformity for $\mathscr{P}$ and if $\mathscr{V}$ is a continuous uniformity for $\mathscr{P}$ then $\mathscr{V}$ is contained in $\mathscr{U}$. A uniformity $\mathscr{U}$ will be termed a fine uniformity if $\mathscr{U}$ is a fine uniformity of some closure space $\mathscr{P}$. The proximity p induced by the fine uniformity $\mathscr{U}$ of a closure space $\mathscr{P}$ will be termed the *Čech proximity* of $\mathscr{P}$ and the unique proximally coarse uniformity proximally equivalent with $\mathscr{U}$ will be termed the *Čech uniformity* of $\mathscr{P}$.

28 A.2. Theorem. *Let $\mathscr{P} = \langle P, u \rangle$ be a closure space. There exists a unique fine uniformity of $\mathscr{P}$. If $\mathscr{U}$ is the fine uniformity of $\mathscr{P}$, then*

(a) *$\mathscr{U}$ consists of all uniformizable neighborhoods of the diagonal of $\mathscr{P} \times \mathscr{P}$, that is, of all $U \subset P \times P$ for which there exists a sequence $\{U_n\}$ of semi-neighborhoods of the diagonal of $\mathscr{P} \times \mathscr{P}$ such that $U \supset U_0$ and $U_{n+1} \circ U_{n+1} \subset U_n$ for each n.*

(b) *A pseudometric d for P is a continuous pseudometric for $\mathscr{P}$ if and only if d is a uniformly continuous pseudometric for $\langle P, \mathscr{U} \rangle$.*

(c) *A cover $\mathscr{X}$ of P is a uniformizable cover of $\mathscr{P}$ if and only if $\mathscr{X}$ is a uniform cover of $\langle P, \mathscr{U} \rangle$.*

(d) *If f is a real-valued relation on P, then the function $f : \mathscr{P} \to \mathsf{R}$ is continuous if and only if the function $f : \langle P, \mathscr{U} \rangle \to \mathsf{R}$ is uniformly continuous.*

Proof. Existence and uniqueness were proved in 24 B.11, statements (a), (b) and (c) were proved in 24 B.14, 24 B.12 and 24 E.3, respectively. If $f : \langle P, \mathscr{U} \rangle \to \mathsf{R}$ is uniformly continuous, then evidently the function $f : \mathscr{P} \to \mathsf{R}$ is continuous. Finally, if $f : \mathscr{P} \to \mathsf{R}$ is continuous and $d = \{\langle x, y \rangle \to |fx - fy|\}$, then d is a continuous pseudometric for $\mathscr{P}$ and hence, by (b), d is a uniformly continuous pseudometric for $\langle P, \mathscr{U} \rangle$. On the other hand, the function $f : \langle P, d \rangle \to \mathsf{R}$ is Lipschitz continuous with bound 1 and hence uniformly continuous. Thus $f : \langle P, \mathscr{U} \rangle \to \mathsf{R}$ is uniformly continuous as the composite ot two uniformly continuous mappings, namely of

$$\mathsf{J} : \langle P, \mathscr{U} \rangle \to \langle P, d \rangle \quad \text{and} \quad f : \langle P, d \rangle \to \mathsf{R}\,.$$

28 A.3. Theorem. *Let $\mathscr{P} = \langle P, u \rangle$ be a closure space. There exists a unique Čech uniformity of $\mathscr{P}$. If $\mathscr{V}$ is the Čech uniformity of P, then*

(a) *$\mathscr{V}$ is the uniformly finest proximally coarse continuous uniformity for $\mathscr{P}$.*

(b) *The finite square uniformizable neighborhoods of the diagonal of $\mathscr{P} \times \mathscr{P}$ form a base for $\mathscr{V}$.*

(c) *A pseudometric d for P is a uniformly continuous pseudometric for $\langle P, \mathscr{V} \rangle$ if and only if d is a totally bounded continuous pseudometric for $\mathscr{P}$.*

(d) *A function f on $\langle P, \mathscr{V} \rangle$ is uniformly continuous if and only if $f : \mathscr{P} \to \mathsf{R}$ is a bounded continuous function.*

Proof. Let $\mathscr{U}$ be the fine uniformity of $\mathscr{P}$. The existence and uniqueness of $\mathscr{V}$ follow from the existence and uniqueness of $\mathscr{U}$ and the fact that any uniformity is proximally equivalent to exactly one proximally coarse uniformity (25 B.9). By 25 B.8 the finite square elements of a proximally coarse semi-uniformity $\mathscr{W}$ form a base for $\mathscr{W}$ and therefore statement (b) follows from 28 A.2 (a). Statements (c) and (d) follow from 28 A.3, (b) and (d), and Theorem 25 B.21.

28 A.4. Theorem. *Let $\mathscr{P} = \langle P, u \rangle$ be a closure space. There exists a unique Čech proximity of $\mathscr{P}$. If p is the Čech proximity of $\mathscr{P}$, then*

(a) *p is the proximally finest continuous uniformizable proximity for $\mathscr{P}$.*

(b) *A function f on $\langle P, p \rangle$ is proximally continuous if and only if the function $f : \mathscr{P} \to \mathsf{R}$ is continuous.*

(c) *X non $p\, Y$ if and only if $X \subset P$, $Y \subset P$ and there exists a continuous function f on $\mathscr{P}$ which is 0 on X, 1 on Y and all its values lie between 0 and 1.*

(d) *The fine uniformity of $\mathscr{P}$ is the uniformly finest uniformity inducing p.*

(e) *A pseudometric d for P is a proximally continuous pseudometric for $\langle P, p \rangle$ if and only if d is a continuous pseudometric for $\mathscr{P}$.*

Proof. Statement (a) is evident, statement (b) follows from 25 A.12 and 28 A.2 (d), statement (c) is an immediate consequence of (b) and Theorem 25 C.5. Statement (d) is obvious and (e) is a consequence of (d), 28 A.2 (b) and 25 B.22. Indeed, if $\mathscr{U}$ is the fine uniformity of $\mathscr{P}$, the continuous pseudometrics for $\mathscr{P}$ coincide with the uniformly continuous pseudometrics for $\langle P, \mathscr{U} \rangle$. Now the statement follows from (d) and 25 B.22.

We proceed to uniformizable closures.

28 A.5. Theorem. *Each of the following conditions is necessary and sufficient for a closure space $\mathscr{P} = \langle P, u \rangle$ to be uniformizable:*

(a) *The fine uniformity of $\mathscr{P}$ induces u.*

(b) *The Čech uniformity of $\mathscr{P}$ induces u.*

(c) *The Čech proximity of $\mathscr{P}$ induces u.*

(d) *If $x \in P$, $\emptyset \neq X \subset P$ and $x \notin uX$, then the distance from x to X is positive for some continuous pseudometric d for $\mathscr{P}$.*

(e) *If $x \in P$ and U is a neighborhood of x in $\mathscr{P}$, then there exists a continuous function f on $\mathscr{P}$ such that $0 \leqq f \leqq 1$, $fx = 1$ and $fy = 0$ for each y in $P - U$.*

(f) *If $x \in P$, $X \subset P$ and $x \notin uX$, then there exists a continuous functions f on $\mathscr{P}$ such that $fx \notin \overline{f[X]}$.*

Proof. I. It follows from definition 28 A.1 that conditions (a), (b) and (c) are equivalent. Evidently (a) is sufficient. We shall prove that (a) is necessary. If u is uniformizable and $\mathscr{W}$ is a uniformity which induces u, then $\mathscr{W}$ is a continuous uniformity for $\mathscr{P}$ and hence $\mathscr{W}$ is uniformly coarser than the fine uniformity $\mathscr{U}$ of $\mathscr{P}$. As a consequence, the closure u, which is induced by $\mathscr{W}$, is coarser then the closure induced by $\mathscr{U}$ which is coarser than u. Thus both uniformities are topologically equivalent.

II. Since continuous pseudometrics for $\mathscr{P}$ coincide with uniformly continuous pseudometrics for $\langle P, \mathscr{U} \rangle$, where $\mathscr{U}$ is the fine uniformity of $\mathscr{P}$, the set of all continuous pseudometrics for $\mathscr{P}$ generates the fine uniformity of $\mathscr{P}$ and hence, by the description 23 B.8 of a semi-uniform closure, condition (d) is equivalent to condition (a).

III. It remains to show, e.g., that (d) implies (e), (e) implies (f) and (f) implies (d). Assuming (d) let U be a neighborhood of a point x of $\mathscr{P}$. If $U = P$ then we can take the constant function $\{x \to 1\}$ as f. If $U \neq P$ then by (d) we can choose a continuous pseudometric d for $\mathscr{P}$ such that the distance from x to $P - U$ is positive, say r. Consider the function $g = \{y \to \operatorname{dist}(y, P - U)\} : \mathscr{P} \to \mathsf{R}$. By 18 A.12 the function g is continuous and clearly $gx = r$, $gy = 0$ for $y \in (P - U)$. Now clearly the function $f = \inf(r^{-1} . g, 1)$ has the required properties. Thus (d) implies (e). The implication (e) $\Rightarrow$ (f) is almost self-evident. Indeed, if $x \notin uX$, then $U = P - X$ is a neighborhood of x in $\mathscr{P}$ and if f is a continuous function on $\mathscr{P}$ such that $fx = 1$ and $f[P - U] \subset (0)$, then evidently $fx \notin \overline{f[X]}$. Finally, assuming (f) let $x \in P$, $\emptyset \neq X \subset P$, $x \notin uX$. By condition (f) there exists a continuous function f on $\mathscr{P}$ such that fx does not belong to the closure of $f[X]$ in R and hence, the distance from fx to $f[X]$ is positive (in R) because $f[X] \neq \emptyset$. The relation $d = \{\langle x, y \rangle \to |fx - fy| \mid \langle x, y \rangle \in (P \times P)\}$ is a continuous pseudometric for $\mathscr{P}$ (by 18 C.10) and the distance from x to X in $\langle P, d \rangle$ is equal to the distance from fx to $f[X]$ in R.

28 A.6. Remark. Uniformizable spaces were introduced by A. Tichonov in 1931 under the name of completely regular spaces. Uniform spaces were introduced by A. Weil in 1937. Tichonov defined uniformizable spaces by equivalent conditions

(e) and (f) of the last theorem. It is to be noted that Tichonov only considered separated uniformizable spaces, and many authors employ the term Tichonov space for separated uniformizable spaces. It seems that the original term completely regular space appears more frequently in the literature than the term uniformizable space introduced by N. Bourbaki.

The theory of uniformizable spaces can be built up independently of the theory of uniform or proximity spaces and, in fact, the concept of a uniformity was introduced only after the theory of uniformizable spaces had been developed. The properties of uniformizable spaces to follow are immediate consequences of the results of sections 23–25. Nevertheless we wish to outline topological proofs which are based upon the purely topological characterizations 28 A.5 (d), (e) and (f), the equivalence of which was proved without any reference to semi-uniform or proximity spaces.

Finally, let us prove that condition (f) of 28 A.5 is equivalent to the following condition

(*) If v is a closure for P such that $f \in \mathbf{C}(\langle P, u\rangle, \mathsf{R})$ implies that $f : \langle P, v\rangle \to \mathsf{R}$ is continuous, then v is finer than u.

If (f) is satisfied, v fulfils the assumption of (*) and $x \in vX$, then $fx \in \overline{f[X]}$ for each continuous function on $\langle P, u\rangle$, and hence $x \in uX$ by (f), which shows that v is finer than u. Conversely assuming (*) we shall prove (f). Let us define a single-valued relation v on P ranging in $\exp P$ such that $x \in vX$ if and only if $fx \in \overline{f[X]}$ for each continuous f on $\langle P, u\rangle$. It is easily seen that v is a closure operation for P and v fulfils the assumption of (*). Thus v is finer than u and hence (f) is fulfilled.

28 A.7. Theorem. *The class of all uniformizable spaces is hereditary and closed under products and sums. Every uniformizable space is a topological regular space and every feebly semi-separated uniformizable space is separated.*

Proof. If $\mathscr{Q}$ is a subspace of a uniformizable space $\mathscr{P}$ and if $\mathscr{U}$ is a uniformity inducing the closure structure of $\mathscr{P}$, then the relativization of $\mathscr{U}$ to $|\mathscr{Q}|$ is a uniformity (24 A.8) inducing the closure structure of $\mathscr{Q}$ (23 D.2). If $\mathscr{P}$ is the product (sum) of a family $\{\mathscr{P}_a\}$ of uniformizable closure spaces and if $\mathscr{U}_a$ is a uniformity inducing the closure structure of $\mathscr{P}_a$ for each a, then the product (sum) of $\{\mathscr{U}_a\}$ is a uniformity (24 A.8) which induces the closure structure of $\mathscr{P}$. We have already proved that every uniformizable space is regular (27 B.7) and topological (24 A.2). Finally, if a uniformizable space $\mathscr{P}$ is feebly semi-separated then $\mathscr{P}$ is separated by 27 B.7 because $\mathscr{P}$ is regular.

Alternate proof. If $\mathscr{Q} = \langle Q, v\rangle$ is a subspace of a uniformizable space $\mathscr{P} = \langle P, u\rangle$ and $x \notin vX$, where $x \in Q, X \subset Q$, then $x \notin uX$ and, as $\mathscr{Q}$ is uniformizable, we can choose a continuous function f on $\mathscr{P}$ such that fx does not belong to the closure of $f[X]$ in R; the domain-restriction g of f to $\mathscr{Q}$ is continuous and $gx = fx$, $g[X] = f[X]$. Hence $gx \notin \overline{g[X]}$. If $\mathscr{P}$ is the product of a family $\{\mathscr{P}_a\}$ of uniformizable spaces and U is a canonical neighborhood of a point $x = \{x_a\}$, $U = \mathbf{E}\{y \mid a \in A' \Rightarrow$

$\Rightarrow \mathrm{pr}_a\, y \in U_a\}$ where A' is a finite subset of A, and if f_a is a continuous function on $\mathscr{P}_a$ such that $f_a x_a = 1$, $f_a[|\mathscr{P}_a| - U_a] \subset (0)$, then the function $f = \{y \to \to \Pi\{f_a\, \mathrm{pr}_a\, y \mid a \in A'\} \mid y \in |\mathscr{P}|\} : \mathscr{P} \to \mathrm{R}$ is continuous on $\mathscr{P}$, $fx = 1$, $fy = 0$ for $y \in (|\mathscr{P}| - U)$. If $\mathscr{P}$ is the sum of a family $\{\mathscr{P}_a\}$ and U is a neighborhood of $\langle a, x\rangle$ in $\mathscr{P}$, then $V = U \cap ((a) \times |\mathscr{P}_a|)$ is also a neighborhood of $\langle a, x\rangle$. Since the subspace $\mathscr{R}_a = (a) \times |\mathscr{P}_a|$ of $\mathscr{P}$ is a homeomorph of $\mathscr{P}_a$ we can choose a continuous function g on $\mathscr{R}_a$ such that $g\langle a, x\rangle = 1$ and g is 0 outside V. If f is the domain-extension of g on $\mathscr{P}$ such that g is 0 outside $|\mathscr{R}_a|$, then f is continuous, $f\langle a, x\rangle = 0$ and f is 0 outside V and hence also outside U. Now let $\mathscr{P}$ be uniformizable. If U is a neighborhood of a point x in $\mathscr{P}$ and f is a continuous function on $\mathscr{P}$ such that $fx = 1$ and f is 0 outside U, then the set $V = f^{-1}[\,]\!]\, 2^{-1}, \to]\!]\,]$ is open and contained in U and the set $W = f^{-1}[\,[\![\, 2^{-1}, \to]\!]\,]$ is a closed neighborhood of x contained in U. Thus $\mathscr{P}$ is both locally closed and locally open, i.e. $\mathscr{P}$ is a topological regular space. The last statement is evident.

28 A.8. Remark. We shall show in 33 D that a separated regular topological space need not be uniformizable; moreover, we shall outline a construction of an infinite separated regular topological space $\mathscr{P}$ such that each continuous function on $\mathscr{P}$ is constant. It is to be noted that first constructions of a separated regular topological space without non-constant continuous function were given by E. Hewitt and J. Novák. Our construction is an adaptation of Novák's construction.

28 A.9. Theorem. *Each of the following two conditions is necessary and sufficient for a closure space $\mathscr{P} = \langle P, u\rangle$ to be uniformizable:*

(a) *There exists a mapping h of $\langle P, u\rangle$ into $[\![\, 0, 1 \,]\!]^{\aleph}$ for some cardinal $\aleph$ such that $x \in uX$ if and only if $hx \in \overline{h[X]}$.*

(b) *There exists a mapping h of $\langle P, u\rangle$ into a uniformizable space $\mathscr{Q}$ such that $x \in uX$ if and only if $hx \in \overline{h[X]}$.*

Proof. Since every metrizable space is uniformizable, the unit interval $[\![\, 0, 1 \,]\!]$ is uniformizable, and by 28 A.7 every cube $[\![\, 0, 1 \,]\!]^{\aleph}$ is also uniformizable. It follows that (a) implies (b). It remains to show that (a) is necessary and (b) is sufficient. I. Suppose that $\langle P, u\rangle$ is uniformizable. By 28 A.5 (condition (f)), there exists a collection $\mathscr{C}$ of continuous functions on $\langle P, u\rangle$ such that each f from $\mathscr{C}$ satisfies the inequality $0 \leqq f \leqq 1$ and that if $x \notin uX$ then $fx \notin \overline{f[X]}$ for some f in $\mathscr{C}$. Consider the mapping h of $\langle P, u\rangle$ into the product space $\mathscr{Q} = [\![\, 0, 1 \,]\!]^{\mathscr{C}}$ which assigns to each $x \in P$ the point $hx = \{fx \mid f \in \mathscr{C}\}$ of $\mathscr{Q}$. Thus h is the reduced product of the family $\{f : \mathscr{P} \to [\![\, 0, 1 \,]\!] \mid f \in \mathscr{C}\}$. Since each $f : \langle P, u\rangle \to [\![\, 0, 1 \,]\!]$, $f \in \mathscr{C}$, is continuous, the mapping h is continuous by 17 C.13, that is, $x \in uX$ implies $hx \in \overline{h[X]}$. Now, conversely, if $x \in P$, $X \subset P$, $x \notin uX$, then there exists an f in $\mathscr{C}$ such that $fx \notin \overline{f[X]}$. Since the projection π_f of $\mathscr{Q}$ onto the f-th coordinate space is continuous and $\pi_f \circ h = f$, in particular $fx = \pi_f hx$ and $f[X] = \pi_f[h[X]]$, we obtain $hx \notin \overline{h[X]}$, which concludes the proof.

II. Suppose that there exists a mapping h of $\langle P, u\rangle$ into a uniformizable space $\mathcal{Q}$ such that $x \in uX$ if and only if $hx \in \overline{h[X]}$. To prove that $\langle P, u\rangle$ is uniformizable, it is enough to show that condition (f) from 28 A.5 is fulfilled. Let $x \in P$, $X \subset P$, $x \notin uX$. By our assumption $hx \notin \overline{h[X]}$. Since $\mathcal{Q}$ is uniformizable, by 28 A.5 (f) there exists a continuous function g on $\mathcal{Q}$ such that $ghx \notin \overline{gh[X]}$. Since h is continuous, the function $f = g \circ h$ is continuous on $\langle P, u\rangle$. Obviously $fx \notin \overline{f[X]}$.

Remark. Let h be a mapping of a space $\langle P, u\rangle$ into a space $\mathcal{Q}$ such that $x \in uX$ if and only if $hx \in \overline{h[X]}$. Evidently, if h is a one-to-one mapping, then h is an embedding. If $\langle P, u\rangle$ is feebly semi-separated, that is, if $x, y \in P$, $x \neq y$ imply $x \notin u(y)$ or $y \notin u(x)$, then h is one-to-one. Indeed, if $x \neq y$, then $x \notin u(y)$ or $y \notin u(x)$, which implies $hx \notin \overline{(hy)}$ or $hy \notin \overline{(hx)}$ and yields $hx \neq hy$.

28 A.10. Corollary. *A closure space $\langle P, u\rangle$ is a separated uniformizable space if and only if $\langle P, u\rangle$ admits an embedding into a cube $[\![0, 1]\!]^{\aleph}$, that is, $\langle P, u\rangle$ is homeomorphic with a subspace of a cube $[\![0, 1]\!]^{\aleph}$.*

Proof. The space $[\![0, 1]\!]$ is separated and hence each subspace of any cube is separated. Thus each subspace of any cube is a separated uniformizable space. Conversely, if $\langle P, u\rangle$ is uniformizable, then (by 28 A.9) there exist a mapping h of $\langle P, u\rangle$ into a cube $[\![0, 1]\!]^{\aleph}$ such that $x \in uX$ if and only if $hx \in \overline{h[X]}$; if in addition, $\langle P, u\rangle$ is separated, then by the remark following 28 A.9 the mapping h is an embedding.

28 A.11. Corollary. *A closure space $\langle P, u\rangle$ is uniformizable if and only if $\langle P, u\rangle$ admits an embedding into the product of an accrete space with a cube $[\![0, 1]\!]^{\aleph}$.*

Proof. As the class of uniformizable spaces is hereditary and completely productive and accrete spaces and the space $[\![0, 1]\!]$ are uniformizable, every space $\langle P, u\rangle$ admitting an embedding under question is necessarily uniformizable. Conversely, let us suppose that $\langle P, u\rangle$ is uniformizable. By Theorem 28 A.9 there exists a mapping h of $\langle P, u\rangle$ into a cube $\mathcal{Q} = [\![0, 1]\!]^{\aleph}$ such that $x \in uX$ if and only if $hx \in \overline{h[X]}$. Let v be the accrete closure for P, I the identity mapping of $\langle P, u\rangle$ onto $\langle P, v\rangle$, and last let f be the reduced product of the mappings h and I, i.e. $\mathbf{D}^*f = \langle P, u\rangle$, $\mathbf{E}^*f = \mathbf{E}^*h \times \langle P, v\rangle$, $fx = \langle hx, x\rangle$ for each $x \in P$. It is obvious that $x \in uX$ if and only if $fx \in \overline{f[X]}$. Since f is a one-to-one mapping, f is an embedding by the remark following 28 A.9.

Remark. Since the class of all uniformizable spaces is completely productive and hereditary and contains all pseudometrizable spaces, each subspace of the product of a family of pseudometrizable spaces is a uniformizable space. On the other hand, by 28 A.11 every uniformizable space is homeomorphic with a subspace of the product of pseudometrizable spaces (every accrete space and the space $[\![0, 1]\!]$ are pseudometrizable). Thus a closure space is uniformizable if and only if it admits an embedding into the product of pseudometrizable spaces. It is to be noted that this result is an immediate consequence of Theorem 24 A.11, which asserts that a semi-uniform space is uniform if and only if it admits a uniform embedding into the product of a family of pseudometrizable uniform spaces, the elementary fact that every uniform embedding is a closure embedding, and Theorem 24 A.8, which asserts that the product of any family of uniform spaces is a uniform space.

B. EXACT OPEN AND EXACT CLOSED SETS

28 B.1. Definition. Let $\langle P, u\rangle$ be a closure space. An *exact closed* (*exact open*) subset of $\langle P, u\rangle$ is a set of the form $f^{-1}[0]$ ($f^{-1}[\mathsf{R} - (0)]$, respectively), where f is a continuous function on $\langle P, u\rangle$.

It should be remarked that exact closed sets are often called *zero-sets* or *Z-sets* and exact open sets are called *cozero-sets* or *N-sets*. The set $f^{-1}[0]$ is often denoted by $Z(f)$ and the set $f^{-1}[\mathsf{R} - (0)]$ by $N(f)$. This notation and terminology will not be used in the sequel.

If $X = f^{-1}[0]$, then $X = |f|^{-1}[0]$ and also $X = (\inf(|f|, 1))^{-1}[0]$. Similarly $f^{-1}[\mathsf{R} - (0)] = (\inf(|f|, 1))^{-1}[\mathsf{R} - (0)]$. It follows that every exact closed (exact open) set X in $\langle P, u\rangle$ is of the form $f^{-1}[0]$ ($f^{-1}[\mathsf{R} - (0)]$), where f is a continuous bounded non-negative function on $\langle P, u\rangle$.

28 B.2. Theorem. *Let $\langle P, u\rangle$ be a closure space. A subset X of $\langle P, u\rangle$ is exact closed if and only if its complement $P - X$ is exact open. Every exact open set is an open $\mathbf{F}_\sigma$-set and every exact closed set is a closed $\mathbf{G}_\delta$-set. The collection of all exact closed sets is closed under finite unions and countable intersections (i.e. is additive and countably multiplicative). The collection of all exact open sets is closed under finite intersections and countable unions (i.e. is multiplicative and countably additive). The sets $\emptyset$ and P are simultaneously exact open and exact closed.*

Proof. I. The first statement is evident.

II. Let X be exact closed, that is, $X = f^{-1}[0]$ for some continuous function f on $\langle P, u\rangle$. The set X is closed as the inverse of a closed set under a continuous mapping. The set (0) is a $\mathbf{G}_\delta$ in R because $(0) = \bigcap\{\,]\!] -1/n, 1/n[\![\; \mid n = 1, 2, \ldots\}$. It follows that $X = \bigcap\{f^{-1}[\,]\!] -1/n, 1/n[\![\,]\}$ is a $\mathbf{G}_\delta$ in $\langle P, u\rangle$. If X is exact open in $\langle P, u\rangle$, then $P - X$ is exact closed, and consequently $P - X$ is a closed $\mathbf{G}_\delta$. It follows that X is an open $\mathbf{F}_\sigma$.

III. Let $\{X_a\}$ be a finite family of exact closed sets. Choose a family $\{f_a\}$ of continuous functions on $\langle P, u\rangle$ such that $X_a = f_a^{-1}[0]$. Clearly $f = \Pi\{f_a\}$ $(= \{x \to \Pi\{f_a x\}\})$ is continuous and $f^{-1}[0] = \bigcup\{X_a\}$. Now let $\{X_a \mid a \in A\}$ be a countable family of exact closed sets in $\langle P, u\rangle$. We may assume $A \subset \mathsf{N}$. Let $\{f_a\}$ be a family of continuous functions such that $X_a = f_a^{-1}[0]$, and moreover, $0 \leqq f_a \leqq 2^{-a}$ for each a in A; such f_a can be chosen because, if g_a is continuous and $X_a = g_a^{-1}[0]$, then $f_a = \inf(|g_a|, 2^{-a})$ possesses the required properties. Consider the series $\Sigma\{f_a \mid a \in A\}$.

Since $0 \leqq f_a \leqq 2^{-a}$, the series is uniformly convergent and its sum f is a continuous function. It is almost self-evident that $f^{-1}[0] = \bigcap\{X_a \mid a \in A\}$.

IV. The invariance of the collection of all exact open sets under finite intersections and countable unions is an immediate consequence of the "dual" assertion for exact closed sets (use the first statement and the de Morgan formula).

V. The constant functions $f_1 = \{x \to 0\}$ and $f_2 = \{x \to 1\}$ are continuous and $P = f_1^{-1}[0] = f_2^{-1}[\mathsf{R} - (0)]$. It follows that P is simultaneously exact closed and exact open. In consequence, the complement $P - P = \emptyset$ of P is also exact open and exact closed.

28 B.3. *If f is a continuous mapping of a space $\mathscr{P}$ into a space $\mathscr{Q}$ and if Y is an exact closed or exact open subset of $\mathscr{Q}$, then $X = f^{-1}[Y]$ possesses the corresponding property in $\mathscr{P}$.* Indeed, if g is a continuous function on $\mathscr{Q}$ then $h = g \circ f$ is a continuous function on $\mathscr{P}$, and if, in addition, $Y = g^{-1}[0]$ or $Y = g^{-1}[\mathsf{R} - (0)]$, then $X = h^{-1}[0]$ or $X = h^{-1}[\mathsf{R} - (0)]$ respectively. *Each closed (open) subset X of a pseudometrizable space $\mathscr{P}$ is exact closed (exact open).* Indeed, if d pseudometrizes $\mathscr{P}$ and X is closed, then $X = f^{-1}[0]$ where $f = \{x \to \text{dist}(x, X)\}$. Combining the two foregoing results we obtain at once that *if f is a continuous mapping of $\mathscr{P}$ into a pseudometrizable space $\mathscr{Q}$ and Y is closed (or open) in $\mathscr{Q}$, then $f^{-1}[X]$ is exact closed (or exact open) in $\mathscr{P}$. In particular, if f is a continuous function on $\mathscr{P}$, then the inverses under f of closed sets (open sets) are exact closed (exact open). If X is simultaneously open and closed in a space $\mathscr{P}$, then X is simultaneously exact open and exact closed in $\mathscr{P}$.* Indeed, the function which is 0 on X and 1 on $|\mathscr{P}| - X$ is continuous and $X = f^{-1}[0] = f^{-1}[\,]\!] -1/2, 1/2 [\![\,]$.

In general, a closed subset (open subset) of a closure space need not be exact closed (exact open). For example, if $\mathscr{P}$ is the product of an uncountable family of at least two-point semi-separated spaces, then each point of $\mathscr{P}$ is closed but no one-point subset of $\mathscr{P}$ is a $\mathbf{G}_\delta$, and consequently no one-point subset of $\mathscr{P}$ is exact closed.

Let us consider the space $\langle \mathsf{R}, u \rangle$ where u is the topological closure operation for R such that $X \subset \mathsf{R}$ is open if and only if, for each x in X, there exists a neighborhood U of x in the space R of reals such that $U \cap \mathsf{Q} \subset X$. Clearly Q is dense in $\langle R, u \rangle$ and each subset X of $\mathsf{R} - \mathsf{Q}$ is closed in $\langle \mathsf{R}, u \rangle$. Moreover, each subset X of $\mathsf{R} - \mathsf{Q}$ is a $\mathbf{G}_\delta$ in $\langle \mathsf{R}, u \rangle$. Indeed, on arranging Q into a sequence $\{q_n\}$ and putting $U_n = (X \cup \mathsf{Q}) - \bigcup\{(q_i) \mid i \leqq n\}$ for $n \in \mathsf{N}$, we obtain a sequence of open subsets of $\langle \mathsf{R}, u \rangle$ such that $\bigcap\{U_n\} = X$. On the other hand, a u-closed $X \subset (\mathsf{R} - \mathsf{Q})$ need not be exact closed; it will suffice to show that $X \subset \mathsf{R}$ is exact closed in $\langle \mathsf{R}, u \rangle$ if and only if X is exact closed in the space R. Indeed, for example $\mathsf{R} - \mathsf{Q}$ is closed in $\langle \mathsf{R}, u \rangle$ but not in R. The identity mapping I of $\langle \mathsf{R}, u \rangle$ onto the space R of reals is continuous. It follows that if f is continuous on R then f is continuous on $\langle \mathsf{R}, u \rangle$. Conversely, from theorem 27 B.10 on extension of mappings into regular spaces it follows at once that each continuous function on $\langle \mathsf{R}, u \rangle$ is also continuous on the space R.

The space $\langle \mathsf{R}, u \rangle$ is not even regular and thus certainly not uniformizable. In the exercises examples are given of a uniformizable space in which there exists a closed $\mathbf{G}_\delta$ which is not exact closed.

Now we proceed to the description of the Čech proximity in terms of exact closed sets.

28 B.4. Theorem. *Two subsets X_1 and X_2 of a closure space $\langle P, u \rangle$ are functionally separated if and only if there exist disjoint exact closed sets Y_1 and Y_2 such that $Y_i \supset X_i$. Stated in other words, two subsets X_1 and X_2 of a closure space $\langle P, u \rangle$ are proximal relative to the Čech proximity of $\langle P, u \rangle$ if and only if the following condition is fulfilled: If Y_i are exact closed in $\langle P, u \rangle$ and $Y_i \supset X_i$, $i = 1, 2$, then $Y_1 \cap Y_2 \neq \emptyset$.*

Proof. Of course, both formulations are equivalent. I. First suppose that X_1 and X_2 are functionally separated (i.e., non-proximal relative to the Čech proximity). From 28 A.7, there exists a continuous function f on $\langle P, u \rangle$ which is 0 on X_1 and 1 on X_2. If $Y_1 = f^{-1}[0]$ and $Y_2 = f^{-1}[1]$, then Y_i are disjoint exact closed sets and clearly $X_i \subset Y_i$. – II. Now suppose that there exist exact closed sets Y_1 and Y_2 such that $Y_1 \cap Y_2 = \emptyset$ and $X_i \subset Y_i$ for $i = 1, 2$. Choose non-negative continuous functions f_i such that $Y_i = f^{-1}[0]$. The function $f_1 + f_2$ is positive. Indeed, if $(f_1 + f_2)x = 0$, then $f_i x = 0$, $i = 1, 2$, and hence $x \in (f_1^{-1}[0] \cap f_2^{-1}[0]) = Y_1 \cap Y_2 = \emptyset$ which is impossible. It follows that the function $g = f_1/(f_1 + f_2)$ is defined and continuous. Now, if $x \in Y_1$ then $gx = 0 / f_2 x = 0$, and if $x \in Y_2$ then $gx = f_1 x / f_1 x = 1$. By 28 A.7 the sets Y_i, and hence also the sets X_i, are functionally separated.

Let us recall that a closure space is topological if and only if it is locally open. Now we shall prove a similar characterization of uniformizable spaces.

28 B.5. Theorem. *Each of the following three conditions is necessary and sufficient for a closure space $\mathscr{P}$ to be uniformizable.*

(a) *$\mathscr{P}$ is locally exact open.*

(b) *$\mathscr{P}$ is a topological space and the collection of all exact open sets in an open base of $\mathscr{P}$.*

(c) *$\mathscr{P}$ is a topological space and the collection of all exact closed sets is a closed base of $\mathscr{P}$.*

Proof. Since $X \subset |\mathscr{P}|$ is exact closed if and only if the complement $|\mathscr{P}| - X$ is exact open, conditions (b) and (c) are equivalent. Obviously (a) implies (b). It remains to show that (a) is necessary and (b) is sufficient.

I. Suppose that the space $\mathscr{P}$ is uniformizable, x is a point of $\mathscr{P}$, and U is a neighborhood of x. By 28A.5 (cond. (e)) we can choose a continuous function f on $\mathscr{P}$ so that $fx = 1$ and $f[|\mathscr{P}| - U] \subset (0)$. Put $V = f^{-1}[\mathsf{R} - (0)]$. The set V is exact open and clearly $x \in V \subset U$.

II. Suppose that condition (b) holds, x is any point of $\mathscr{P}$ and U is any neighborhood of x in $\mathscr{P}$. It is to be proved that there exists a continuous function f on $\mathscr{P}$

such that $fx = 1$ and $f[|\mathscr{P}| - U] \subset (0)$. Since $\mathscr{P}$ is topological and exact open sets form an open base for the space $\mathscr{P}$, we can choose an exact open set V so that $x \in V \subset U$. Let g be a continuous function on $\mathscr{P}$ such that $V = g^{-1}[\mathsf{R} - (0)]$. Since $gx \neq 0$, the function $f = g/gx$ is defined and clearly f possesses all the required properties.

C. UNIFORMIZABLE MODIFICATION

By Definition 24 B.13 the uniformizable modification v of a closure operation u is the finest uniformizable closure coarser than u. The space $\langle P, v\rangle$ will be termed the uniformizable modification of $\langle P, u\rangle$. Evidently, $u = v$ if and only if u is uniformizable. By 24 B.16 the uniformizable modification of a closure space $\mathscr{P}$ is the unique uniformizable space $\mathscr{P}$ such that $|\mathscr{P}| = |\mathscr{Q}|$ and that a mapping of $\mathscr{P}$ into any uniformizable space is continuous if and only if the mapping $f : \mathscr{Q} \to \mathsf{E}^*f$ is continuous.

It is clear (and it was stated in 24 B.11) that if $\mathscr{U}$ is the fine uniformity of a closure space $\langle P, u\rangle$ and v is the closure induced by $\mathscr{U}$, then v is the uniformizable modification of u; clearly $\mathscr{U}$ is the fine uniformity of $\langle P, v\rangle$. Hence and from the fact that the fine uniformity, the Čech uniformity and the Čech proximity of a given closure space are topologically equivalent, we obtain the following result.

28 C.1. Theorem. *Each of the following conditions is necessary and sufficient for a uniformizable closure space $\mathscr{Q}$ to be the uniformizable modification of a given closure space $\mathscr{P}$:*

(a) *The fine uniformities of $\mathscr{P}$ and $\mathscr{Q}$ coincide.*
(b) *The Čech uniformities of $\mathscr{P}$ and $\mathscr{Q}$ coincide.*
(c) *The Čech proximities of $\mathscr{P}$ and $\mathscr{Q}$ coincide.*

28 C.2. Corollary. *Let $\mathscr{U}$ be a fine uniformity (Čech uniformity, Čech proximity) for a set P and let Γ be the set of all closures u such that $\mathscr{U}$ is the fine uniformity (Čech uniformity, Čech proximity) of $\langle P, u\rangle$. There exists a unique coarsest element v of Γ, and v is the uniformizable modification of each element of Γ.*

A uniformizable closure space is uniquely determined by the collection of all exact open sets (28 B.5). We shall describe the uniformizable modification of a given closure space $\mathscr{P}$ by the collection of all exact open sets of $\mathscr{P}$.

28 C.3. Theorem. *Let $\mathscr{X}$ be the collection of all exact open sets of a closure space $\langle P, w\rangle$ and let Γ be the set of all closures u such that $\mathscr{X}$ is the collection of all exact open sets of $\langle P, u\rangle$. There exists a unique uniformizable closure v in Γ. The closure v is the coarsest element of Γ and $\mathscr{X}$ is an open base for $\langle P, v\rangle$. The set Γ is the set of all closures u such that v is the uniformizable modification of u.*

Proof. I. Notice that the Čech proximity p of a closure space $\langle P, w\rangle$ is completely determined by the collection $\mathscr{X}$ of all exact open sets. Indeed, by 28 B.4 X_1 non $p\, X_2$ if and only if there exist exact open sets Y_i such that $X_i \subset P - Y_i$, $i = 1, 2$, and $(P - Y_1) \cap (P - Y_2) = \emptyset$. As a consequence, two spaces have the same Čech proximity if and only if the collections of exact open sets coincide. — II. It follows from I that Γ is the set of all closures u for P such that the Čech proximity of $\langle P, u\rangle$ coincides with the Čech proximity of $\langle P, w\rangle$. Let v be the uniformizable modification of w. Again by I and by 28 C.1 the closure v belongs to Γ, and v is the uniformizable modification of a closure u if and only if $u \in \Gamma$. The uniformizable modification v of a closure u is always coarser than u and $v = u$ if and only if u is uniformizable. Thus v is the unique uniformizable closure of Γ and v is the coarsest element of Γ. Finally, $\mathscr{X}$ is an open base for $\langle P, v\rangle$ by 28 B.5 because v is uniformizable.

Remark. One can prove the last theorem without any reference to proximity spaces or uniformity spaces. By 28 B.5 a space $\mathscr{P}$ is uniformizable if and only if $\mathscr{P}$ is topological and exact open sets of $\mathscr{P}$ form an open base for $\mathscr{P}$. As a consequence, a space $\mathscr{Q}$ is the uniformizable modification of a space $\mathscr{P}$ if and only if the exact open sets of $\mathscr{P}$ and $\mathscr{Q}$ coincide, $\mathscr{Q}$ is topological and the exact open sets form an open base for $\mathscr{Q}$. Now, given w, the collection $\mathscr{X}$ is a base for a topological space $\langle P, v\rangle$ and clearly v belongs to Γ. The remainder is left to the reader.

28 C.4. *Let $\mathscr{Y}$ be the collection of all exact closed sets of a closure space $\langle P, w\rangle$ and let Γ be the set of all closures u such that $\mathscr{X}$ is the collection of all exact closed sets of $\langle P, u\rangle$. There exists a unique uniformizable closure v in Γ. The closure v is a coarsest element of Γ and $\mathscr{X}$ is a closed base for $\langle P, v\rangle$. The set Γ is the set of all closures u such that v is the uniformizable modification of u.*

Proof. A set is exact open in $\langle P, u\rangle$ if and only if its complement in P is exact closed in $\langle P, u\rangle$. Apply 28 C.3.

28 C.5. *Let $\langle Q, v\rangle$ be a subspace of a closure space $\langle P, u\rangle$. If u_1 is the uniformizable modification of u, then the relativization v_1 of u_1 to Q is a uniformizable closure coarser than v but v_1 need not be the finest uniformizable closure coarser than v (compare with the corresponding result* 17 A.6 *for the topological modification). Moreover if $\langle Q, v\rangle$ is a uniformizable subspace of a space $\langle P, u\rangle$, then v is the uniformizable modification of itself but v need not be a relativization of the uniformizable modification of $\langle P, u\rangle$.* For example let $\langle P, u\rangle$ be an infinite separated space such that every continuous function on $\langle P, u\rangle$ is constant (see ex. 1), Q be an infinite isolated subset of P, and let v be the relativization of u to Q. Then v is a discrete closure and thus certainly a uniformizable closure, the uniformizable modification u_1 of u is the accrete closure for P and the relativization v_1 of u_1 to Q is also an accrete closure. Thus $v \neq v_1$.

28 C.6. It is easily seen that the uniformizable modification of the sum of a family $\{\mathscr{P}_a\}$ of closure spaces coincides with the sum of the family $\{\mathscr{Q}_a\}$ where $\mathscr{Q}_a$ is the uniformizable modification of $\mathscr{P}_a$, stated in other words, the uniformizable modification commutes with the operation of forming sums. Denoting by $\mathfrak{v}\mathscr{P}$ the uniformizable modification of a space $\mathscr{P}$ we can write

$$\mathfrak{v}\,\Sigma\{\mathscr{P}_a\} = \Sigma\{\mathfrak{v}\mathscr{P}_a\}\,.$$

On the other hand it seems nothing is known about the commutativity of the uniformizable modification and the operation of forming products.

28 C.7. We have introduced the concept of the uniform modification of a semi-uniformity $\mathscr{U}$ (as the uniformly finest uniformity uniformly coarser than $\mathscr{U}$) and the concept of the uniformizable modification of a proximity p (as the proximally finest uniformizable proximity proximally coarser than p), and we have proved that the proximity q induced by the uniform modification $\mathscr{V}$ of a semi-uniformity $\mathscr{U}$ is the uniformizable modification of the proximity induced by $\mathscr{U}$ (25 C.2). On the other hand, if a closure u is induced by a proximity p then the uniformizable modification q of p induces a uniformizable closure v which is coarser than u but which need not be the finest uniformizable closure coarser than u. For example, we shall construct a proximity p inducing the closure structure of the space R of reals such that the uniformizable modification of p is the proximally accrete closure for $|\mathsf{R}|$. Let p consist of all $\langle X, Y\rangle$ such that either X and Y are proximal in R or both X and Y are infinite. It is easily seen that p is a proximity inducing the closure structure of R (p is even the proximally coarsest proximity with this property). If x and y are any two points of R and U and V are neighborhoods of x and y respectively, then $U p V$ because both sets are infinite, and so certainly $U q V$. As a consequence $(x)\, q\, (y)$ for each x and y in R.

29. NORMAL SPACES

A uniformizable space is said to be normal if every two disjoint closed sets are functionally separated, i.e. are distant with respect to the Čech proximity of the space. It turns out that every pseudometrizable space as well as every generalized ordered space is normal (29 B.1), and that a uniformizable space need not be normal. While all the classes of spaces considered up to now were hereditary and closed under finite products, the class of all normal spaces is not hereditary and the product of two normal spaces need not be normal. On the other hand, normal spaces have many significant properties, e.g., normal spaces are characterized among all the uniformizable spaces by each of the following two important properties: every continuous function on a closed subspace of a normal space is the restriction of a continuous function on the whole space (the Urysohn-Tietze extension theorem 29 A.12); if $\mathcal{U}$ is an open point-finite cover then there exists an open cover $\{V_U \mid U \in \mathcal{U}\}$ such that $\overline{V}_U \subset U$ for each U in $\mathcal{U}$ (Theorem 29 C.1 on shrinkability of open covers).

Subsection A is concerned with various characterizations of normal spaces. In the second subsection we shall prove that every generalized ordered space is normal and we shall show, by various examples, that a subspace of a normal space need not be normal and the product of two normal spaces need not be normal. The closing subsection C examines properties of covers of normal spaces. The most profound results are Theorem 29 C.8 asserting that every open cover of a pseudometrizable space has a σ-discrete locally finite open refinement, and Theorem 29 C.1 asserting that for each point-finite open cover $\mathcal{U}$ of a normal space there exists an open cover $\{V_U \mid U \in \mathcal{U}\}$ such that the closure of each V_U, $U \in \mathcal{U}$, is contained in U. As an immediate consequence we obtain that a cover $\mathcal{U}$ of a normal space is uniformizable if and only if $\mathcal{U}$ has an open locally finite refinement.

A. CHARACTERIZATIONS OF NORMAL SPACES

29 A.1. Definition. Given a closure space $\mathcal{P}$, the proximity $\mathbf{E}\{\langle X, Y\rangle \mid \overline{X} \cap \overline{Y} \neq \emptyset\}$ for $\mathcal{P}$ will be called the *Wallman proximity* of $\mathcal{P}$. A closure *space* $\mathcal{P}$ will be called *normal* if the Wallman proximity of $\mathcal{P}$ is uniformizable and induces the closure of $\mathcal{P}$.

We begin with a discussion of the definition of normal spaces.

29 A.2. *The Wallman proximity of a closure space $\mathscr{P}$ is a continuous proximity for $\mathscr{P}$, and it is proximally finer than the Čech proximity of $\mathscr{P}$. The Wallman proximity coincides with the Čech proximity if and only if the Wallman proximity is uniformizable.*

Proof. By definition a proximity p for $\mathscr{P}$ is a continuous proximity if the closure induced by p is coarser than the closure of $\mathscr{P}$, that is, if $x \in \overline{X}$ implies $(x)\ p X$. Since $x \in \overline{X}$ implies $\overline{(x)} \cap \overline{X} \neq \emptyset$, the Wallman proximity is continuous. To prove that the Wallman proximity of $\mathscr{P}$ is proximally finer than the Čech proximity p we must show that $\overline{X} \cap \overline{Y} \neq \emptyset$ implies $X\ p\ Y$, i.e. that $\overline{X} \cap \overline{Y} \neq \emptyset$ implies that X and Y are not functionally separated. But this is obvious, for if f is a continuous function on $\mathscr{P}$ which is 0 on X and 1 on Y, then f is 0 on $\overline{X}$ and 1 on $\overline{Y}$ and this implies $\overline{X} \cap \overline{Y} = \emptyset$. Since the Čech proximity of $\mathscr{P}$ is the proximally finest continuous uniformizable proximity for $\mathscr{P}$ and the Wallman proximity is always proximally finer than the Čech proximity, the last assertion follows.

Corollary. *Given a closure space $\mathscr{P}$, the Wallman proximity of $\mathscr{P}$ coincides with the Čech proximity of $\mathscr{P}$ if and only if $\overline{X} \cap \overline{Y} = \emptyset$ implies that X and Y are functionally separated.*

Some generalities will be needed. By definition a proximity p for a closure space $\mathscr{P}$ induces the closure structure of $\mathscr{P}$ if and only if $x \in \overline{X} \Leftrightarrow (x)\ pX$. The implication $x \in \overline{X} \Rightarrow (x)\ pX$ means that p is a continuous proximity for $\mathscr{P}$. Thus a continuous proximity p for $\mathscr{P}$ induces the closure of $\mathscr{P}$ if and only if $(x)\ pX$ implies $x \in \overline{X}$. It follows that the Wallman proximity induces the closure structure of the space if and only if

$$(*) \qquad \overline{(x)} \cap \overline{X} \neq \emptyset \quad \text{implies} \quad x \in \overline{X}.$$

Let us recall that if the closure structure of a space $\mathscr{P}$ is induced by a proximity, then it is induced by the proximity $\mathsf{E}\{\langle X, Y\rangle \mid (\overline{X} \cap Y) \cup (X \cap \overline{Y}) \neq \emptyset\}$ which is the finest proximity inducing the closure, and the closure of a space $\mathscr{P}$ is induced by a proximity if and only if it is induced by a semi-uniformity, which is equivalent, by **23 B.3**, with the implication

$$(**) \qquad x \in \overline{(y)} \quad \text{implies} \quad y \in \overline{(x)}.$$

It is to be noted that $(**)$ also follows from $(*)$ directly. Indeed, if $x \in \overline{(y)}$, then $\overline{(y)} \cap \overline{(x)} \neq \emptyset$ and by $(*)$ $y \in \overline{(x)}$. It is easy to show by examples that $(**)$ does not imply $(*)$. For instance, if $\mathscr{P}$ is the three-point set $(0, 1, 2)$ endowed with the (quasi-discrete) closure operation u satisfying the equalities $u(0) = |\mathscr{P}|$, $u(1) = (0, 1)$, $u(2) = (0, 2)$, then $\mathscr{P}$ satisfies $(**)$ but not $(*)$. Thus the closure u is induced by a proximity but not by the Wallman proximity.

29 A.3. Theorem. *Let $\mathscr{P}$ be a closure space. Each of the following conditions is necessary and sufficient for the closure of $\mathscr{P}$ to be induced by the Wallman proximity of $\mathscr{P}$:*

(a) *If $x \in \mathscr{P}$, $X \subset |\mathscr{P}|$, $\overline{(x)} \cap \overline{X} \neq \emptyset$, then $x \in \overline{X}$.*

(b) *If $x, y \in \mathscr{P}$, $x \in \overline{(y)}$ and U is a neighborhood of y, then U is a neighborhood of x.*

(c) *If $x, y \in \mathscr{P}$ and $x \in \overline{(y)}$, then $U \subset |\mathscr{P}|$ is a neighborhood of y if and only if U is a neighborhood of x.*

In particular, if $\mathscr{P}$ is semi-separated (i.e. each one-point set is closed), then the closure structure of $\mathscr{P}$ is induced by the Wallman proximity. If $\mathscr{P}$ is a topological semi-uniformizable space, then the closure of $\mathscr{P}$ is induced by the Wallman proximity.

Proof. I. It has already been shown that condition (a) is both necessary and sufficient for the closure structure of $\mathscr{P}$ to be induced by the Wallman proximity. It remains to prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a). Suppose that (a) holds, $x \in \overline{(y)}$ and U is a neighborhood of y; then $y \notin \overline{|\mathscr{P}| - U}$ and hence $\overline{(y)} \cap \overline{|\mathscr{P}| - U} = \emptyset$ by (a), in particular, $x \notin \overline{|\mathscr{P}| - U}$ which means that U is a neighborhood of x. Now suppose (b) holds and $x_1 \in \overline{(y_1)}$. From condition (b) it follows at once that $y_1 \in \overline{(x_1)}$. Indeed, if U is a neighborhood of y_1, then U is a neighborhood of x_1 by (b), in particular, $x_1 \in U$, that is, $(x_1) \cap U \neq \emptyset$ which implies $y_1 \in \overline{(x_1)}$. Now (c) follows from (b) by twofold application of (b), namely to $x = x_1$, $y = y_1$ and $x = y_1$ and $y = x_1$. Finally, suppose (c) holds and $\overline{(x)} \cap \overline{X} \neq \emptyset$. Choose a point y in $\overline{(x)} \cap \overline{X}$. If U is a neighborhood of x, then U is a neighborhood of y by (c) and hence $U \cap X \neq \emptyset$. It follows that $x \in \overline{X}$.

II. If $\mathscr{P}$ is semi-separated, then $\overline{(x)} = (x)$ for each x and condition (b) is automatically fulfilled.

III. If $\mathscr{P}$ is a topological semi-uniformizable space, $x \in \overline{(y)}$ and U is an open neighborhood of y, then $y \in \overline{(x)}$ since $\mathscr{P}$ is semi-uniformizable and hence $x \in U$; but U is open and hence U is a neighborhood of x. Since open neighborhoods of y form a local base at y ($\mathscr{P}$ is topological) the condition (b) follows.

Now we return to normal spaces. It has already been observed that the Wallman proximity is uniformizable if and only if it coincides with the Čech proximity. By Theorem 25 B.2 a proximity p for a set $\mathscr{P}$ is uniformizable if and only if X_1 non pX_2 implies that $U_1 \cap U_2 = \emptyset$ for some proximal neighborhoods U_i of X_i $(i = 1, 2)$. (Recall that U is a proximal neighborhood of X if and only if $(|\mathscr{P}| - U)$ non pX.) Applying this theorem to Wallman proximities we obtain that the Wallman proximity of a closure space $\mathscr{P}$ is uniformizable if and only if the following condition is fulfilled: if $\overline{X} \cap \overline{Y} = \emptyset$, then the sets $\overline{X}$ and $\overline{Y}$ are separated, i.e. there exist neighborhoods U of $\overline{X}$ and V of $\overline{Y}$ such that $U \cap V = \emptyset$. Indeed by the definition, a subset U of a closure space $\mathscr{P}$ is a proximal neighborhood relative to the Wallman proximity of a subset X of $\mathscr{P}$ if and only if $\overline{|\mathscr{P}| - U} \cap \overline{X} = \emptyset$ which means that U is a neighborhood of $\overline{X}$. Combining this result with 29 A.3 we obtain from definition 29 A.1 the following fundamental result:

29 A.4. Theorem. *A closure space $\mathscr{P}$ is normal if and only if the following two conditions are fulfilled:*

(a) *if* $X \subset |\mathscr{P}|$, $Y \subset |\mathscr{P}|$, $\overline{X} \cap \overline{Y} = \emptyset$, *then the sets* X *and* Y *are separated;*

(b) $x \in |\mathscr{P}|$, $X \subset |\mathscr{P}|$, $\overline{(x)} \cap \overline{X} \neq \emptyset$ *imply* $x \in \overline{X}$.

Condition (b) *may be replaced by equivalent conditions* (b) *or* (c) *from* 29 A.3.

Remark. It is to be noted that the foregoing theorem is not trivial. Its proof depends essentially upon rather profound theorems, mainly on Theorem 25 B.2 which gives a simple necessary and sufficient condition for a proximity to be uniformizable, and upon the description of a uniformity in terms of uniformly continuous pseudometrics, that is, certain uniformly continuous functions on the product; the latter theorem is a consequence of the metrization lemma 18 B.10. Nevertheless, without analysing the proof it is clear that the proof cannot be trivial: from assumptions (a) and (b) which do not refer to R explicitly there follows existence of sufficiently many continuous functions.

Corollary. *A closure space* $\mathscr{P}$ *is normal if and only if it is a semi-uniformizable topological space such that every two disjoint closed sets are separated.*

Proof. If $\mathscr{P}$ is normal then $\mathscr{P}$ is uniformizable (by definition) and hence $\mathscr{P}$ is semi-uniformizable and topological. The condition (a) of the theorem implies that every two disjoint closed sets are separated. Conversely, if $\mathscr{P}$ is topological and every two disjoint closed subsets are separated, then the condition (a) of the theorem is fulfilled, and if $\mathscr{P}$ is also semi-uniformizable, then condition (b) of the theorem follows from the last assertion of 29 A.3. By the theorem, $\mathscr{P}$ is normal.

For convenience we shall prove some restatements of the condition in the corollary.

29 A.5. Theorem. *Each of the following conditions is necessary and sufficient for a semi-uniformizable topological space* $\mathscr{P}$ *to be normal:*

(a) *Every two disjoint closed subsets of* $\mathscr{P}$ *are separated.*

(b) *If* U *is a neighborhood of a closed subset* X *of* $\mathscr{P}$, *then there exists a neighborhood* V *of* X *such that* $\overline{V} \subset U$.

(b′) *For every closed subset* X *of* $\mathscr{P}$ *the collection of all closed neighborhoods of* X *is a base of the neighborhood system at* X.

(c) *If* U *and* V *are open subsets of* $\mathscr{P}$ *such that* $U \cup V = |\mathscr{P}|$ *then there exist closed subsets* X *and* Y *such that* $X \subset U$, $Y \subset V$ *and* $X \cup Y = |\mathscr{P}|$.

Corollary. *Every pseudometrizable space is normal.*

Proof of Corollary. By 20 A.4 every two disjoint closed subsets of a pseudometrizable space are separated.

Proof of 29 A.5. Condition (a) is simultaneously necessary and sufficient by virtue of the corollary of 29 A.4. Since conditions (b) and (b′) are obviously equivalent, it will suffice to prove (a) ⇒ (b) ⇒ (c) ⇒ (a). Write P instead of $|\mathscr{P}|$.

I. Suppose (a) holds and U is a neighborhood of a closed subset X of $\mathscr{P}$. Since $\mathscr{P}$ is topological, $Y = \overline{P - U}$ is closed. Clearly $X \cap Y = \emptyset$. By (a) we can choose neighborhoods U_1 of X and U_2 of Y such that $U_1 \cap U_2 = \emptyset$. Now U_1 is a neighborhood of X and $\overline{U}_1 \subset \overline{P - U_2} \subset P - Y$ for $Y \subset \operatorname{int} U_2$. Since $P - Y \subset U$, we obtain $\overline{U}_1 \subset U$.

II. Now suppose (b) and let U and V be open subsets of $\mathscr{P}$ such that $U \cup V = P$. The sets $P - U$ and $P - V$ are closed and disjoint and V is a neighborhood of $P - U$. By (b) we can choose a neighborhood U_1 of $P - U$ such that $\overline{U}_1 \subset V$. Put $Y = \overline{U}_1$ and $X = P - \operatorname{int} U_1$. Since $\mathscr{P}$ is topological the sets X and Y are closed. Clearly $X \cup Y = P$, $Y = \overline{U}_1 \subset V$ and $X = P - \operatorname{int} U_1 \subset P - (P - U) \subset U$.

III. Finally suppose (c) and consider two disjoint closed subsets X and Y. Put $U = P - Y$, $V = P - X$. The sets U and V are open and $U \cup V = P$. By (c) there exist closed sets $X_1 \subset U$ and $Y_1 \subset V$ such that $X_1 \cup Y_1 = P$. Clearly $P - X_1$ and $P - Y_1$ are disjoint neighborhoods of Y and X respectively.

Theorem 29 A.5 admits the following generalization.

29 A.6. Theorem. *Each of the following conditions is necessary and sufficient for a topological semi-uniformizable space $\mathscr{P}$ to be normal.*

(a) *Every open finite cover of $\mathscr{P}$ possesses a closed finite refinement.*

(b) *If $\{V_a\}$ is an open finite cover of $\mathscr{P}$ then there exists a closed cover $\{X_a\}$ of $\mathscr{P}$ such that $X_a \subset V_a$ for each a.*

(c) *Every open finite cover of $\mathscr{P}$ is semi-uniformizable.*

(d) *Every open finite cover of $\mathscr{P}$ is uniformizable.*

Proof. It will be shown that (d) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Leftrightarrow$ (a), (b) is necessary and sufficient and implies (d). Evidently (d) $\Rightarrow$ (c), (b) $\Rightarrow$ (a) and (b) is sufficient, because (b) implies the condition (c) of 29 A.5. Write P instead of $|\mathscr{P}|$.

I. To prove (c) $\Rightarrow$ (b), suppose (c) holds and consider an open finite cover $\{V_a\}$ of P. Since $\{V_a\}$ is semi-uniformizable, by definition there exists a semi-neighborhood U of the diagonal such that the cover $\{U[x] \mid x \in P\}$ refines $\{V_a\}$. Let Y_a be the set of all $x \in P$ such that $U[x] \subset V_a$ and put $X_a = \overline{Y}_a$. Since $\mathscr{P}$ is topological, the sets X_a are closed, and $\{X_a\}$ is a closed cover of P because $\{Y_a\}$ covers P. Since U is a semi-neighborhood of the diagonal, we have $\overline{Y}_a \subset U[Y_a] \subset V_a$ which shows that $X_a \subset V_a$ for each a.

II. To prove (a) $\Rightarrow$ (b), suppose that (a) holds and consider an open finite cover $\{V_a\}$ of P. By (a) there exists a closed finite refinement $\mathscr{X}$ of $\{V_a\}$. Let X_a be the union of all $X \in \mathscr{X}$, $X \subset V_a$. Clearly $\{X_a\}$ has the required properties.

III. The necessity of (b) follows by induction from the condition (c) of the foregoing Theorem 29 A.5.

IV. It remains to show that (b) implies (d). Suppose (b). We have proved that (b) is sufficient and hence we can suppose that the space $\mathscr{P}$ is normal, i.e. that every two disjoint closed subsets of P are functionally separated. Let $\{V_a\}$ be an open finite cover of P. By (b) there exists a closed cover $\{X_a\}$ such that $X_a \subset V_a$ for each a. Choose a family $\{f_a\}$ of non-negative continuous functions on $\mathscr{P}$ such that $f_a x = 1$ if $x \in X_a$ and $f_a x = 0$ if $x \in (P - V_a)$. Such a family exists because X_a and $P - V_a$

are functionally separated for each a. Consider the function

$$d = \{\langle x, y\rangle \to \Sigma\{|f_a x - f_a y|\}\}$$

on $P \times P$. Clearly d is a continuous pseudometric for $\mathscr{P}$. It is easy to show that d is subordinated to $\{V_a\}$ with $r = 1$, that is, each open 1-sphere is contained in some V_a. Indeed, if $x \in P$, then $x \in X_a$ for some a; now, if $y \in (P - V)$, then $|f_a x - f_a y| = |1 - 0| = 1$, and hence $d(x, y) \geqq 1$, which implies that the open 1-sphere about x is contained in V_a. By **24 E.11**, $\{V_a\}$ is uniformizable. The proof is complete.

29 A.7. *A closed subspace of a normal space is a normal space. The sum of any family of normal spaces is a normal space.*

Proof. I. Let X be a closed subspace of a normal space P. Since the class of all topological semi-uniformizable spaces is hereditary, to prove that X is normal it is enough to show that every two disjoint closed subsets of X are separated in X. If X_1 and X_2 are closed and disjoint in X, then X_1 and X_2 possess the same property in P and hence, P being normal, the sets X_1 and X_2 are separated in P and thus in X. – II. Now let P be the sum of a family $\{P_a\}$ of normal spaces. By **17 B.2** and **26 B.1** P is a semi-uniformizable topological space since all P_a are such spaces. To prove P is normal, by **29 A.5** it remains to show that every two disjoint closed subsets X_1 and X_2 of P are separated. If P'_a is the image under the canonical embedding of P_a into P, for each a, then $\{P'_a\}$ is a disjoint open cover of P. For each a, the sets $X_{1a} = X_1 \cap P'_a$ and $X_{2a} = X_2 \cap P'_a$ are closed in P'_a. Since each P'_a is normal, X_{1a} and X_{2a} are separated in P'_a for each a. Thus we can choose families $\{U_{1a}\}$ and $\{U_{2a}\}$ such that $X_{ia} \subset U_{ia} \subset P'_a$, $U_{1a} \cap U_{2a} = \emptyset$ and U_{ia} is open in P'_a for each a. If $U_i = \bigcup\{U_{ia}\}$, then U_i are open in P, $U_1 \cap U_2 = \emptyset$ and $X_i \subset U_i$, which shows that X_1 and X_2 are separated.

On the other hand, we will show that the product of two normal spaces need not be normal (**29 B.3**), and that a subspace of a normal space need not be normal (**29 B.6**). Thus the class of all normal spaces is neither productive nor hereditary. From this fact it will follow that a uniformizable space need not be normal; indeed, the class of all uniformizable spaces is hereditary and productive and it contains the class of all normal spaces. Before proceeding to the examples in question we shall strengthen the first result of the foregoing theorem.

29 A.8. Theorem. *If X is a $\mathbf{F}_\sigma$-subset of a normal space P, then the subspace X of P is normal.*

The proof of the theorem is based on the following lemma.

29 A.9. Lemma. *Every two semi-separated $\mathbf{F}_\sigma$-subsets of a normal space are separated.*

Indeed, clearly it is enough to show that every two disjoint closed subsets of X are separated, and this fact is obtained from the lemma as follows. Let X_1 and X_2 be two disjoint closed subsets of X. By **20 A.3** they are semi-separated in X and hence in P. Since clearly both sets X_i are $\mathbf{F}_\sigma$-subsets of P, they are separated in P (by the lemma), and hence in X, which establishes the theorem.

Proof of 29 A.9. Let X^1 and X^2 be two semi-separated $\mathbf{F}_\sigma$-subsets of a normal space P. By definition,

$$(\overline{X^1} \cap X^2) \cup (X^1 \cap \overline{X^2}) = \emptyset \quad \text{and} \quad X^i = \bigcup\{F_n^i \mid n \in \mathrm{N}\}$$

where F_n^i are closed. It will suffice to construct sequences $\{U_n^i \mid n \in \mathrm{N}\}$, $i = 1, 2$, of open sets such that

$$F_n^i \subset U_n^i \quad \text{and} \quad U_n^i \cap U_m^j = \emptyset$$

for $i \neq j$ and for each $n, m \in \mathrm{N}$. Indeed, if $U^i = \bigcup\{U_n^i \mid n \in \mathrm{N}\}$, then U^i are open, $U^i \supset X^i$ $(i = 1, 2)$ and $U^1 \cap U^2 = \emptyset$ because $U^i \cap U_n^j = \emptyset$ and $U^i \cap \bigcup\{U_n^j\} = \emptyset$.

The existence of such sequences $\{U_n^i\}$ will be proved by induction. Let $k \in \mathrm{N}$ and let $\{U_n^i \mid n < k\}$ be sequences of open sets such that

A) $F_n^i \subset U_n^i$ for each $n < k$, $i = 1, 2$.

B) $\overline{U_n^i} \cap X^j = \emptyset$ for each $n < k$, $i, j = 1, 2$, $i \neq j$.

C) $\overline{U_n^i} \cap \overline{U_m^j}$ for each $n, m < k$, $i \neq j$.

We shall find U_k^i, $i = 1, 2$, such that the conditions remain true with k replaced by $k + 1$. Since P is topological, the set $\bigcup\{\overline{U_n^2} \mid n < k\} \cup \overline{X^2}$ is closed and clearly disjoint with X^1 (by B and $\overline{X^2} \cap X^1 = \emptyset$), and hence with the closed set $F_k^1 \subset X^1$. Since P is normal, there exists an open neighborhood U_k^1 of F_k^1 such that

$$\overline{U_k^1} \cap (\bigcup\{\overline{U_n^2} \mid n < k\} \cup \overline{X^2}) = \emptyset\,.$$

Similarly, the set $\overline{X^1} \cup \bigcup\{\overline{U_n^1} \mid n \leqq k\}$ is closed and disjoint with X^2 and hence with the closed set F_k^2. Since P is normal, there exists an open neighborhood U_k^2 of F_k^2 such that

$$\overline{U_k^2} \cap (\bigcup\{\overline{U^1} \mid n \leqq k\} \cup \overline{X^1}) = \emptyset\,.$$

Obviously the conditions A) – C) are fulfilled with k replaced by $k + 1$.

In conclusion we shall prove some important characterizations of a normal space which will imply the Urysohn-Tietze extension theorem.

Recall that the relativization of the Čech proximity (uniformity) of a closure space $\mathscr{P}$ to a subspace $\mathscr{Q}$ of $\mathscr{P}$ is proximally (uniformly) coarser than the Čech proximity (uniformity) of $\mathscr{Q}$, and it follows from the next theorem that the relativization of the Čech proximity (uniformity) may be strictly coarser than the Čech proximity (uniformity) of a closed subspace.

29 A.10. Theorem. *Each of the following two conditions is necessary and sufficient for a topological semi-uniformizable space $\mathscr{P}$ to be normal:*

(a) *The Čech uniformity of each closed subspace of $\mathscr{P}$ is the relativization of the Čech uniformity of $\mathscr{P}$.*

(b) *The Čech proximity of each closed subspace of $\mathscr{P}$ is the relativization of the Čech proximity of $\mathscr{P}$.*

Proof. Evidently the conditions are always equivalent. Suppose (b); to prove that $\mathscr{P}$ is normal, by 29 A.2, corollary, it is enough to show that each two disjoint closed sets X and Y are functionally separated, i.e. are distant relative to the Čech proximity of $\mathscr{P}$. Let X and Y be closed and disjoint and consider the subspace $\mathscr{Q} = X \cup Y$ of $\mathscr{P}$. Since both X and Y are simultaneously closed and open in $\mathscr{Q}$, they are necessarily functionally separated in $\mathscr{Q}$ (the function which is 0 on X and 1 on Y is continuous) and therefore, by (b), they are distant with respect to the Čech proximity of $\mathscr{P}$. Conversely, if $\mathscr{P}$ is normal and X and Y are distant relative to the Čech proximity of a closed subspace $\mathscr{Q}$ of $\mathscr{P}$, then the sets $X_1 = \overline{X}^{\mathscr{Q}}$ and $Y_1 = \overline{Y}^{\mathscr{Q}}$ are disjoint, and $\mathscr{Q}$ being closed in $\mathscr{P}$, the sets X_1 and Y_1 are closed in $\mathscr{P}$. Since $\mathscr{P}$ is normal, the sets X_1 and Y_1 are distant with respect to the Čech uniformity of $\mathscr{P}$.

29 A.11. Theorem. *In order that a topological semi-uniformizable space $\mathscr{P}$ be normal it is necessary and sufficient that every bounded continuous function on any closed subspace of $\mathscr{P}$ have a continuous domain-extension on $\mathscr{P}$.*

Proof. If the condition is fulfilled and X and Y are disjoint closed subsets of $\mathscr{P}$, then clearly the function f on the subspace $Q = X \cup Y$ of $\mathscr{P}$, which is 0 on X and 1 on Y, is continuous and bounded; and if g is a continuous extension of f on $\mathscr{P}$, then g is 0 on X and 1 on Y and therefore X and Y are functionally separated in $\mathscr{P}$. By 29 A.2, $\mathscr{P}$ is normal. Conversely, if $\mathscr{P}$ is normal and f is a bounded continuous function on a closed subspace $\mathscr{Q}$ of $\mathscr{P}$, then f is uniformly continuous with respect to the Čech uniformity of $\mathscr{Q}$, which is the relativization of the Čech uniformity of $\mathscr{P}$ (by 29 A.10) and therefore, by 25 F.2, f is the restriction of a uniformly continuous function on $\langle|\mathscr{P}|, \mathscr{U}\rangle$ where $\mathscr{U}$ is the Čech uniformity of $\mathscr{P}$.

29 A.12. Urysohn-Tietze extension theorem. *Every continuous function on a closed subspace of a normal space $\mathscr{P}$ has a continuous domain-extension on $\mathscr{P}$.*

Proof. For bounded functions the theorem follows from 29 A.11. The general case is easily reduced to this case, see ex. 4.

B. EXAMPLES

Here we shall show that a subspace of a normal space need not be normal and the product of two normal spaces need not be normal. Up to now we only know one example of a normal space, namely, by Corollary of 29 A.5, every pseudometrizable space is normal. The class of all pseudometrizable spaces is hereditary and closed under countable products and therefore we must look for a different kind of normal spaces.

29 B.1. Theorem. *Every generalized ordered space is normal.*

Proof. Let u be a generalized closure for a monotone ordered set $\langle P, \leqq\rangle$. Since u is a separated (27 A.5 (a)) topological closure (15 A.14), to prove that u is normal it is enough to show that every two disjoint closed subsets X_1 and X_2 of $\langle P, u\rangle$ are separated. Suppose that X_1 and X_2 are disjoint closed subsets of $\langle P, u\rangle$.

I. Consider the relation

$$\varrho = \mathsf{E}\{\langle x, y\rangle \mid [\![x, y]\!] \cup [\![y, x]\!] \subset P - (X_1 \cup X_2)\} .$$

It is easily seen that ϱ is an equivalence on $G = P - (X_1 \cup X_2)$. Let $f : G/\varrho \to G$ be a mapping such that $fR \in R$ for each equivalence class $R \in G/\varrho$.

II. We shall construct families $\{U_i(x) \mid x \in X_i\}$, $i = 1, 2$, such that $U_i(x)$ is a neighborhood of x and $U_i = \bigcup\{U_i(x) \mid x \in X_i\}$ $i = 1, 2$, are disjoint neighborhoods of X_1 and X_2.

III. Fix $i = 1, 2$ and let $j \neq i, j = 1, 2$ (thus $i = 1$ and $j = 2$, or $i = 2$ and $j = 1$). Let x be a fixed element of X_i. The neighborhood $U_i(x)$ will have the form $U_i^-(x) \cup \cup U_i^+(x)$. For the sake of brevity we shall write simply U, U^+ and U^-. First we shall define U^- : (a) If the point x is left-isolated, then $U^- = (x)$. (b) If each neighborhood of x contains at least one $y \in X_i$, $y < x$, then $U^- =]\!] y, x]\!]$ where y is a point of X_i, such that $y < x$ and $X_j \cap]\!] y, x [\![= \emptyset$. Such a point y exists because $P - X_j$ is a neighborhood of x. (c) In the remaining case, the point x is not left-isolated and there exists a neighborhood V of x such that x is the least element of $V \cap X_i$ (i.e. $[\![\leftarrow, x]\!] \cap V \cap X_i = (x)$). Since $P - X_j$ is a neighborhood of x, the set $W = V - X_j$ is a neighborhood of x. If $y \in W$, $y < x$, then clearly $y \in G$. Since x is not left-isolated we can choose a $y < x$, $y \in G \cap W$. The point y belongs to an equivalence class $\varrho[y]$ and we define $U^- =]\!] z, x]\!]$, where $z = f\varrho[y]$. Since $y < x$ and $\langle z, y\rangle \in \varrho$, we find that $z < x$. The set U^+ is defined similarly; moreover, it is enough to replace left-isolated, $y < x$, least element, U^- by right-isolated, $x < y$, greatest element, U^+, respectively.

IV. We must show that $U_1 \cap U_2 = \emptyset$. Clearly, $U_1 \cap X_2 = U_2 \cap X_1 = \emptyset$. If $y \in U_i \cap G$ $(i = 1, 2)$ then one of the following three cases obtains: (a$_i$) there exist $x_k \in X_i$, $k = 1, 2$, such, that $\varrho[y] \subset]\!] x_1, x_2 [\![\subset P - X_j$; (b$_i$) $y > f\varrho[y]$, and there exists $\inf \{x \mid x \in (X_1 \cup X_2),\ x > f\varrho[y]\}$ and it belongs to X_i; (c$_i$) $y < f\varrho[y]$ and $\sup \{x \mid x \in (X_1 \cup X_2),\ x < f\varrho[y]\} \in X_i$. The straightforward verification is left to the reader. From this it follows at once that $U_1 \cap U_2 = \emptyset$. The proof is complete.

A subspace of a generalized ordered space is a generalized ordered space and therefore, by the preceding theorem, each subspace of a generalized ordered space is normal. On the other hand the product of two generalized ordered spaces need not be normal. We shall need the following simple proposition which is often used to prove that a space is not normal.

29 B.2. *If $\mathscr{P}$ is a normal space of a density character $\mathfrak{m} \geqq \aleph_0$ and if X is a closed discrete subspace of $\mathscr{P}$, then*

$$\exp \operatorname{card} X \leqq (\exp \aleph_0)^{\mathfrak{m}} .$$

In particular, if $\mathscr{P}$ is a normal space with a countably infinite density character and X is a closed discrete subspace of $\mathscr{P}$, then

$$\exp \operatorname{card} X \leqq (\exp \aleph_0)^{\aleph_0} = \exp \aleph_0 .$$

Proof. Let us consider a dense subspace Y of $\mathscr{P}$ of cardinal $\mathfrak{m}$. By 27 A.8, corollary, the cardinal of $\mathbf{C}(\mathscr{P}, \mathsf{R})$ is at most $(\exp \aleph_0)^{\mathfrak{m}} = (\text{card } \mathsf{R})^{\mathfrak{m}}$ which is the cardinal of the set of all functions on Y. Now let X be a closed discrete subspace of $\mathscr{P}$. If $Z \subset X$, then Z is closed in X and hence, X being closed in $\mathscr{P}$, Z is closed in $\mathscr{P}$. Since $\mathscr{P}$ is normal, every two disjoint closed subsets of $\mathscr{P}$ are functionally separated; therefore, for each $Z \subset X$, we can choose a continuous function f_Z on $\mathscr{P}$ which is 0 on Z and 1 on $X - Z$. Clearly $\{Z \to f_Z\}$ is a one-to-one mapping of $\exp X$ into $\mathbf{C}(\mathscr{P}, \mathsf{R})$. Consequently the cardinal of $\exp X$ is at most that of $\mathbf{C}(\mathscr{P}, \mathsf{R})$ which is, as has been proved above, at most $(\exp \aleph_0)^{\mathfrak{m}}$.

29 B.3. Example. Let $\mathscr{P} = \langle \mathsf{R}, u \rangle$ denote the set of reals endowed with the closure of the right-approximation; i.e., given an x, the collection $\mathbf{E}\{ [\![x, y [\![\mid x < y\}$ is a local base at x in $\mathscr{P}$. By 29 B.1 the space $\mathscr{P}$ is normal. We shall prove that the product space $\mathscr{P} \times \mathscr{P}$ is not normal. Consider the set X consisting of all pairs $\langle x, -x \rangle$, $x \in \mathsf{R}$. It is easily seen that X is closed in $\mathscr{P} \times \mathscr{P}$, e.g. it is enough to prove that X is closed in $\mathsf{R} \times \mathsf{R}$; the diagonal D of $\mathsf{R} \times \mathsf{R}$ is closed in $\mathsf{R} \times \mathsf{R}$ and $\{\langle x, y \rangle \to \to \langle x, -y \rangle\} : \mathsf{R} \times \mathsf{R} \to \mathsf{R} \times \mathsf{R}$ is a homeomorphism which carries X into D. Next, X is a discrete subspace of $\mathscr{P} \times \mathscr{P}$. Indeed, the set $U = [\![x, \to]\!] \times [\![-x, \to]\!]$ is a neighborhood of $\langle x, -x \rangle$ in $\mathscr{P} \times \mathscr{P}$, and $U \cap X = (\langle x, -x \rangle)$ because $x < y$ implies $-y < -x$. Thus $\mathscr{P} \times \mathscr{P}$ contains a closed discrete subspace of cardinal $\exp \aleph_0$. Now it follows from 29 B.2 that $\mathscr{P} \times \mathscr{P}$ is not normal provided that the density character of $\mathscr{P}$ is countable. Clearly the set Q of rational numbers is dense in $\mathscr{P}$ and hence $\mathsf{Q} \times \mathsf{Q}$ is dense in $\mathscr{P} \times \mathscr{P}$. Since Q is countable, $\mathsf{Q} \times \mathsf{Q}$ is also countable, and the space $\mathscr{P}$ is not normal.

To prove that the class of all normal spaces is not hereditary we must look for yet another kind of normal spaces, because each subspace of a generalized ordered space is a generalized ordered space and so certainly a normal space.

29 B.4. *The following condition is sufficient for a regular topological space $\mathscr{P}$ to be normal: each net in $\mathscr{P}$ has an accumulation point.*

Proof. Assuming that some disjoint closed subsets X_1 and X_2 are not separated we shall derive a contradiction. Let $\mathscr{X}_i$ be the neighborhood system of X_i in $\mathscr{P}$ and let $\mathscr{X} = [\mathscr{X}_1] \cap [\mathscr{X}_2]$ $(= \mathbf{E}\{U_1 \cap U_2 \mid U_i \in \mathscr{X}_i\})$. By our assumption $\mathscr{X}$ is a proper filter of sets on $\mathscr{P}$. Let x be an accumulation point of a net $N = \langle \{N_X \mid X \in \mathscr{X}\}, \supset \rangle$, where $N_X \in X$. The net N is frequently in each neighborhood of X_i, $i = 1, 2$, and therefore, $\mathscr{P}$ being regular and X_i being closed, $x \in X_i$. Thus $X_1 \cap X_2 \neq \emptyset$ which contradicts our assumption $X_1 \cap X_2 = \emptyset$.

29 B.5. Remark. A space $\mathscr{P}$ satisfying the condition of 29 B.4 is said to be compact. Thus 29 B.4 can be stated as follows: every compact topological regular space is normal. It should be remarked that compact spaces will be investigated in Section 41. By 15 B.16 every order-complete ordered space is compact. We shall need the following important result: The product $\mathscr{P}$ of a family $\{\mathscr{P}_a \mid a \in A\}$ of compact spaces is a compact space. Let N be a net in $\mathscr{P}$. By 15 ex. 5, there exists a generalized subnet

M of N such that M is an ultranet in $\mathcal{P}$, i.e., M is eventually in X or in $|\mathcal{P}| - X$ for each $X \subset |\mathcal{P}|$. For each a in A let $M_a = \mathrm{pr}_a \circ M$. Evidently M_a is an ultranet in $\mathcal{P}_a$ for each a. Choose an x in $\mathcal{P}$ such that $\mathrm{pr}_a\, x$ is an accumulation point of M_a in $\mathcal{P}_a$ for each a. Since M_a in an ultranet, $\mathrm{pr}_a\, x$ is a limit point of M_a and therefore x is a limit point of M. Thus x is an accumulation point of N.

29 B.6. Every separated uniformizable space is a subspace of a normal space, namely of a homeomorph of a cube $[\![0, 1]\!]^{\aleph}$; this latter is normal by 29 B.4 because it is uniformizable and, by 29 B.5, each net has an accumulation point. On the other hand, a separated uniformizable space need not be normal because the product of two separated normal spaces is uniformizable but need not be normal (by 29 B.3). Thus the class of all normal spaces is not hereditary.

Further examples seem to be in place.

29 B.7. *For each ordinal α let T_α denote the set of all ordinals less than α, endowed with the order closure. By* 29 B.1 *every T_α is a normal space. It is easily seen that T_α is a subspace of each T_β, $\alpha < \beta$. We shall prove that*

(a) *the product space $\mathcal{P} = T_{\omega_1} \times T_{\omega_1+1}$ is not normal, and*

(b) *the product space $\mathcal{Q} = T_{\omega_1+1} \times T_{\omega_1+1}$ is normal.*

Thus we obtain further example of two normal spaces whose product is not normal, and an example of a normal space (namely $\mathcal{Q}$) containing a non-normal space, namely $\mathcal{P}$.

The ordered set T_{ω_1+1} is order-complete and therefore, by 29 B.5, the space T_{ω_1+1} and hence, again by 29 B.5, the space $\mathcal{Q}$, fulfils the condition of 29 B.4. Since $\mathcal{Q}$ is uniformizable, $\mathcal{Q}$ is normal by 29 B.4.

To prove that the space $\mathcal{P}$ is not normal it is sufficient to show that the sets $X = \mathbf{E}\{\langle \alpha, \omega_1 \rangle \mid \alpha < \omega_1\} = T_{\omega_1} \times (\omega_1)$ and $Y = \mathbf{E}\{\langle \alpha, \alpha \rangle \mid \alpha < \omega_1\}$ are not separated in $\mathcal{P}$. Indeed, the sets X and Y are obviously disjoint, X is closed as the product of two closed sets, namely T_{ω_1} and (ω_1), and Y is closed by 27 A.7 because T_{ω_1+1} is separated. To show that X and Y are not separated it is sufficient to prove that $\overline{U} \cap Y \neq \emptyset$ for each neighborhood U of X. Let U be a neighborhood of X. By induction one can easily prove that there exists a sequence $\{\langle \alpha_n, \beta_n \rangle\}$ in U such that $\alpha_n \leqq \beta_n \leqq \alpha_{n+1}$ for each $n \in \mathrm{N}$. Let α be the least upper bound of the sequence $\{\alpha_n\}$ in T_{ω_1}. The least upper bound exists because T_{ω_1}, as each well-ordered set, is boundedly order-complete, and each countable family in T_{ω_1} is bounded because ω_1 is not cofinal with ω_0. By definition of the order closure, the sequence $\{\alpha_n\}$ converges to α in T_{ω_1}. Clearly $\alpha = \sup\{\beta_n\}$, and hence the sequence $\{\beta_n\}$ converges to α in T_{ω_1}. It follows that the sequence $\{\langle \alpha_n, \beta_n \rangle\}$ converges to $\langle \alpha, \alpha \rangle$ in $\mathcal{P}$. But $\langle \alpha_n, \beta_n \rangle \in U$ for each n and hence $\langle \alpha, \alpha \rangle \in \overline{U}$. Thus $Y \cap \overline{U} \neq \emptyset$, which completes the proof.

Remark. If α is any ordinal which contains no countable cofinal subset, then $T_\alpha \times T_{\alpha+1}$ is not normal. The proof of 29 B.7 applies.

Now we shall show that a non-normal space may be obtained from a normal space by removing one point.

29 B.8. Let $\mathscr{P}$ and $\mathscr{Q}$ be closure spaces such that $\mathscr{Q}$ is infinite, the cardinal of $\mathscr{P}$ is greater than that of $\mathscr{Q}$, $\mathscr{P}$ has exactly one cluster point, say x, and the complements of neighborhoods of x are finite, and the space $\mathscr{Q}$ also has exactly one accumulation point, say y, and complements of neighborhoods of y are finite.

Evidently each net in $\mathscr{P}$ has a constant subnet or converges to x and therefore each net in $\mathscr{P}$ has an accumulation point. The same argument yields that each net in $\mathscr{Q}$ has an accumulation point. By 29 B.5 each net in the product space $\mathscr{P} \times \mathscr{Q}$ has an accumulation point. Since both $\mathscr{P}$ and $\mathscr{Q}$ are regular, the space $\mathscr{P} \times \mathscr{Q}$ is also regular. By 29 B.4 the product space $\mathscr{P} \times \mathscr{Q}$ is normal. We shall prove that the subspace $\mathscr{R}$ of $\mathscr{P} \times \mathscr{Q}$, whose underlying set is $(|\mathscr{P}| \times |\mathscr{Q}|) - (\langle x, y\rangle)$, is not normal. Consider the sets

$$X = (|\mathscr{P}| - (x)) \times (y), \quad Y = (x) \times (|\mathscr{Q}| - (y)).$$

Clearly $X \cap Y = \emptyset$ and it is easily seen that both sets are closed in $\mathscr{R}$, and we shall show that X and Y are not separated in $\mathscr{R}$. It is sufficient to show that the closure of any neighborhood U of Y intersects X, and this will be proved by showing that $(z) \times (|\mathscr{Q}| - (y))$ is contained in U for some z in $|\mathscr{P}| - (x)$ because then the point $\langle z, y\rangle$ belongs to both X and the closure of U. For each t in $|\mathscr{Q}| - (y)$ there exists a finite subset F_t of $|\mathscr{P}|$ such that $(|\mathscr{P}| - F_t) \times (t) \subset U$. The cardinal of the set $F = \bigcup\{F_t \mid t \in |\mathscr{Q}|, t \neq y\}$ is at most that of $|\mathscr{Q}|$, and hence less than the cardinal of $|\mathscr{P}|$. Thus the set $|\mathscr{P}| - F$ is infinite and any point $z \in (|\mathscr{P}| - F)$, $z \neq x$, has the required properties.

29 B.9. The proof of the fact that the space $\mathscr{R}$ of 29 B.8 is not normal can be applied in a more general situation: Let $\mathscr{P}$ and $\mathscr{Q}$ be semi-separated spaces, $x \in |\mathscr{P}|$, y be a cluster point of $\mathscr{Q}$ and suppose that the singleton (x) is not the intersection of a family of its neighborhoods such that the cardinal of the index set is at most the cardinal of $|\mathscr{Q}|$. Then the disjoint closed subsets X and Y of $\mathscr{R}$ are not separated in $\mathscr{R}$, where X, Y and $\mathscr{R}$ are defined as in 29 B.8. For example, the space $(T_{\omega_1+1} \times T_{\omega_0+1}) - (\langle\omega_1, \omega_0\rangle)$ is not normal, and more generally, $(T_{\alpha+1} \times T_{\beta+1}) - (\langle\alpha, \beta\rangle)$ is not normal whenever β is a limit ordinal and $\beta > \alpha$ is not cofinal with α.

Remark. The space $\mathscr{R}$ of 29 B.8 can be used for a construction of a regular topological space which is not uniformizable. This construction requires the concept of the quotient of a closure space under an equivalence; this will be introduced in Section 33, where the construction in question and also a construction of an infinite regular separated topological space without non-constant continuous functions will be given.

29 B.10. Non-normality of products with one point removed. By 29 B.5 each cube $\mathscr{P} = [\![0, 1]\!]^{\aleph}$ is a normal space. On the other hand we shall show that the subspace $\mathscr{P} - (x)$ of $\mathscr{P}$ is not normal for each $x \in |\mathscr{P}|$ whenever $\aleph \geqq \exp \aleph_0$. First we shall show that removing some point from a Cantor space $2^{\aleph}$, $\aleph \geqq \exp \aleph_0$, we obtain a space which is not normal; more precisely

(a) *Let* 2 *denote the set* $(0, 1)$ *endowed with the discrete closure* γ, *and let* 0 *be the point of the product space* $2^{\aleph}$ *(the so-called Cantor space) such that each of its coordinates is zero. If* $\aleph \geqq \exp \aleph_0$, *then the subspace* $\mathscr{P}$ *of* $2^{\aleph}$, *whose underlying set is* $2^{\aleph} - (0)$, *is not normal.*

It is evident that we may assume that $\aleph = \exp \aleph_0$. Let A be a set of cardinal $\exp \aleph_0$ and let $\mathscr{P} = 2^A - (0)$. Let us consider the set X consisting of all points x^b, $b \in A$, where $\mathrm{pr}_a x^b = 0$ if $a \neq b$ and $\mathrm{pr}_b x^b = 1$. It is easily seen that X is a closed discrete subspace of $\mathscr{P}$. Evidently the cardinal of X is $\exp \aleph_0$. By 22 ex. 1, the density character of the space 2^A and therefore of $\mathscr{P}$ is $\aleph_0$. By 29 B.2 the space $\mathscr{P}$ is not normal.

(b) *Let* $\{\mathscr{P}_a \mid a \in A\}$ *be a family of semi-separated spaces such that each* $\mathscr{P}_a$ *has at least two points. If* $\mathscr{P}$ *is the product of* $\{\mathscr{P}_a\}$, x *is a point of* $\mathscr{P}$ *and the cardinal of* A *is at least* $\exp \aleph_0$, *then the subspace* $\mathscr{P} - (x)$ *of* $\mathscr{P}$ *is not normal.*

Let us choose a point y such that $\mathrm{pr}_a x \neq \mathrm{pr}_a y$ for each a. Let f be a mapping of the Cantor space 2^A into $\mathscr{P}$ which assigns to each z the point whose a-th coordinate is $\mathrm{pr}_a x$ if $\mathrm{pr}_a z = 0$ and $\mathrm{pr}_a y$ if $\mathrm{pr}_a z = 1$. Evidently f is an embedding and $f0 = x$. Since clearly $\mathbf{E}f$ is closed in $\mathscr{P}$, $\mathbf{E}f - (x)$ is closed in $|\mathscr{P}| - (x)$. Finally, $\mathbf{E}f - (x)$ is not normal because $2^A - (0)$ is not normal. As a consequence, $|\mathscr{P}| - (x)$ is not normal.

C. COVERS OF NORMAL SPACES

By theorem 29 A.6, if $\{U_a\}$ is a finite open cover of a normal space $\mathscr{P}$ then there exists an open cover $\{V_a\}$ of $\mathscr{P}$ such that the closure of V_a is contained in U_a for each a. This result permits the following essential generalization.

29 C.1. Theorem. *Let* $\{U_a \mid a \in A\}$ *be a point-finite open cover of a normal space* $\mathscr{P}$. *There exists an open cover* $\{V_a \mid a \in A\}$ *of* $\mathscr{P}$ *such that* $\overline{V}_a \subset U_a$ *for each* a *in* A.

Remark. By 29 A.6 the property of normal spaces stated in 29 C.1 characterizes normal spaces in the class of all topological semi-uniformizable spaces.

Proof. I. Let $\{X_a \mid a \in A\}$ be an open cover of $\mathscr{P}$ such that $X_a \subset U_a$ for each a and let $b \in A$. We shall prove that there exists an open cover $\{Y_a\}$ such that $Y_a = X_a$ for each $a \neq b$ and $\overline{Y}_b \subset X_b$. Let X be the union of all X_a, $a \neq b$. Clearly, (X, X_b) is an open cover of $\mathscr{P}$ and therefore, $\mathscr{P}$ being normal, we can choose an open set Y_b such that $\overline{Y}_b \subset X_b$ and (X, Y_b) is a cover of $\mathscr{P}$. Clearly $\{Y_a\}$ has the required property. The remaining part of the proof does not depend on normality and may be applied in a more general situation (see ex. 5).

II. Let us consider the set $\mathscr{F}$ of all open covers $\{X_a \mid a \in A\}$ of $\mathscr{P}$ such that, for each a in A, either $\overline{X}_a \subset U_a$ or $X_a = U_a$. For each $\{X_a\}$ and $\{Y_a\}$ in $\mathscr{F}$ let $\{X_a\} \prec$ $\prec \{Y_a\}$ if and only if $X_a \supset Y_a$ for each a and $X_a = Y_a$ whenever $\overline{X}_a \subset U_a$. Clearly the relation $\prec$ orders $\mathscr{F}$. It follows from I that any maximal element $\{X_a\}$ of $\langle \mathscr{F}, \prec \rangle$ is the required refinement of $\{U_a\}$. Indeed, assuming that $\overline{X}_b$ is not contained in U_b, then $X_b = U_b$ (because $\{X_a\} \in \mathscr{F}$) and clearly the cover $\{Y_a\}$ of I follows $\{X_a\}$ in $\langle \mathscr{F}, \prec \rangle$; but, clearly, $\{X_a\}$ does not follow $\{Y_a\}$ and hence $\{X_a\}$ is not a maximal element, which contradicts our assumption. Thus the proof will be complete if we show that there exists at least one maximal element of $\langle \mathscr{F}, \prec \rangle$. To prove the existence of a maximal element it is sufficient to show that each non-void monotone set in $\langle \mathscr{F}, \prec \rangle$ has an upper bound.

III. Let $\mathscr{G}$ be a non-void monotone set in $\langle \mathscr{F}, \prec \rangle$. Thus, if $\mathscr{X} \in \mathscr{G}$, $\mathscr{Y} \in \mathscr{G}$ then $\mathscr{X} \prec \mathscr{Y}$ or $\mathscr{Y} \prec \mathscr{X}$. For brevity we shall write $\mathscr{X} = \{\mathscr{X}_a \mid a \in A\}$ for each $\mathscr{X}$ in $\mathscr{G}$. For each a in A let $Z_a = \bigcap\{\mathscr{X}_a \mid \mathscr{X} \in \mathscr{G}\}$. We shall prove that $\mathscr{Z} = \{Z_a\}$ is an upper bound of $\mathscr{G}$ in $\langle \mathscr{F}, \prec \rangle$. If $a \in A$, then either $\mathscr{X}_a = U_a$ for each $\mathscr{X}$ in $\mathscr{G}$ or $\overline{\mathscr{X}_a} \subset U_a$ for some $\mathscr{X}$ in $\mathscr{G}$ and then $\mathscr{Y}_a = \mathscr{X}_a$ for each $\mathscr{Y} \succ \mathscr{X}$ and hence $Z_a = \overline{\mathscr{X}_a}$. It follows that the set Z_a is open, $Z_a = U_a$ or $\overline{Z}_a \subset U_a$, and finally, if $\mathscr{X} \in \mathscr{G}$ then $Z_a \subset \mathscr{X}_a$, and $\overline{\mathscr{X}_a} \subset U_a$ implies that $Z_a = \mathscr{X}_a$. As a consequence it remains to prove that $\{Z_a\}$ is cover of $\mathscr{P}$ and this follows from our assumption that $\{U_a\}$ is point-finite. Assume that a point x of $\mathscr{P}$ belongs to no Z_a. There exists a finite subset B of A such that no U_a, $a \in A - B$, contains x. It is self-evident that, for each b in B, there exists an $\mathscr{X}^b$ in $\mathscr{G}$ such that $\mathscr{X}^b_b$ does not contain x. Since $\{\mathscr{X}^b \mid b \in B\}$ is a finite family in $\mathscr{G}$ and $\mathscr{G}$ is monotone, we can choose an $\mathscr{X}$ in $\mathscr{G}$ following each $\mathscr{X}^b$, $b \in B$. Since $\mathscr{X}_b \subset \mathscr{X}^b_b$, the set $\mathscr{X}_b$ does not contain x for any b in B. If $a \in A - B$, then U_a does not contain x, and hence $\mathscr{X}_a$ $(\subset U_a)$ does not contain x. Thus no $\mathscr{X}_a$, $a \in A$, contains x; this contradicts our assumption that $\mathscr{X}$ is a cover of $\mathscr{P}$. The proof is complete.

29 C.2. Theorem. *Every locally finite open cover of a normal space is uniformizable.*

The proof follows from 29 C.1 and the following lemma.

29 C.3. *Let $\{X_a \mid a \in A\}$ be a locally finite cover of a closure space $\mathscr{P}$ and let $\{Y_a \mid a \in A\}$ be a cover of $\mathscr{P}$ such that Y_a and $|\mathscr{P}| - X_a$ are functionally separated for each a in A. Then $\{X_a\}$ is a uniformizable cover of $\mathscr{P}$.*

Indeed, if $\{U_a\}$ is a locally finite open cover of normal space $\mathscr{P}$, then by virtue of 29 C.1 there exists a cover $\{Y_a\}$ of $\mathscr{P}$ such that $\overline{Y}_a \subset U_a$ for each a. Since $\mathscr{P}$ is normal, the sets $\overline{Y}_a$ and $|\mathscr{P}| - U_a$ are functionally separated for each a. By 29 C.3 the cover $\{U_a\}$ is uniformizable.

Proof of 29 C.3. For each a in A let us choose a non-negative continuous function f_a such that $f_a x = 1$ if $x \in Y_a$ and $f_a x = 0$ if $x \in |\mathscr{P}| - X_a$; consider the relations $d_a = \{\langle x, y\rangle \to |f_a x - f_a y| \mid \langle x, y\rangle \in \mathscr{P} \times \mathscr{P}\}$. Each d_a is a continuous pseudometric for $\mathscr{P}$. Let $d = \Sigma\{d_a \mid a \in A\}$. It will be shown that d is a continuous pseudometric for $\mathscr{P}$ which is subordinated to the cover $\{X_a\}$; by 24 E.11 it will follow that the cover $\{X_a\}$ is uniformizable.

Clearly d is a pseudometric for $\mathscr{P}$. To prove that d is a continuous pseudometric it is enough to show that d is a continuous function on $\mathscr{P} \times \mathscr{P}$. Let $\langle x, y\rangle$ be any point of $\mathscr{P} \times \mathscr{P}$. Since $\{X_a\}$ is locally finite and each f_a vanishes on $|\mathscr{P}| - X_a$, we can choose a finite subset A_0 of A and neighborhoods U of x and V of y such that all f_a, $a \in A - A_0$, vanish on both sets U and V, and hence $d_a z = 0$ if $z \in U \times V$ and $a \in A - A_0$. It follows that if $\langle x', y' \rangle \in U \times V$, then

$$d\langle x', y' \rangle = \Sigma\{d_a\langle x', y' \rangle \mid a \in A_0\} .$$

Thus the restriction of d to $U \times V$ is the sum of a finite family of continuous functions, namely of restrictions of d_a to $U \times V$ with a in A_0. Therefore the restriction of d to $U \times V$ is a continuous function. We have proved that each point of $\mathscr{P} \times \mathscr{P}$ possesses a neighborhood W such that the restriction of d to W is continuous; thus d is continuous on $\mathscr{P} \times \mathscr{P}$.

It remains to prove that d is subordinated to $\{X_a\}$, that is, each open 1-sphere (relative to d) is contained in some X_a, $a \in A$. Let x be any point of $\mathscr{P}$. Choose an a in A so that $x \in Y_a$. By the choice of f_a, $d_a\langle x, y \rangle < 1$ implies $y \in X_a$. Indeed, $d_a\langle x, y \rangle = |f_a x - f_a y|$, $f_a x = 1$ and $y \in (|\mathscr{P}| - X_a)$ implies $f_a y = 0$. It follows that $d\langle x, y \rangle < 1$ implies $y \in X_a$. Indeed, $d\langle x, y \rangle < 1$ implies $d_a\langle x, y \rangle < 1$ because $d_a \leqq d$.

If a cover $\mathscr{U}$ of a space is refined by a uniformizable cover, then $\mathscr{U}$ is uniformizable. If follows from 29 C.2 that a cover $\mathscr{U}$ of a normal space is uniformizable provided that it is refined by an open locally finite cover. It seems appropriate to prove here that any uniformizable cover of any space is refined by a locally finite open cover. The proof of this fact is rather complicated. We begin with a definition.

29 C.4. Definition. A family $\{X_a \mid a \in A\}$ of subsets of a closure space $\mathscr{P}$ is said to be *σ-point-finite* or *σ-discrete* if there exists $\{A_n \mid n \in \mathsf{N}\}$ such that $A = \bigcup\{A_n\}$ and each family $\{X_a \mid a \in A_n\}$, $n \in \mathsf{N}$, is respectively point-finite or discrete.

Let us recall that a family $\{X_a \mid a \in A\}$ of sets is called *σ-locally finite* if $A = \bigcup\{A_n\}$ such that each family $\{X_a \mid y \in A_n\}$ is locally finite.

The definitions of locally countable and point-countable families are evident. Any σ-locally finite family is σ-point-finite, and therefore point-countable, but need not be locally countable. In the converse direction, a locally countable family need not be σ-point-finite, and therefore a point-countable family need not be σ-point-finite.

For example, let $\mathscr{P}$ be the space T_{ω_1} of countable ordinals and let us consider the family $\{\leqq [x] \mid x \in \mathscr{P}\}$. This family is locally countable ($\mathsf{E}\{\alpha \mid \alpha \leqq \beta\}$ is a required neighborhood of β) but not σ-point-finite.

29 C.5. Theorem. *If $\mathscr{U}$ is a uniformizable cover of a closure space $\mathscr{P}$, then there exists a locally finite σ-discrete open refinement $\{X_a \mid a \in A\}$ of $\mathscr{U}$ and a cover $\{Y_a \mid a \in A\}$ of $\mathscr{P}$ such that the sets Y_a and $|\mathscr{P}| - X_a$ are functionally separated for each a in A.*

Combining 29 C.5 with 29 C.3 we immediately obtain the following fundamental characterization of uniformizable covers.

29 C.6. Theorem. *In order that a cover $\mathscr{U}$ of a closure space $\mathscr{P}$ be uniformizable it is necessary and sufficient that there exist covers $\{X_a \mid a \in A\}$ and $\{Y_a \mid a \in A\}$ of $\mathscr{P}$ such that $\{X_a\}$ is a locally finite refinement of $\mathscr{U}$ and the sets Y_a and $|\mathscr{P}| - X_a$ are functionally separated for each a in A (moreover, the sets X_a may be required open).*

Combining 29 C.5 with 29 C.2 the following fundamental characterization of uniformizable covers of normal spaces is immediately evident.

29 C.7. Theorem. *A cover of normal space is uniformizable if and only if it is refined by a locally finite open cover.*

The proof of Theorem 29 C.5 is an immediate consequence of the following theorem stating one of the most profound properties of pseudometrizable spaces.

29 C.8. Theorem. *Each interior cover of a pseudometrizable space is refined by a σ-discrete locally finite open cover.*

Indeed, let $\mathscr{U}$ be a uniformizable cover of a closure space $\langle P, u\rangle$. By 24 E.11 there exists a continuous pseudometric d for $\langle P, u\rangle$ which is subordinated to $\mathscr{U}$, that is, the identity mapping of $\langle P, u\rangle$ onto $\langle P, d\rangle$ is continuous and each open 1-sphere is contained in a member or an element of $\mathscr{U}$. Clearly $\mathscr{U}$ can be regarded as an interior cover of the pseudometric space $\langle P, d\rangle$. Indeed, if $x \in P$, then the open 1-sphere about x is contained in some member or element U of $\mathscr{U}$, and hence x belongs to the interior of U in $\langle P, d\rangle$. Now let $\{X_a\}$ be a σ-discrete locally finite open refinement of the cover $\mathscr{U}$ of $\langle P, d\rangle$ (such a cover exists by virtue of 29 C.8). Since the space $\langle P, d\rangle$ is normal, there exists a cover $\{Y_a\}$ of $\langle P, d\rangle$ such that the sets Y_a and $P - X_a$ are functionally separated (in $\langle P, d\rangle$) for each $a \in A$. Since the identity mapping of $\langle P, u\rangle$ onto $\langle P, d\rangle$ is continuous, the covers $\{X_a\}$ and $\{Y_a\}$ of $\langle P, u\rangle$ possess the required properties.

Proof of 29 C.8. Let $\{U_a \mid a \in A\}$ be an open cover of a pseudometric space $\langle P, d\rangle$. Let $\leqq$ be a well-order for the set A. For each $k = 1, 2, \ldots$ and each $a \in A$ let

$$X_{ak} = \mathbf{E}\{x \mid \operatorname{dist}(x, P - U_a) \geqq k^{-1},\ b < a \Rightarrow x \notin U_b\},$$

$$X_k = \bigcup\{X_{ak} \mid a \in A\}.$$

We shall prove that $\{X_{ak} \mid a \in A, k \in (\mathsf{N} - (0))\}$ is a cover of P. Given $x \in P$, choose the least a such that $x \in U_a$; the distance from x to $P - U_a$ is positive (remember that U_a is open) and therefore greater than k^{-1} for some $k = 1, 2, \ldots$. Since the point x belongs to no U_b with $b < a$, x belongs to X_{ak}. Next, notice that no open $1/2k$-sphere about a point intersects two different X_{ak}. Indeed if $a < b$ then the distance from X_{ak} to X_{bk} is greater than or equal to the distance from X_{ak} to $P - U_{ak}$ $(\supset X_{bk})$, which is at at least k^{-1}. Finally, let V_{ak} be the open $1/3k$-sphere about X_{ak}, Y_k the closed $1/4k$-sphere about X_k and $W_{ak} = V_{ak} - \bigcup\{Y_j \mid j = 1, \ldots, k - 1\}$. We shall prove that $\{W_{ak}\}$ is the required refinement of $\{U_a\}$. Each set W_{ak} is open as the difference of an open set V_{ak} and a closed set $\bigcup\{Y_j \mid j = 1, \ldots, k - 1\}$. Evidently, $W_{ak} \subset U_a$ for each a and k and hence $\{W_{ak}\}$ refines $\{U_a\}$. Next, each family $\{W_{ak} \mid a \in A\}$ is discrete. It is enough to show that the family $\{V_{ak} \mid a \in A\}$ is discrete and this follows from the fact that no open $1/6k$-sphere intersects two distinct V_{ak} (remember that no open $1/2k$-sphere intersects two distinct X_{ak}, and V_{ak} is the open $1/3k$-sphere about X_{ak}). Now we shall prove that $\{W_{ak}\}$ is locally finite. Given x in P let us choose a k such that $x \in X_k$, and let us consider the open 1/4-sphere U about x. Since $U \subset Y_k$, U intersects no W_{aj} with $j > k$. Since the families $\{W_{aj} \mid a \in A\}$ are discrete we can choose neighborhoods G_j of $x, j = 1, \ldots, k$, such that G_j intersects at most one W_{aj}. Then clearly $U \cap \bigcap\{G_j \mid j \leqq k\}$ is a neighborhood of x which intersects at most k distinct W_{aj}. It remains to show that $\{W_{ak}\}$ is a cover of P, and to prove this it is sufficient to demonstrate that $Y_k \subset \bigcup\{W_{aj} \mid a \in A,\ j \leqq k\}$ because $\{X_k\}$ and hence $\{Y_k\}$ covers P. The inclusion is evident for $k = 1$ and therefore it is sufficient to show that $Y_k - Y_{k-1} \subset \bigcup\{W_{aj} \mid a \in A,\ j \leqq k\}$ for each $k > 1$. However, this is also evident.

30. HEREDITARY AND PERFECT NORMALITY. PARACOMPACTNESS

It has already been shown that a subspace of a normal space need not be normal. In the first subsection hereditarily normal spaces, i.e., spaces such that every subspace is normal, will be studied. An important class of hereditarily normal spaces is formed by the so-called perfectly normal spaces which are defined as normal spaces each open subset of which is an $\mathbf{F}_\sigma$. In the second subsection we shall derive the Bing-Nagata-Smirnov metrization theorem. Subsection C is concerned with the development of properties of paracompact spaces (semi-uniformizable and topological spaces such that every open cover is uniformizable). Every paracompact space is normal but a normal space need not be paracompact, and moreover, a perfectly normal space need not be paracompact. Neither a subspace of a paracompact space nor the product of two paracompact spaces need be paracompact.

In subsection D we show that the product of a metrizable space with a hereditarily paracompact space need not be normal; however, the product of a perfectly normal space with a hereditarily normal space is hereditarily normal.

In the final subsection E we shall study locally determined and relatively locally determined collections of sets in a paracompact or hereditarily paracompact space.

A. HEREDITARILY NORMAL AND PERFECTLY NORMAL SPACES

30 A.1. Definition. A closure space $\mathscr{P}$ is said to be *hereditarily normal* if each subspace of $\mathscr{P}$ is normal.

30 A.2. *Every hereditarily normal space is normal and every pseudometrizable or generalized ordered space is hereditarily normal.*

Proof. The subspace $\mathscr{P}$ of a space $\mathscr{P}$ is homeomorphic with the space $\mathscr{P}$, and consequently each hereditarily normal space is normal. Since the classes of all pseudometrizable spaces and of all generalized ordered spaces are hereditary and they are contained in the class of all normal spaces (corollary of 29 A.5 and 29 B.1) the rest of the statement follows.

30 A.3. Theorem. *The class of all hereditarily normal spaces is hereditary and closed under sums. The product of two hereditarily normal spaces may fail to be a normal space.*

Proof. Obviously each subspace of a hereditarily normal space is hereditarily normal. If $\mathscr{P}$ is the sum of a family $\{\mathscr{P}_a\}$ of spaces, then each subspace of $\mathscr{P}$ is the sum of a family $\{\mathscr{Q}_a\}$, where each $\mathscr{Q}_a$ is a subspace of $\mathscr{P}_a$ (of course, $\mathscr{Q}_a$ may be empty). Since the sum of normal spaces is a normal space, invariance under sums follows. Let $\mathscr{P}$ be the ordered set R endowed with the closure of right-approximation. By 29 B.3 the product space $\mathscr{P} \times \mathscr{P}$ is not normal, and by 30 A.2 the space $\mathscr{P}$ is hereditarily normal.

30 A.4. Theorem. *Each of the following two conditions is necessary and sufficient for a topological space $\mathscr{P}$ to be hereditarily normal:*

(a) *Every open subspace of $\mathscr{P}$ is normal.*

(b) *The space $\mathscr{P}$ is semi-uniformizable and every two semi-separated subsets of $\mathscr{P}$ are separated.*

Proof. Obviously condition (a) is necessary. If condition (b) is fulfilled, then $\mathscr{P}$ is normal, because every two disjoint closed sets are semi-separated. Since condition (b) is hereditary, it is possessed by each subspace when possessed by the space, thus the space satisfying (b) is hereditarily normal. It remains to prove that (a) implies (b). Suppose (a) is true and X_1 and X_2 are two semi-separated subsets of $\mathscr{P}$, that is, $(\overline{X}_1 \cap X_2) \cup (X_1 \cap \overline{X}_2) = \emptyset$. We must show that the sets X_1 and X_2 are separated. Now consider the subspace $\mathscr{Q} = |\mathscr{P}| - (\overline{X}_1 \cap \overline{X}_2)$ of $\mathscr{P}$. The space $\mathscr{P}$ being topological, $\mathscr{Q}$ is open in $\mathscr{P}$. By our assumption $\mathscr{Q}$ is normal. The sets $Y_i = |\mathscr{Q}| \cap \overline{X}_i$ are relatively closed (i.e. closed in $\mathscr{Q}$). It is easy to show that $Y_1 \cap Y_2 = \emptyset$. The space $\mathscr{Q}$ being normal, there exist open sets U_1 and U_2 in $\mathscr{Q}$ such that $Y_i \subset U_i$ and $U_1 \cap U_2 = \emptyset$. But $\mathscr{Q}$ is open in $\mathscr{P}$ and consequently the sets U_i, being open in $\mathscr{Q}$, are open in $\mathscr{P}$; this means that the sets Y_1 and Y_2, and hence also the sets $X_1 \subset Y_1$ and $X_2 \subset Y_2$, are separated in $\mathscr{P}$.

Now we proceed to an investigation of perfectly normal spaces.

30 A.5. Definition. A closure space $\mathscr{P}$ is said to be *perfectly normal* if $\mathscr{P}$ is normal and each open subset of $\mathscr{P}$ is an $\mathbf{F}_\sigma$, or equivalently, if $\mathscr{P}$ is normal and each closed subset of $\mathscr{P}$ is a $\mathbf{G}_\delta$.

30 A.6. Theorem. *Every perfectly normal space is hereditarily normal. The class of all perfectly normal spaces is hereditary and closed under sums. The product of two perfectly normal spaces need not be normal.*

Proof. I. Every perfectly normal space is hereditarily normal by theorem 29 A.8 asserting that an $\mathbf{F}_\sigma$-subspace of a normal space is normal and theorem 30 A.4 asserting that a space is hereditarily normal provided that each open subspace is normal. — II. If $\mathscr{P}$ is perfectly normal and $\mathscr{Q}$ is subspace of $\mathscr{P}$, then $\mathscr{Q}$ is normal since $\mathscr{P}$ is hereditarily normal by I, and if U is an open subset of $\mathcal{Q}$, then there exists an open subset V of $\mathcal{P}$ such that $V \cap |\mathcal{Q}| = U$ (because $\mathcal{P}$ is topological); since $\mathcal{P}$ is perfectly normal, V is an $\mathbf{F}_\sigma$ in $\mathcal{P}$ and hence $U = V \cap |\mathcal{Q}|$ is an $\mathbf{F}_\sigma$ in $\mathcal{Q}$. – III. The proof of invariance under sums is left to the reader. – IV. The set R of reals endowed with the closure of right-approximation is perfectly normal (it is normal by 29 B.1 and each open set is an $\mathbf{F}_\sigma$ by 22 ex. 2). The product $\mathsf{R} \times \mathsf{R}$ is not normal by 29 B.3.

By Definition 28 B.1, a subset X of a closure space $\mathcal{P}$ is called exact closed (exact open) if $X = f^{-1}[0]$ ($X = f^{-1}[\mathsf{R} - (0)]$) for some continuous function f on $\mathcal{P}$. For properties of exact closed and exact open sets consult subsection 28 B. By 28 B.2 every exact closed set is a closed $\mathbf{G}_\delta$ and every exact open set is an open $\mathbf{F}_\sigma$. If the space is normal then also the converse is true, as states the proposition which follows.

30 A.7. *If $\mathcal{P}$ is a normal space then a subset X of $\mathcal{P}$ is exact closed if and only if X is a closed $\mathbf{G}_\delta$-subset of $\mathcal{P}$; and a subset X of $\mathcal{P}$ is exact open if and only if X is an open $\mathbf{F}_\sigma$-subset of $\mathcal{P}$.*

Proof. The "only if" parts have already been proved. I. Let X be a closed $\mathbf{G}_\delta$ in a normal space $\mathcal{P}$ and $X = \bigcap\{U_n \mid n \in \mathsf{N}\}$ where U_n are open. The space $\mathcal{P}$ being normal, for each n the sets X and $|\mathcal{P}| - U_n$ are functionally separated. By 28 B.4 we can choose exact closed sets Z_n, $n \in \mathsf{N}$, such that $X \subset Z_n \subset U_n$ for each n. Now clearly $X = \bigcap\{Z_n\}$. The set X is exact closed by 28 B.2 as the countable intersection of exact closed sets.

II. If X is an open $\mathbf{F}_\sigma$, then $|\mathcal{P}| - X$ is a closed $\mathbf{G}_\delta$, and by I, $|\mathcal{P}| - X$ is exact closed which implies that X is exact open.

It follows from 30 A.7 that each closed subset (each open subset) of a perfectly normal space is exact closed (exact open, respectively). We shall prove somewhat more.

30 A.8. Theorem. *Each of the following conditions is necessary and sufficient for a semi-uniformizable topological space $\mathcal{P}$ to be perfectly normal:*

(a) *Each closed subset of $\mathcal{P}$ is exact closed.*

(b) *Each open subset of $\mathcal{P}$ is exact open.*

Proof. Clearly conditions (a) and (b) are equivalent, as it has been noted above, and proposition 30 A.7 implies that (a) is necessary. It is now sufficient to show that, for instance, (a) is sufficient. Suppose (a). We must prove that the space is normal, and since $\mathcal{P}$ is a semi-uniformizable topological space, by virtue of 29 A.5 it is enough to show that every two disjoint closed subsets of $\mathcal{P}$ are functionally separated. But this follows from the proposition 28 B.4 which asserts that every two disjoint exact closed sets are functionally separated.

30 A.9. *Every pseudometrizable space is perfectly normal.*

Proof. By 28 B.3 every closed subset of a pseudometrizable space is exact closed.

B. METRIZATION

By definition a closure space $\langle P, u\rangle$ is pseudometrizable if the closure u is induced by a pseudometric. Thus the pseudometrizable spaces are defined by requiring the existence of certain real-valued relations. On the other hand, by 24 A.4, a space $\langle P, u\rangle$ is pseudometrizable if and only if the closure u is induced by a uniformity with a countable base; this characterization does not depends on the space of reals, but it uses the concept of a uniformity. Here we shall give a purely topological characterization of pseudometrizable spaces which is due to R. Bing, J. Nagata and J. Smirnov.

30 B.1. Metrization Theorem. *Each of the following two conditions is necessary and sufficient for a regular topological space $\mathscr{P}$ to be pseudometrizable:*

(a) *$\mathscr{P}$ possesses a σ-discrete open base.*

(b) *$\mathscr{P}$ possesses a σ-locally finite open base.*

30 B.2. Corollary. *In order that a regular topological space $\mathscr{P}$ be a pseudometrizable space with a countable density character it is necessary and sufficient that $\mathscr{P}$ have a countable open base.*

Proof. Let $\mathscr{P}$ be a regular topological space. If $\mathscr{P}$ has a countable open base, then $\mathscr{P}$ is pseudometrizable by 30 B.1 because evidently every countable family is σ-discrete, and $\mathscr{P}$ has a countable density character because the density character is always less than or equal to the total character. Conversely, if $\mathscr{P}$ is a pseudometrizable space with a countable density character, then the total character of $\mathscr{P}$ is countable by 22 A.8. It is to be noted that this part of the proof can also be derived directly from 30 B.1; it is sufficient to notice that if a closure space has the density character $\mathfrak{m} \geqq \aleph_0$ and $\{U_a \mid a \in A\}$ is a locally finite family of non-void open sets, then the cardinal of A is at most $\mathfrak{m}$ (see 22 ex. 2).

Proof of 30 B.1. Every σ-discrete family is σ-locally finite and hence (a) implies (b). It will be shown that (a) is necessary and (b) is sufficient. The necessity of (a) is an immediate consequence of Theorem 29 C.8. Indeed, suppose that $\mathscr{P}$ is pseudometrizable and choose a pseudometric d inducing the closure structure of $\mathscr{P}$. For each $n = 1, 2, \ldots$ let $\mathscr{U}_n$ be the cover of $\langle|\mathscr{P}|, d\rangle$ consisting of all open $1/n$-spheres and let $\mathscr{V}_n$ be a σ-discrete open refinement of $\mathscr{U}_n$. Clearly the union $\mathscr{V}$ of all $\mathscr{V}_n$ is a σ-discrete collection of open sets. It is easy to show that $\mathscr{V}$ is an open base for $\mathscr{P}$. Indeed, if W is a neighborhood of a point x, then some r-sphere U about x is contained in W. Choose an n so that $2/n < r$ and a V in $\mathscr{V}_n$ such that $x \in V$. Clearly $x \in V \subset U \subset W$. The proof of the sufficiency of condition (b) is based on the following lemma which will be needed later and which implies that a regular topological space satisfying condition (b) is normal.

30 B.3. Lemma. *Let X_1 and X_2 be two subsets of a closure space $\mathscr{P}$ such that, for each collection $\mathscr{U}$ of subsets of $\mathscr{P}$ which interiorly covers $X_1 \cup X_2$ (i.e. the* *interiors cover $X_1 \cup X_2$), there exists a σ-locally finite family which also interiorly covers $X_1 \cup X_2$ and refines $\mathscr{U}$. If the sets (x) and X_i are separated provided that $x \in X_j$, $j \neq i$, then the sets X_1 and X_2 are separated. In particular, if $\mathscr{P}$ is a regular topological space such that each open cover of $\mathscr{P}$ has a σ-locally finite open refinement, then every two disjoint closed sets are separated and hence $\mathscr{P}$ is normal.*

Proof. I. Let us suppose that $\mathscr{P}$, X_1 and X_2 fulfil the assumptions of the lemma. For each x in X_i, $i = 1, 2$, let us choose a neighborhood U_x of x such that $\overline{U}_x \cap X_j = \emptyset$ where $j \neq i$. The collection $\mathscr{U}$ of all U_x, $x \in X_1 \cup X_2$, interiorly covers $X_1 \cup X_2$. By our assumption there exists a σ-locally finite family $\mathscr{V}$ of subsets of $\mathscr{P}$ which interiorly covers $X_1 \cup X_2$ and which also refines $\mathscr{U}$, i.e., each $V \in \mathscr{V}$ is contained in some $U \in \mathscr{U}$. Let $\mathscr{V} = \bigcup\{\mathscr{V}_n \mid n \in \mathrm{N}\}$ where each $\mathscr{V}_n$ is locally finite. For $n \in \mathrm{N}$ and $i = 1, 2$ let $\mathscr{V}_n^i$ be the collection of all $V \in \mathscr{V}_n$ intersecting X_i, and let V_n^i be the union of the collection $\mathscr{V}_n^i$. It is easy to see that

A) The sequence $\{V_n^i \mid n \in \mathrm{N}\}$ interiorly covers X_i, $i = 1, 2$.

B) For each $i, j = 1, 2$, $i \neq j$, $\overline{V_n^i} \cap X_j = \emptyset$.

Property A) is proved as follows: if $x \in X_i$, then $x \in \operatorname{int} V$ for some V in $\mathscr{V}$ (because $\mathscr{V}$ interiorly covers X_i); thus $V \in \mathscr{V}_n$ for some $n \in \mathrm{N}$. Since $V \cap X_i \neq \emptyset$, $V \in \mathscr{V}_n^i$ and hence $V \subset V_n^i$ showing that $x \in \operatorname{int} V \subset \operatorname{int} V_n^i$. The proof of B) follows from the fact that each $\mathscr{V}_n^i$ is locally finite. Since $\mathscr{V}_n^i$ is locally finite we have $\overline{V_n^i} = \bigcup\{\overline{V} \mid V \in \mathscr{V}_n^i\}$ by 14 B.17, and consequently to prove $\overline{V_n^i} \cap X_j = \emptyset$ it is enough to show that $\overline{V} \cap X_j = \emptyset$ for each V in $\mathscr{V}_n^i$. Let $V \in \mathscr{V}_n^i$. By our assumption $V \subset U_x$ for some $x \in X_1 \cup X_2$. If $x \in X_j$, then $\overline{U}_x \cap X_i = \emptyset$ and hence $\overline{V} \cap X_i = \emptyset$; this contradicts the fact that $V \cap X_i \neq \emptyset$ which follows from $V \in \mathscr{V}_n^i$. It follows that $x \in X_i$ and hence $\overline{V} \subset \overline{U}_x \subset |\mathscr{P}| - X_j$ by the choice of U_x.

Finally, for each $i, j = 1, 2$, $i \neq j$, put

$$W_n^i = V_n^i - \bigcup\{V_k^j \mid k \leqq n\}, \quad W^i = \bigcup\{W_n^i \mid n \in \mathrm{N}\}.$$

It follows from conditions A) and B) at once that the sequence $\{W_n^i \mid n \in \mathrm{N}\}$, $i = 1, 2$, interiorly covers X_i and hence W^i is a neighborhood of X_i. Indeed, we see from B) that $|\mathscr{P}| - \bigcup\{V_k^j \mid k \leqq n\}$ is a neighborhood of each point of X_i, and consequently W_n^i is a neighborhood of a point x of X_i if and only if V_n^i is a neighborhood of x; from A) we find that $\{W_n^i \mid n \in \mathrm{N}\}$ interiorly covers X_i.

The proof is completed by showing that $W^1 \cap W^2 = \emptyset$. If $x \in W^1 \cap W^2$, then $x \in W_k^1 \cap W_l^2$ for some $k, l \in \mathrm{N}$; but this is impossible. Indeed, if for instance $k \geqq l$, then $x \in W_k^1$ implies $x \notin V_i^2$ for $i \leqq k$; in particular $x \notin V_l^2$ and hence $x \notin W_l^2$ because $W_l^2 \subset V_l^2$. Similarly, $k \leqq l$ leads to a contradiction.

II. The second statement follows at once from the first one. Suppose that each open cover of a regular topological space $\mathscr{P}$ has a σ-locally finite open refinement, and let X_1 and X_2 be disjoint closed sets. Since $\mathscr{P}$ is regular, each point of X_i is separated from the second set. If $\mathscr{U}$ is a collection which interiorly covers $X_1 \cup X_2$, then the collection $\mathscr{U}_1$ consisting of the interiors of sets from $\mathscr{U}$ is a collection of open sets ($\mathscr{P}$ was assumed to be topological) which covers $X_1 \cup X_2$; if $\mathscr{U}_2$ consists of all members of $\mathscr{U}_1$ and of the set $|\mathscr{P}| - (X_1 \cup X_2)$, then $\mathscr{U}_2$ is an open cover of $\mathscr{P}$. If $\mathscr{W}$ is a σ-locally finite open refinement of $\mathscr{U}_2$ and $\mathscr{V}$ is the set of all $W \in \mathscr{W}$, $W \cap (X_1 \cup X_2) \neq \emptyset$, then $\mathscr{V}$ possesses all the required properties.

30 B.4. Proof of 30 B.1, conclusion. It remains to verify the sufficiency of condition (b). Suppose (b) is true, $\mathscr{V}$ is a σ-locally finite open base for $\mathscr{P}$ and $\{\mathscr{V}_n \mid n \in \mathsf{N}\}$ is a sequence of locally finite families such that the union of $\{\mathscr{V}_n\}$ is $\mathscr{V}$.

I. First we shall prove that the space $\mathscr{P}$ is normal. By virtue of Lemma 30 B.3 it is sufficient to show that each open cover of $\mathscr{P}$ has a σ-locally finite open refinement. If $\mathscr{U}$ is an open cover of $\mathscr{P}$ and if $\mathscr{V}'$ is the collection of all $V \in \mathscr{V}$ which are contained in some element of $\mathscr{U}$, then $\mathscr{V}'$ is a σ-locally finite family of open sets, and as is easily seen, $\mathscr{V}'$ is a cover of $\mathscr{P}$. Indeed, if $x \in U \in \mathscr{U}$, then $x \in V \subset U$ for some V in $\mathscr{V}$ because $\mathscr{V}$ is an open base for $\mathscr{P}$, and hence $x \in V \in \mathscr{V}'$.

II. The space $\mathscr{P}$ is perfectly normal. Since $\mathscr{P}$ is normal (by I) it remains to show that each open subset U of $\mathscr{P}$ is an $\mathbf{F}_\sigma$. Let $\mathscr{V}'$ be the set of all $V \in \mathscr{V}$ such that $\overline{V} \subset U$ and let $\mathscr{V}'_n = \mathscr{V}_n \cap \mathscr{V}'$. Put $X_n = \bigcup\{\overline{V} \mid V \in \mathscr{V}'_n\}$, $n \in \mathsf{N}$. Clearly $\bigcup\{X_n\} = U$. Since each collection $\mathscr{V}'_n$ is locally finite, each set X_n is closed as the union of a locally finite family of closed sets.

III. *Construction of a pseudometric inducing the closure structure of $\mathscr{P}$.* Since $\mathscr{P}$ is perfectly normal (by II), each open subset of $\mathscr{P}$ is exact open, and consequently we can choose a family $\{f_V \mid V \in \mathscr{V}\}$ of non-negative continuous functions on $\mathscr{P}$ such that $f^{-1}[\mathsf{R} - (0)] = V$ for each V in $\mathscr{V}$. Since each family $\{f_V \mid V \in \mathscr{V}_n\}$ is locally finite, for each $n \in \mathsf{N}$ the real-valued relation

$$d_n = \{\langle x, y\rangle \to \Sigma\{|f_V x - f_V y| \mid V \in \mathscr{V}_n\} \mid \langle x, y\rangle \in \mathscr{P} \times \mathscr{P}\}$$

is a continuous pseudometric for $\mathscr{P}$. Consider the continuous pseudometric

$$d = \Sigma\{\inf(d_n, 2^{-n}) \mid n \in \mathsf{N}\}$$

for $\mathscr{P}$. It is easy to see that d induces the closure of $\mathscr{P}$. Since d is continuous, clearly $x \in \overline{X}$ implies that the d-distance from x to X is zero. Now let $x \notin \overline{X}$. Since $\mathscr{V}$ is an open base, there exists a V in $\mathscr{V}$ such that $x \in V \subset |\mathscr{P}| - X$. The set V belongs to some $\mathscr{V}_n$, say to $\mathscr{V}_k$. By the choice of the function f_V, $f_V x > 0$ and $f_V y = 0$ for y in X. It follows that the d_k-distance from x to X is at least $|f_V x - 0| = f_V x > 0$, and finally, the d-distance of x to X is at least $\min(2^{-k}, f_V x) > 0$.

30 B.5. Example. For each ordinal α let T_α be the ordered space of ordinals β, $\beta < \alpha$. It follows from 30 B.2 that every T_α with $\alpha < \omega_1$ is metrizable; indeed it is clear that T_α with $\alpha < \omega_1$ has a countable base. The spaces T_α with $\alpha \geqq \omega_1 + 1$ are not metrizable because they are of an uncountable local character (because the local character at ω_1 is $\aleph_1$). There remains the case $\alpha = \omega_1$. We shall prove that T_{ω_1} is not metrizable. Assuming that T_{ω_1} is metrizable we find from 30 B.1 that there exists a σ-locally finite open base $\mathscr{U}$; let $\mathscr{U} = \bigcup\{\mathscr{U}_n \mid n \in \mathsf{N}\}$ where each $\mathscr{U}_n$ is locally finite. By 17—18, ex. 5, each $\mathscr{U}_n$ is necessarily finite and hence $\mathscr{U}$ is countable. Thus T_{ω_1} has a countable total character, but this is impossible because the density character of T_{ω_1} is uncountable (because every countable set has an upper bound in T_{ω_1}). If follows that T_{ω_1} is a locally metrizable hereditarily normal space which is not metrizable.

C. PARACOMPACT SPACES

By 29 A.6 a semi-uniformizable topological space is normal if and only if every finite open cover is uniformizable. By 29.C 7, 8 every open cover of a pseudometrizable space is uniformizable. On the other hand an open cover of a (hereditarily) normal space need not be uniformizable (e.g. we shall show in 30 C.5 that the open cover $\{ \llbracket \leftarrow, \alpha \rrbracket \mid \alpha < \omega_1\}$ of T_{ω_1} is not uniformizable).

30 C.1. Definition. A *paracompact space* (also called a *fully normal space*) is defined to be a topological semi-uniformizable space such that every open cover is uniformizable.

30 C.2. Theorem. *Every paracompact space is normal and any pseudometrizable space is paracompact.* – See 29 A.6, 29 C.7, 8.

30 C.3. Theorem. *In order that a closure space $\mathscr{P}$ be paracompact it is necessary and sufficient that $\mathscr{P}$ be a regular topological space such that every open cover of $\mathscr{P}$ has a locally finite open refinement.*

Proof. Since every paracompact space is normal, the condition is necessary by 29 C.5. Conversely, if the condition is fulfilled, then the space $\mathscr{P}$ is normal by Lemma 30 B.3, whereupon paracompactness of $\mathscr{P}$ follows from Theorem 29 C.2 which asserts that every locally finite open cover of a normal space is uniformizable.

Remark. *If $\mathscr{P}$ is a separated topological space such that every open cover is refined by a locally finite open cover, then $\mathscr{P}$ is paracompact.* According to the foregoing theorem it is enough to show that $\mathscr{P}$ is regular. Assuming that X is closed in $\mathscr{P}$ and $y \in |\mathscr{P}| - X$ let us choose a family $\{U_x \mid x \in X\}$ such that each U_x is an open neighborhood of x and the closure of U_x does not contain y, and also take an open locally finite refinement $\mathscr{V}$ of the open cover of $|\mathscr{P}|$ consisting of the set $|\mathscr{P}| - X$ and all the sets U_x, $x \in X$. If U is the union of all $V \in \mathscr{V}$, $V \cap X \neq \emptyset$, then evidently U is an open neighborhood of X, the closure of U does not contain y because $\mathscr{V}$ is locally finite and thus closure-preserving, and if $V \in \mathscr{V}$, $V \cap X \neq \emptyset$, then necessarily $V \subset U_x$ for some x, and hence $y \in |\mathscr{P}| - \overline{U}_x \subset |\mathscr{P}| - \overline{V}$. Thus y and X are separated and the space $\mathscr{P}$ is regular.

We know that the uniformizability of a cover of a space can be described by means of covers, neighborhoods of the diagonal, pseudometrics, mappings into metrizable spaces, and partitions of the unity. Applying these to paracompactness we obtain the following various characterizations of paracompact spaces. It is to be noted that several modifications of the characterization 30 C.3 will be given in the closing part of the subsection.

30 C.4. Theorem. *Each of the following conditions is necessary and sufficient for a topological semi-uniformizable space $\mathscr{P}$ to be paracompact:*

(a) *Every open cover of $\mathscr{P}$ has an open star-refinement.*

(b) *For each open cover $\mathscr{U}$ of $\mathscr{P}$ there exists a continuous pseudometric subordinated to $\mathscr{U}$.*

(c) *For each open cover $\mathscr{U}$ of $\mathscr{P}$ there exists a locally finite partition of unity subordinated to $\mathscr{U}$.*

(d) *Every open cover of $\mathscr{P}$ is refined by a cover consisting of all open spheres for some continuous pseudometric d for $\mathscr{P}$.*

(e) *Every open cover of $\mathscr{P}$ is refined by a cover $\{f^{-1}[V] \mid V \in \mathscr{V}\}$ where f is a continuous mapping into a pseudometrizable space $\mathscr{R}$ and $\mathscr{V}$ is an open cover of $\mathscr{R}$.*

Remark. To prove that (a) is sufficient it is enough to notice that, given an open cover $\mathscr{U}$ of $\mathscr{P}$, it follows from (a) that there exists a sequence $\{\mathscr{U}_n\}$ of open covers of $\mathscr{P}$ such that $\mathscr{U}_{n+1}$ is a star-refinement of $\mathscr{U}_n$ and $\mathscr{U}_0 = \mathscr{U}$.

30 C.5. Examples. (a) The space T_{ω_1+1} is paracompact (by 30 C.3) because it is normal, as an ordered space, and every open cover has a finite subcover (17 ex. 5). On the other hand T_{ω_1+1} is not metrizable because it is not of a countable local character at ω_1. – (b) The subspace T_{ω_1} of T_{ω_1+1} is not paracompact because the open cover $\mathscr{U} = \{ [\![\leftarrow, \alpha]\!] \mid \alpha < \omega_1\}$ has no locally finite refinement. Indeed, by 17 ex. 5 every locally finite family of non-void subsets of T_{ω_1} is finite and no finite subfamily of $\mathscr{U}$ is a cover. Thus a subspace of a paracompact space need not be paracompact. Remember that T_{ω_1} is hereditarily normal and locally metrizable (30 B.4). On the other hand:

(b) *Every paracompact locally pseudometrizable space is pseudometrizable.*

Proof. Let $\mathscr{U}$ be an interior cover of a paracompact space $\mathscr{P}$ such that each subspace $U \in \mathscr{U}$ of $\mathscr{P}$ is pseudometrizable. Let $\mathscr{V}$ be a locally finite open refinement of $\mathscr{U}$. Clearly each element of $\mathscr{V}$ is pseudometrizable. By 30 B.1 there exists a family $\{\mathscr{W}_V \mid V \in \mathscr{V}\}$ such that each $\mathscr{W}_V$ is a σ-locally finite open base for the subspace V of $\mathscr{P}$. Let $\mathscr{W}_V = \bigcup\{\mathscr{W}_{Vn} \mid n \in \mathrm{N}\}$ such that each $\mathscr{W}_{Vn}$ is locally finite. Put $\mathscr{W} = \bigcup\{\mathscr{W}_V \mid V \in \mathscr{V}\}$, $\mathscr{W}_n = \bigcup\{\mathscr{W}_{Vn} \mid V \in \mathscr{V}\}$. Clearly $\mathscr{W}$ is an open base for $\mathscr{P}$, $\mathscr{W} = \bigcup\{\mathscr{W}_n \mid n \in \mathrm{N}\}$ and each collection $\mathscr{W}_n$ is locally finite. Thus $\mathscr{W}$ is a σ-locally finite open base for $\mathscr{P}$. Since $\mathscr{P}$ is paracompact and so certainly regular and topological, $\mathscr{P}$ is pseudometrizable by 30 B.1.

Now we proceed to an examination of properties of paracompact spaces.

30 C.6. Theorem. *If $\mathscr{P}$ is a paracompact space, then the fine uniformity of $\mathscr{P}$ consists of all neighborhoods of the diagonal of the product space $\mathscr{P} \times \mathscr{P}$.*

Proof. It suffices to show that if U is a neighborhood of the diagonal then there exists a uniformizable neighborhood V of the diagonal such that $V \subset U$. Since U is a neighborhood of the diagonal and $\mathscr{P}$ is topological we can choose an open cover $\mathscr{W}$ of $\mathscr{P}$ such that $W \times W \subset U$ for each W in $\mathscr{W}$. Since $\mathscr{P}$ is paracompact, $\mathscr{W}$ is a uniformizable cover of $\mathscr{P}$ and therefore there exists a uniformizable neighborhood of the diagonal V such that the cover $\{V[x] \mid x \in \mathscr{P}\}$ refines $\mathscr{W}$. Clearly $V \subset U$.

30 C.7. Theorem. *If $\mathscr{P}$ is a paracompact space and $\mathscr{Q}$ is a closed subspace of $\mathscr{P}$, then the fine uniformity of $\mathscr{Q}$ is the relativization of the fine uniformity of $\mathscr{P}$.*

Proof. It is sufficient to show that every uniformizable cover $\mathscr{V}$ of $\mathscr{Q}$ is refined by a cover $[\mathscr{U}] \cap |\mathscr{Q}|$ where $\mathscr{U}$ is some uniformizable cover of $\mathscr{P}$. We may assume that $\mathscr{V}$ is an open cover of $\mathscr{Q}$. For each V in $\mathscr{V}$ let V' be an open subset of $\mathscr{P}$ such that $V' \cap |\mathscr{Q}| = V$, and let $\mathscr{U}$ be the collection of sets consisting of the set $|\mathscr{P}| - |\mathscr{Q}|$ and all sets V', $V \in \mathscr{V}$. Clearly $\mathscr{U}$ is an open cover such that $[\mathscr{U}] \cap |\mathscr{Q}|$ refines $\mathscr{V}$. Since $\mathscr{P}$ is paracompact, $\mathscr{U}$ is uniformizable.

30 C.8. Corollary. *If d is a bounded continuous pseudometric for a closed subspace $\mathscr{Q}$ of a paracompact space $\mathscr{P}$, then d is the relativization of a continuous pseudometric for $\mathscr{P}$.*

Proof. Use Theorem 25 F.1 asserting that every bounded uniformly continuous pseudometric on a subspace of a uniform space $\mathscr{R}$ is the relativization of a uniformly continuous pseudometric for $\mathscr{R}$.

It is not true that if the fine uniformity of a subspace $\mathscr{Q}$ of a closure space $\mathscr{P}$ is the relativization of the fine uniformity of $\mathscr{P}$, then $\mathscr{Q}$ is closed in $\mathscr{P}$ (ex. 8). However

30 C.9. Theorem. *If a subspace $\mathscr{Q}$ of a separated closure space $\mathscr{P}$ is paracompact and the fine uniformity of $\mathscr{Q}$ is the relativization of the fine uniformity of $\mathscr{P}$, then $|\mathscr{Q}|$ is closed in $\mathscr{P}$.*

Proof. Suppose that there exists a point $x \in \overline{|\mathscr{Q}|} - |\mathscr{Q}|$ and let $\mathscr{U}$ be the neighborhood system at x in $\mathscr{P}$. Since $\mathscr{P}$ is separated we have $\bigcap\{\overline{U} \mid U \in \mathscr{U}\} = (x)$, and hence $\mathscr{V} = \{|\mathscr{Q}| - \overline{U} \mid U \in \mathscr{U}\}$ is an open cover of $\mathscr{Q}$. It is easily seen that there exists no uniformizable cover $\mathscr{W}$ of $\mathscr{P}$ such that $|\mathscr{Q}| \cap [\mathscr{W}]$ refines $\mathscr{V}$. By virtue of 29 C.5 it is sufficient to show that $|\mathscr{Q}| \cap [\mathscr{W}]$ refines $\mathscr{V}$ for no locally finite open cover $\mathscr{W}$ of $\mathscr{P}$; however, this is evident.

In a normal space every two disjoint closed sets are separated. Now we shall prove that paracompact spaces possess an essentially stronger property.

30 C.10. Theorem. *If $\{X_a \mid a \in A\}$ is a locally finite (discrete) family of subsets of a paracompact space $\mathscr{P}$, then there exists a locally finite (discrete) open family $\{U_a \mid a \in A\}$ such that $X_a \subset U_a$ for each a in A; furthermore, one may take $U_a = U[X_a]$, where U is an appropriate open neighborhood of the diagonal of $\mathscr{P} \times \mathscr{P}$.*

Proof. Assuming that $\{X_a\}$ is locally finite (discrete) we can choose an open cover $\mathscr{V}$ of $\mathscr{P}$ such that each V in $\mathscr{V}$ intersects only a finite number of X_a's (at most one member of $\{X_a\}$). Let W be a uniformizable neighborhood of the diagonal such that the cover $\{W[x] \mid x \in \mathscr{P}\}$ refines $\mathscr{V}$ and let U be a symmetric open neighborhood of the diagonal such that $U \circ U \subset W$. Put $U_a = U[X_a]$ for each a in A. It is easily seen that the family $\{U_a \mid a \in A\}$ has the required properties. Indeed, if $x \in \mathscr{P}$ then $W[x]$ intersects only a finite number of X_a's (at most one member of $\{X_a\}$), because $\{W[x]\}$ refines $\mathscr{V}$, and clearly $U[x] \cap U[X] \neq \emptyset$ implies $W[x] \cap X \neq \emptyset$, and consequently $U[x]$ intersects only a finite number of U_a's (at most one member of $\{U_a\}$).

Remark. The property of paracompact spaces stated in the preceding theorem does not characterize paracompact spaces among all normal spaces; uniformizable spaces with this property are called *systematically normal.*

By example 30 C.5 a subspace of a paracompact space need not be paracompact.

30 C.11. *Every closed subspace of a paracompact space is paracompact. If $\mathscr{Q}$ is a subspace of a paracompact space $\mathscr{P}$ and if, for each neighborhood V of $|\mathscr{Q}|$ in $\mathscr{P}$, there exists a paracompact subspace $\mathscr{R}$ of $\mathscr{P}$ such that $|\mathscr{Q}| \subset |\mathscr{R}| \subset V$, then $\mathscr{Q}$ is paracompact.*

Proof. I. Let $\mathscr{Q}$ be a closed subspace of a paracompact space $\mathscr{P}$. If $\mathscr{U}$ is an open cover of $\mathscr{Q}$, then we can choose a family $\{U' \mid U \in \mathscr{U}\}$ such that each U' is open in $\mathscr{P}$ and $U' \cap |\mathscr{Q}| = U$. The collection $\mathscr{V}$ consisting of the set $|\mathscr{P}| - |\mathscr{Q}|$ and all the sets U', $U \in \mathscr{U}$ is an open cover of $\mathscr{P}$, and clearly $[\mathscr{V}] \cap \mathscr{Q} = \mathscr{U}$. Since $\mathscr{V}$ is a uniformizable cover of $\mathscr{P}$, $\mathscr{U}$ is a uniformizable cover of $\mathscr{Q}$ and hence $\mathscr{Q}$ is paracompact.

II. Using the assumptions of the second statement, let $\mathscr{U}$ be any open cover of $\mathscr{Q}$ and let us choose a family $\{U' \mid U \in \mathscr{U}\}$ such that each U' is open and $U' \cap |\mathscr{Q}| = U$ for each U in $\mathscr{U}$. Consider the neighborhood $V = \bigcup\{U' \mid U \in \mathscr{U}\}$ of $|\mathscr{Q}|$ and take a paracompact subspace $\mathscr{R}$ of $\mathscr{P}$ such that $|\mathscr{Q}| \subset |\mathscr{R}| \subset V$. The open cover $\{U' \cap \cap |\mathscr{R}| \mid U \in \mathscr{U}\}$ has a locally finite open refinement $\mathscr{V}$. Clearly $[\mathscr{V}] \cap |\mathscr{Q}|$ is a locally finite refinement of $\mathscr{U}$.

30 C.12. Theorem. *Each of the following conditions is necessary and sufficient for a semi-uniformizable topological space $\mathscr{P}$ to be paracompact:*

(a) *$\mathscr{P}$ is regular and every open cover of $\mathscr{P}$ has an open σ-locally finite refinement.*

(b) *$\mathscr{P}$ is regular and every open cover of $\mathscr{P}$ has a locally finite refinement (not necessarily open or closed).*

(c) *Every open cover of $\mathscr{P}$ has a closed locally finite refinement.*

(d) *$\mathscr{P}$ is regular and every open cover of $\mathscr{P}$ has a locally finite open refinement.*

(e) *If $\{U_a \mid a \in A\}$ is an open cover of $\mathscr{P}$, then there exists a locally finite open cover $\{G_a \mid a \in A\}$ of $\mathscr{P}$ such that $\bar{G}_a \subset U_a$ for each a.*

Proof. Condition (d) is necessary and sufficient by 30 C.3. Notice that each of the conditions (c) and (e) implies regularity; given $x \in \mathscr{P}$ and an open neighborhood of x, consider the cover $(U, |\mathscr{P}| - \overline{(x)})$. Evidently (e) implies (a). It remains to show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e).

I. Suppose (a) and let $\mathscr{U}$ be an open cover of $\mathscr{P}$. Choose a σ-locally finite open refinement $\{V_b \mid b \in B\}$ of $\mathscr{U}$; let $B = \bigcup\{B_n \mid n \in \mathrm{N}\}$ such that each $\{V_b \mid b \in B_n\}$ is locally finite. Put $W_n = \bigcup\{V_b \mid b \in B_n\}$; evidently $\{W_n\}$ is an open cover of $\mathscr{P}$. For each b in B let $n(b)$ be an integer n such that $b \in B_n$; and put $X_b = V_b - \bigcup\{W_i \mid i < n(b)\}$. We shall prove that $\{X_b \mid b \in B\}$ is a locally finite refinement of the cover $\{V_b\}$ and hence of $\mathscr{U}$. Given an x in $\mathscr{P}$, if k is the smallest integer such that $x \in W_k$ and if $x \in V_b$, $b \in B_k$, then evidently $x \in X_b$ and consequently $\{X_a\}$ is a cover. Since $X_b \subset V_b$, $\{X_b\}$ is a refinement of $\{V_b\}$. It remains to show that $\{X_b\}$ is locally finite. Let $x \in |\mathscr{P}|$ and k be an integer such that $x \in W_k$; choose a family $\{G_i \mid i \leqq k\}$ such that G_i is a neighborhood of x intersecting only a finite number of V_b's, $b \in B_i$. Now if $G = W_k \cap \bigcap\{G_i \mid i \leqq k\}$ then clearly G intersects no X_b, $b \in B_l$, $l > k$, and only a finite number of X_b's, $b \in B_i$, for each $i \leqq k$; thus G intersects only a finite number of X_a's.

II. The implication (b) $\Rightarrow$ (c) is almost evident. Given an open cover $\mathscr{U}$ of $\mathscr{P}$, we can choose an open cover $\mathscr{V}$ so that $\{\overline{V} \mid V \in \mathscr{V}\}$ refines $\mathscr{U}$ (because $\mathscr{P}$ is regular) and then, by (b), a locally finite refinement $\mathscr{X}$ of $\mathscr{V}$. Clearly $\{\overline{X} \mid X \in \mathscr{X}\}$ is a locally finite closed refinement of $\mathscr{U}$ (each $X \in \mathscr{X}$ is contained in some $V \in \mathscr{V}$ and each V is contained with its closure in some $U \in \mathscr{U}$).

III. Suppose (c). Let $\{U_a \mid a \in A\}$ be an open cover of $\mathscr{P}$. By (c) there exists a locally finite closed refinement $\{X_b \mid b \in B\}$ of $\{U_a\}$. Choose a family $\{a(b) \mid b \in B\}$ such that $X_b \subset U_{a(b)}$ for each b and then a family $\{V_x \mid x \in \mathscr{P}\}$ such that V_x is an open neighborhood of x intersecting only a finite number of X_b's, and such that $V_x \cap X_b \neq \emptyset$ implies $V_x \subset U_{a(b)}$. Again by (c) we can choose a closed locally finite refinement $\{Y_c \mid c \in C\}$ of $\{V_x \mid x \in \mathscr{P}\}$. Evidently each Y_c intersects only a finite number of X_b's. For each b in B let C_b be the set of all c such that $X_b \cap Y_c \neq \emptyset$. Clearly the family $\{C_b \mid b \in B\}$ is point-finite. Put $Z_b = \bigcup\{Y_c \mid c \in C_b\}$ and consider the cover $\{Z_b \mid b \in B\}$ of $\mathscr{P}$ which is locally finite because $\{Y_c\}$ is locally finite and $\{C_b \mid b \in B\}$ is point-finite (also see 14 ex. 4). Finally, put $W_b = |\mathscr{P}| - \bigcup\{Y_c \mid c \in (C - C_b)\}$. Since $\{Y_c\}$ is locally finite and hence closure-preserving, the sets W_b are open and evidently $X_b \subset W_b \subset Z_b$, which implies that $\{W_b \mid b \in B\}$ is a locally finite cover of $\mathscr{P}$. It remains to show that $\{W_b\}$ refines $\{U_a\}$; it is enough to prove that $Z_b \subset U_{a(b)}$ for each b. If $c \in C_b$, then $Y_c \cap X_b \neq \emptyset$ and hence, $\{Y_c\}$ being a refinement of $\{V_x\}$, there exists an x in $\mathscr{P}$ such that $Y_c \subset V_x$; thus $V_x \cap X_b \neq \emptyset$ and hence $V_x \subset U_{a(b)}$ so that $Y_c \subset U_{a(b)}$. As a consequence $Z_b = \bigcup\{Y_c \mid c \in C_b\} \subset U_{a(b)}$, which completes the proof.

IV. Finally, suppose (d) and let $\{U_a \mid a \in A\}$ be an open cover of $\mathscr{P}$. First let us choose an open cover $\mathscr{V}$ of $\mathscr{P}$ such that $\{\overline{V} \mid V \in \mathscr{V}\}$ refines $\{U_a\}$, and then an open locally finite refinement $\mathscr{W}$ of $\mathscr{V}$. For each W in $\mathscr{W}$ let $a = a(W)$ be an element of A such that $W \subset V \subset \overline{V} \subset U_a$ for some V in $\mathscr{V}$, and put $G_a = \bigcup\{W \mid W \in \mathscr{W}, a = a(W)\}$. Obviously $\{G_a\}$ is an open locally finite refinement of $\{U_a\}$ and $\overline{G}_a \subset U_a$ for each a.

It has already been shown that a closed subspace of a paracompact space is paracompact. Using the foregoing theorem, condition (a), we shall prove essentially more.

30 C.13. *If Q is a F_σ-subset of a paracompact space $\mathscr{P}$ then the subspace Q is paracompact.*

Proof. Let $Q = \bigcup\{X_n \mid n \in \mathsf{N}\}$, where each X_n is closed in $\mathscr{P}$, and let $\{U_a\}$ be an open cover of Q. Choose a family $\{V_a\}$ of open sets in $\mathscr{P}$ such that $V_a \cap Q = U_a$ for each a. Since clearly, for each $k \in \mathsf{N}$, $(|\mathscr{P}| - X_k) \cup \bigcup\{V_a\} = \mathscr{P}$ by 30 C.12, condition (d), there exists an open locally finite cover $\{W_b^k \mid b \in B_k\}$ of $\mathscr{P}$ such that, for each b, either $W_b^k \subset V_a$ for some a or $W_b^k \subset (|\mathscr{P}| - Y_k)$. Let B_k' be the set of all $b \in B_k$ such that $W_b^k \subset V_a$ for some a and let $B = \bigcup\{B_k' \mid k \in \mathsf{N}\}$. Clearly $\{W_b^k \cap Q \mid b \in B\}$ is a σ-locally finite open refinement of $\{U_a\}$. By 30 C.12, condition (a), Q is paracompact.

30 C.14. Example. If $\mathscr{P}$ is the ordered set of reals endowed with the closure of right-approximation, then every open cover of $\mathscr{P}$ has a countable subcover (22 ex. 2) and hence is paracompact by theorem 30 C.12, condition (a). But the product $\mathscr{P} \times \mathscr{P}$ is not paracompact because $\mathscr{P} \times \mathscr{P}$ is not normal by 29 B.6. Thus the product of two paracompact (hereditarily normal) spaces need not be paracompact.

D. HEREDITARILY PARACOMPACT SPACES

30 D.1. Definition. A closure space is said to be *hereditarily paracompact* if each of its subspaces is paracompact.

30 D.2. *Every pseudometrizable space is hereditarily paracompact.* – Every subspace of a pseudometrizable space is pseudometrizable and every pseudometrizable space is paracompact (30 C.2).

30 D.3. *A closure space is hereditarily paracompact if and only if each of its open subspaces is paracompact.* – 30 C.11.

30 D.4. *Every perfectly normal paracompact space is hereditarily paracompact.*

Proof. By 30 C.13 every F_σ-subspace of a paracompact space is paracompact. Apply 30 D.3.

Evidently every hereditarily paracompact space is hereditarily normal, but by 30 C.5 (b) a hereditarily normal space need not be paracompact. Now we shall prove that the product of two hereditarily paracompact spaces need not be paracompact nor even normal. Moreover, one of the coordinate spaces may be pseudometrizable. It is to be noted that the product of two pseudometrizable spaces is pseudometrizable and hence hereditarily normal.

30 D.5. Examples. We begin with a construction which, applied to a hereditarily paracompact space, again leads to a hereditarily paracompact space.

(a) Let $\langle P, u\rangle$ be a closure space and X be a subset of P. Let us define a closure operation v for P such that $vY = (uY) \cap (P - X)$ if $Y \subset P - X$ and $vY = Y \cup ((P - X) \cap uY)$ if $Y \subset X$, i.e., $vY = Y \cup (uY - X)$ for each Y. It is easily seen that X is a discrete open subspace of $\langle P, v\rangle$, $P - X$ is closed in $\langle P, v\rangle$ and the relativizations of u and v to $P - X$ coincide. We shall say that $\langle P, v\rangle$ is obtained from $\langle P, u\rangle$ by making X discrete. Next we shall need the following properties.

(α) If $\mathscr{R}$ is a subspace of $\langle P, u\rangle$ and $\mathscr{Q}$ is obtained from $\mathscr{R}$ by making the set $|\mathscr{R}| \cap X$ discrete, then $\mathscr{Q}$ is a subspace of $\langle P, v\rangle$.

(β) If u is topological then v is also topological.

(γ) If u is topological, then V is open in $\langle P, v\rangle$ if and only if $V = U \cup X'$, where U is open in $\langle P, u\rangle$ and $X' \subset X$, and

(δ) If $P - X$ is a $\mathbf{G}_\delta$ in $\langle P, v\rangle$ then $P - X$ is a $\mathbf{G}_\delta$ in $\langle P, u\rangle$, and hence if X is not an $\mathbf{F}_\sigma$ in $\langle P, u\rangle$, then X is not an $\mathbf{F}_\sigma$ in $\langle P, v\rangle$.

(ε) If $\langle P, u\rangle$ is separated and hereditarily paracompact, then $\langle P, v\rangle$ is also hereditarily paracompact.

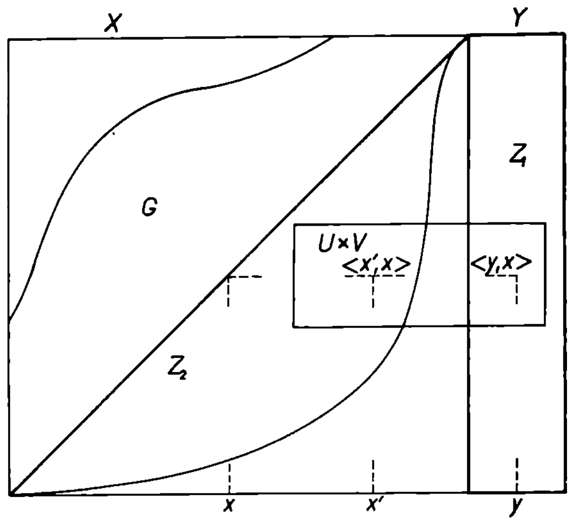

Fig. 2.

The proof of (α), (β), (γ) and (δ) is simple and therefore left to the reader. To prove (ε), according to (α) it is enough to show that $\langle P, v\rangle$ is paracompact whenever $\langle P, u\rangle$ is separated and hereditarily paracompact. Since u is separated and v is finer than u, v is separated and therefore, by 30 C.3, it is enough to show that every open cover $\{V_a\}$ of $\langle P, v\rangle$ has an locally finite open refinement. Choose a family $\{U_a\}$ of open sets in $\langle P, u\rangle$ such that $V_a = U_a \cup (X \cap V_a)$ for each a (this is possible by (γ)). Put $Q = \bigcup\{U_a\}$. Since the subspace $\mathscr{Q}$, where $|\mathscr{Q}| = Q$, of $\langle P, u\rangle$ is paracompact, we can choose a locally finite open rafinement $\mathscr{W}$ of the open cover $\{U_a\}$ of the subspace $\mathscr{Q}$ of $\langle P, u\rangle$. Clearly $\mathscr{W}$ is a locally finite collection in $\langle P, v\rangle$ ($Q \supset P - X$ and every point of X is isolated in $\langle P, v\rangle$). Now if $\mathscr{G}$ consists of all the sets of $\mathscr{W}$ and all singletons (x), $x \in P - Q$, then clearly $\mathscr{G}$ is a locally finite open cover of $\langle P, v\rangle$ which refines $\{V_a\}$.

(b) Now let $\mathscr{P}$ be the space obtained from the unit interval $[\![0, 1]\!]$ by making the set of all irrationals discrete and let $\mathscr{Q}$ be a subspace of R consisting of all irrationals of $[\![0, 1]\!]$. Thus spaces $\mathscr{P}$ and $\mathscr{Q}$ are both hereditarily paracompact. We shall prove that the product space $\mathscr{P} \times \mathscr{Q}$ is not normal. Let $X = |\mathscr{Q}|$ and $Y = |\mathscr{P}| - X$; thus $X(Y)$ is the set of all irrational (rational) numbers of $[\![0, 1]\!]$. Now consider (see fig. 2) the subsets $Z_1 = Y \times X$, $Z_2 = \mathbf{E}\{\langle x, x\rangle \mid x \in X\}$ of $\mathscr{P} \times \mathscr{Q}$. Clearly Z_1 and Z_2 are disjoint. Next, Z_1 is closed as the product of two closed sets, and Z_2 is closed because Z_2 is closed in $]\!] 0, 1 [\![\times \mathscr{Q}$ by 27 A.7 ($\mathscr{Q}$ is a subspace of $]\!] 0, 1 [\![$, $]\!] 0, 1 [\![$ is separated) and the closure structure of $\mathscr{P} \times \mathscr{Q}$ is finer than the closure structure of $]\!]0, 1[\![\times \mathscr{Q}$. Finally, we shall prove that Z_1 and Z_2 are not separated in $\mathscr{P} \times \mathscr{Q}$. It is enough to show that the closure of any open neighborhood G of Z_2 intersects Z_1. The space $\mathscr{Q}$ is metrizable as a subspace of a metrizable space; we shall now apply the relativization of the metric for R to $|\mathscr{Q}|$. For each positive integer n let X_n be the set of all $x \in X$ such that the set $(x) \times U$, where U is the open n^{-1}-sphere about x in $\mathscr{Q}$, is contained in G. Clearly $\bigcup\{X_n\} = X$. Since the irrationals of $[\![0, 1]\!]$ are not $\mathbf{F}_\sigma$ in $[\![0, 1]\!]$ (cf. 22 ex. 7), by (a) (δ), X is not an $\mathbf{F}_\sigma$ in $\mathscr{P}$, and therefore $\overline{X}_k \cap Y \neq \emptyset$ for some k. Choose an y in $\overline{X}_k \cap Y$ and then choose an $x \in X_k$ such that $|x - y| < (2k)^{-1}$. We shall prove that every neighborhood of the point $\langle y, x\rangle \in Z_1$ intersects G and hence $\langle y, x\rangle \in \overline{G} \cap Z_1$. If $U \times V$ is any canonical neighborhood of $\langle y, x\rangle$ in $\mathscr{P} \times \mathscr{Q}$, then U is a neighborhood of y in $\mathscr{P}$ and hence we can choose an x' in $U \cap X_k$ so that $|x' - y| < (2k)^{-1}$. Thus $|x - x'| \leqq |x - y| + |y - x'| \leqq k^{-1}$ and hence, since $x' \in X_k$, the point $\langle x', x\rangle$ belongs to G. However, $\langle x', x\rangle \in U \times V$ because $x' \in U$ and $x \in V$. Thus $(U \times V) \cap G \neq \emptyset$.

30 D.6. Theorem. *Each of the following two conditions is necessary and sufficient for a subset X of a separated hereditarily paracompact space $\mathscr{P}$ to be closed:*

(a) *The fine uniformity of the subspace X of $\mathscr{P}$ is a relativization of the fine uniformity of $\mathscr{P}$.*

(b) *Every bounded continuous pseudometric on the subspace X of $\mathscr{P}$ is a relativization of a continuous pseudometric for $\mathscr{P}$.*

Proof. Condition (a) is necesary by 30 C.7 and sufficient by 30 C.9. Both conditions are equivalent by 25 F.2, 3.

30 D.7. *Suppose that $\mathscr{P}$ is hereditarily paracompact and $\{X_a \mid a \in A\}$ is a family of subsets which is locally finite in the subspace $X = \bigcup\{X_a\}$ of $\mathscr{P}$. Then there exists a family $\{U_a\}$ of open subsets of $\mathscr{P}$ which is locally finite in the subspace $U = \bigcup\{U_a\}$ such that $U_a \supset X_a$ for each a. Moreover, U_a can be chosen so that $U_a \supset \overline{X}_a \cap U$. If X_a are open in X then U_a can be chosen so that $U_a \cap X = X_a$.*

Proof. Let G be the union of all open subsets H of $\mathscr{P}$ such that $H \cap X_a \neq \emptyset$ for only a finite number of a's. Clearly $X \subset G$ and $\{X_a\}$ is a locally finite family in G, and consequently the family $\{\overline{X}_a \cap G\}$ is also locally finite in G. The subspace G being paracompact, by 30 C.10 there exists a locally finite family $\{V_a\}$ of open subsets of the space G such that $V_a \supset \overline{X}_a \cap G$. In the general case we can put $U_a = V_a$, while in the case that the X_a are open in X we choose open sets W_a in $\mathscr{P}$ such that $W_a \cap X = X_a$ and then clearly we may put $U_a = V_a \cap W_a$.

30 D.8. *In order that a regular topological space $\mathscr{P}$ be paracompact and perfectly normal it is necessary and sufficient that for each family $\{U_a\}$ of open sets there exists a σ-locally finite family $\{V_b\}$ of open sets such that $\{V_b\}$ refines $\{U_a\}$ and $\bigcup\{V_b\} = \bigcup\{U_a\}$.*

Proof. I. Suppose the condition holds. Clearly $\mathscr{P}$ is paracompact (30 C.12, condition (a)). If U is any open subset of $\mathscr{P}$ and $\{U_x \mid x \in U\}$ is a family such that each U_x is an open neighborhood of x and the closure of U_x is contained in U, and if $\{V_b\}$ is a σ-locally finite family of open subsets of $\mathscr{P}$ such that $\{V_b\}$ refines $\{U_x\}$ and $\bigcup\{V_b\} = \bigcup\{U_x\} = U$, then clearly $\bigcup\{\overline{V}_b\}$ is an F_σ-set in $\mathscr{P}$ and $\bigcup\{\overline{V}_b\} = U$.

II. Conversely, suppose that $\mathscr{P}$ is paracompact and perfectly normal. Given a family $\{U_a\}$ of open subsets of $\mathscr{P}$, put $U = \bigcup\{U_a\}$ and choose a sequence $\{G_n\}$ of open subsets of $\mathscr{P}$ such that $\overline{G_n} \subset U$ and $\bigcup\{G_n\} = U$ ($\mathscr{P}$ is perfectly normal and U is open). For each n the family $\{U_a \cap \overline{G}_n \mid a \in A\}$ is an open cover of the paracompact space $\overline{G}_n$ (a closed subspace of a paracompact space) and therefore $\{U_n \cap \overline{G}_n\}$ is refined by a locally finite open cover $\{W_{\langle n,c\rangle} \mid c \in C_n\}$. Evidently $\{W_{\langle n,c\rangle} \mid c \in C_n\}$ is locally finite in $\mathscr{P}$. For each b in $B = \Sigma\{C_n \mid n \in \mathsf{N}\}$, $b = \langle n, c\rangle$, put $V_b = G_n \cap W_b$. Clearly $\{V_b \mid b \in B\}$ has the required properties.

Remark. This constitutes a new proof of 30 D.4.

By example 30 D.4 (b) the product of a metrizable space and a hereditarily paracompact space need not be normal. Now we shall prove the following

30 D.9. Theorem. *The product of a metrizable space and a paracompact perfectly normal space is paracompact and perfectly normal (and by* 30 D.4 *hereditarily paracompact).*

Proof. Let $\mathscr{P}$ be paracompact and perfectly normal and let $\mathscr{Q}$ be metrizable. By 29 B.1 the space $\mathscr{Q}$ has a σ-discrete open base $\{U_{na} \mid a \in A_n, n \in \mathsf{N}\}$ where each family $\{U_{na} \mid a \in A_n\}$ is discrete. — I. First we shall prove that each open subset G of $\mathscr{P} \times \mathscr{Q}$ is an F_σ. For each $n \in \mathsf{N}$ and each $a \in A_n$ let V_{na} be the union of all open subsets V of P such that $V \times U_{na} \subset G$. Evidently the family $\{V_{na} \times U_{na} \mid n \in \mathsf{N}, a \in A_n\}$ is σ-discrete, and hence σ-locally finite in $\mathscr{P} \times \mathscr{Q}$. It is easily seen that $\bigcup\{V_{na} \times U_{na}\} = G$ and hence, each $V_{na} \times U_{na}$ being F_σ in $\mathscr{P} \times \mathscr{Q}$ as the product of two F_σ-sets, G is clearly an F_σ. — II. We shall prove that $\mathscr{P} \times \mathscr{Q}$ is paracompact. Let $\{G_b \mid b \in B\}$ be an open cover of $\mathscr{P} \times \mathscr{Q}$. For each $n \in \mathsf{N}$, $a \in A_n$ and $b \in B$ let $V_{b,n,a}$ stand for the union of all open sets V in $\mathscr{P}$ such that $V \times U_{na} \subset G_b$. Put $V^*_{na} = \bigcup\{V_{b,n,a} \mid b \in B\}$. By 30 D.7, for each $n \in \mathsf{N}$ and $a \in A_n$ we can choose a σ-locally finite family $\{W_{n,a,c} \mid c \in C_{na}\}$ of open subsets of $\mathscr{P}$ whose union is V^*_{na}, such that $\{W_{n,a,c} \mid c \in C_n\}$ refines $\{V_{b,n,a} \mid b \in B\}$. Now it is easily seen that the family $\{W_{n,a,c} \times U_{na} \mid n \in \mathsf{N}, a \in A_n, c \in C_n\}$ is a σ-locally finite cover of $\mathscr{P} \times \mathscr{Q}$ which refines $\{G_b \mid b \in B\}$. By 30 C.12, condition (a), the space $\mathscr{P} \times \mathscr{Q}$ is paracompact.

In conclusion we shall show that, except for the trivial case where $\mathscr{Q}$ is discrete, the conditions imposed on $\mathscr{P}$ in the foregoing theorem are necessary for the product space $\mathscr{P} \times \mathscr{Q}$ to be paracompact and hereditarily normal.

30 D.10. Theorem. *Let $\mathscr{P}$ be a closure space and let $\mathscr{Q}$ be a metrizable space. In order that the product space $\mathscr{P} \times \mathscr{Q}$ be paracompact and hereditarily normal it is necessary and sufficient that either*

(a) *$\mathscr{P}$ is paracompact and perfectly normal, or*

(b) *$\mathscr{P}$ is paracompact hereditarily normal and $\mathscr{Q}$ is discrete.*

Proof. Condition (a) is sufficient by 30 D.9, and (b) is sufficient because if $\mathscr{Q}$ is discrete, then the product space $\mathscr{P} \times \mathscr{Q}$ coincides with the sum $\sum\{\mathscr{P} \mid x \in \mathscr{Q}\}$. The proof will be complete if we show that the assumption that $\mathscr{P} \times \mathscr{Q}$ (and hence $\mathscr{P}$) is paracompact and hereditarily normal, and that neither the condition (a) nor condition (b) is fulfilled, leads to a contradiction. Since $\mathscr{P}$ is normal but not perfectly normal, there exists an open subset G of $\mathscr{P}$ which is not an $\mathbf{F}_\sigma$ in $\mathscr{P}$. Since the metrizable space $\mathscr{Q}$ is not discrete we can choose a cluster point y of $\mathscr{Q}$ and a one-to-one sequence $\{y_n\}$ converging to y in $\mathscr{Q}$ such that $y_n \neq y$ for each n. Put $F = |\mathscr{P}| - G$ and consider the subspace $\mathscr{R} = \mathscr{P} \times \mathscr{Q} - (F \times (y))$ of $\mathscr{P} \times \mathscr{Q}$; thus $\mathscr{R}$ is normal by our assumption. We shall prove that the closed disjoint subsets $X = F \times \mathbf{E}\{y_n\}$ and $Y = G \times (y)$ of $\mathscr{R}$ are not separated in $\mathscr{R}$ which will be a contradiction. Since X and Y are disjoint and closed in $\mathscr{R}$, they are semi-separated in $\mathscr{R}$ and hence in $\mathscr{P} \times \mathscr{Q}$. Since the space $\mathscr{P} \times \mathscr{Q}$ is hereditarily normal, by 30 A.4, condition (b), the sets X and Y are separated in $\mathscr{P} \times \mathscr{Q}$, and hence there exist open subsets U and V of $\mathscr{P} \times \mathscr{Q}$ such that $X \subset U$, $Y \subset V$, $U \cap V = \emptyset$. For each $n \in \mathbf{N}$ let W_n be the set of all x in $\mathscr{P}$ such that $\langle x, y_n \rangle \in U$. Clearly each W_n is an open set in $\mathscr{P}$ containing F. Since $G = |\mathscr{P}| - F$ is not an $\mathbf{F}_\sigma$-set, we obtain that $\bigcap\{W_n\} - F = G - \bigcup\{|\mathscr{P}| - W_n\} \neq \emptyset$. Choosing a point x in this intersection we obtain $\langle x, y \rangle \in \overline{U}$; on the other hand $\langle x, y \rangle \in Y$, and consequently the neighborhood V of $\langle x, y \rangle$ must intersect U; this contradicts our assumption $U \cap V = \emptyset$.

E. FEEBLE LOCALIZATION

Let $\mathscr{X}$ be a collection of subsets of a space $\mathscr{P}$. By 21 A.12 we say that a subset Y of $\mathscr{P}$ feebly locally at $x \in \mathscr{P}$ belongs to $\mathscr{X}$ if there exists a neighborhood U of x such that $U \cap Y \in \mathscr{X}$, and we say that Y feebly locally (relatively feebly locally) belongs to $\mathscr{X}$ if Y feebly locally at x belongs to $\mathscr{X}$ for each $x \in \mathscr{P}$ ($x \in Y$). Evidently, each $X \in \mathscr{X}$ feebly locally belongs to $\mathscr{X}$, and if Y feebly locally belongs to $\mathscr{X}$ then Y relatively feebly locally belongs to $\mathscr{X}$. By 21 A.12 we say that $\mathscr{X}$ is feebly locally determined (relatively feebly locally determined) if a subset Y of $\mathscr{P}$ belongs to $\mathscr{X}$ whenever it feebly locally (relatively feebly locally) belongs to $\mathscr{X}$. E.g. in any space the collection of all open sets as well as the collection of all closed sets is feebly locally determined.

It turns out that paracompactness and hereditary paracompactness are sufficient conditions for some important collections of sets to be feebly locally determined or relatively feebly locally determined.

30 E.1. Theorem. *In a paracompact space the collections of all $\mathbf{F}_\sigma$-sets, $\mathbf{G}_\delta$-sets, exact open sets, exact closed sets, exact $\mathbf{F}_\sigma$-sets and exact $\mathbf{G}_\delta$-sets are feebly locally determined. In a hereditarily paracompact space the collection of all $\mathbf{G}_\delta$-sets as well as the collection of sets of the form $U \cap X$, when U is open and X is an $\mathbf{F}_\sigma$, are relatively feebly locally determined.*

Remark. In a normal space the collection of all $\mathbf{F}_\sigma$-sets as well as the collection of all $\mathbf{G}_\delta$-sets need not be feebly locally determined (ex. 8).

The proof will be given in a sequence of propositions, interesting in themselves, which can be applied in more general situations. It is convenient to introduce the following concept.

30 E.2. Definition. We shall say that a *collection* $\mathscr{X}$ of subsets of a closure space $\mathscr{P}$ is *rich* if each point x of $\mathscr{P}$ has a neighborhood U such that

($*$) if $F \subset V \subset U$, F is closed, V is open, then there exists a $Y \subset |\mathscr{P}|$ such that $F \subset Y \subset V$ and $X \in \mathscr{X}$ implies $X \cap Y \in \mathscr{X}$.

It is to be noted that in any closure space the following collections are rich: every hereditary collection (i.e. $Y \subset X \in \mathscr{X} \Rightarrow Y \in \mathscr{X}$), the collection of all closed sets, the collection of all open sets, the collection of all $\mathbf{G}_\delta$-sets, $\mathbf{F}_\sigma$-sets, Borel sets, Baire sets, etc. Next, if a neighborhood U of x has the property ($*$), then every subset of U has the property ($*$).

30 E.3. *Suppose that $\mathscr{X}$ is a feebly locally determined, additive and rich collection of subsets of a regular topological space $\mathscr{P}$. Then $\mathscr{X}$ is closed under locally finite unions, i.e. if a family $\{X_a\}$ in $\mathscr{X}$ is locally finite in $\mathscr{P}$, then $\bigcup\{X_a\} \in \mathscr{X}$.*

Proof. There exists a family $\{U_x \mid x \in \mathscr{P}\}$ such that each U_x is an open neighborhood of x with property 30 E.2 ($*$) such that the set A_x of all indices a with $U_x \cap X_a \neq \emptyset$ is finite. Since $\mathscr{P}$ is regular we can choose a family $\{V_x \mid x \in \mathscr{P}\}$ such that each V_x is a neighborhood of x and $\overline{V}_x \subset U_x$. Since $\overline{V}_x$ is closed we can choose a family $\{Y_x\}$ such that $\overline{V}_x \subset Y_x \subset U_x$ and $X \in \mathscr{X}$ implies $X \cap Y_x \in \mathscr{X}$. Clearly $Y_x \cap \bigcup\{X_a\} = \bigcup\{Y_x \cap X_a \mid a \in A_x\}$ and hence $Y_x \cap \bigcup\{X_a\} \in \mathscr{X}$ for each x in $\mathscr{P}$. But Y_x is a neighborhood of x and $\mathscr{X}$ is feebly locally determined and thus $\bigcup\{X_a\} \in \mathscr{X}$.

30 E.4. *Suppose that $\mathscr{X}$ is a rich collection of subsets of a paracompact space $\mathscr{P}$ such that the union of any locally finite family in $\mathscr{X}$ belongs to $\mathscr{X}$. Then the collection $\mathscr{X}$ is feebly locally determined (and, of course, additive).*

Proof. Assuming that a set X feebly locally belongs to $\mathscr{X}$, we can choose a family $\{U_x \mid x \in \mathscr{P}\}$ such that U_x is a neighborhood of x and $X \cap U_x \in \mathscr{X}$ for each x. Next, choose a family $\{V_x \mid x \in \mathscr{P}\}$ such that each V_x is a neighborhood of x with property 30 E.2 ($*$). Since $\mathscr{P}$ is paracompact we can choose locally finite open covers $\{W_x \mid x \in \mathscr{P}\}$ and $\{G_x \mid x \in \mathscr{P}\}$ such that $\overline{W}_x \subset G_x \subset U_x \cap V_x$ for each x. Finally, choose a family $\{Y_x \mid x \in \mathscr{P}\}$ such that $\overline{W}_x \subset Y_x \subset G_x$ and $Y \in \mathscr{X}$ implies $Y \cap Y_x \in \mathscr{X}$ for each x. Now $Y_x \cap X = Y_x \cap (U_x \cap X) \in \mathscr{X}$ for each x and hence, $\{Y_x \cap X \mid x \in \mathscr{P}\}$ being locally finite, $X = \bigcup\{Y_x \cap X \mid x \in \mathscr{P}\} \in \mathscr{X}$.

30 E.5. *Suppose that $\mathscr{X}$ is a collection of subsets of a closure space $\mathscr{P}$ and let $\mathscr{X}_\sigma$ be the collection of all countable unions of sets of $\mathscr{X}$, and $\mathscr{X}_\delta$ be the collection of all countable intersections of sets of $\mathscr{X}$. We know that $\mathscr{X}_\sigma$ is additive, and if $\mathscr{X}$ is multiplicative, then $\mathscr{X}_\sigma$ is multiplicative; $\mathscr{X}_\delta$ is multiplicative and if $\mathscr{X}$ is additive then $\mathscr{X}_\delta$ is also additive. Supposing that $\mathscr{P}$ is paracompact, if $\mathscr{X}$ is feebly locally* *determined, rich and additive, then $\mathcal{X}_\sigma$ has the same properties, and if, in addition $\mathcal{X}$ is multiplicative, then $\mathcal{X}_\delta$ also has these properties.*

Proof. The statement concerning $\mathcal{X}_\sigma$ follows from 30 E.3, 30 E.4 and the fact that if $\mathcal{X}$ is closed under locally finite unions then $\mathcal{X}_\sigma$ is also closed under locally finite unions (see 22 ex. 11). To prove the statement concerning $\mathcal{X}_\delta$, by 30 E.4 it is enough to show that $\mathcal{X}_\delta$ is closed under locally finite unions. Suppose that $\{X_a \mid a \in A\}$ is a locally finite family in $\mathcal{X}_\delta$. Since $\mathcal{P}$ is paracompact we can choose a locally finite cover $\{Y_b \mid b \in B\}$ of $\mathcal{P}$ such that the set A_b of all $a \in A$ with $X_a \cap Y_b \neq \emptyset$ is finite for each b in B, and $Y \in \mathcal{X}$ implies $Y \cap Y_b \in \mathcal{X}$ (see the proof of 30 E.4). For each b in B let $Z_b = \bigcup\{X_a \cap Y_b \mid a \in A_b\}$. We have $Z_b \in \mathcal{X}_\delta$ (because $\mathcal{X}$, and hence $\mathcal{X}_\delta$ is additive), $\bigcup\{Z_b \mid b \in B\} = \bigcup\{X_a \mid a \in A\}$, $Z_b \subset Y_b \in \mathcal{X}$, $\{Z_b \mid b \in B\}$ is locally finite. By 22 ex. 11, $\bigcup\{Z_b\}$, and hence $\bigcup\{X_a\}$, belongs to $\mathcal{X}_\delta$.

30 E.6. Proof of 30 E.1. Since, in any space, the collection of all open (closed) sets is rich, additive, multiplicative and feebly locally determined, the statements concerning $\mathbf{F}_\sigma$-sets and $\mathbf{G}_\delta$-sets follow from 30 E.5. Now to prove the statements concerning exact closed and exact open sets it is enough to recall that in a normal space, and thus in any paracompact space, a set X is exact closed (exact open) if and only if X is a closed $\mathbf{G}_\delta$-set (open $\mathbf{F}_\sigma$-set). Finally, the statements concerning exact $\mathbf{G}_\delta$-sets and exact $\mathbf{F}_\sigma$-sets again follow from 30 E.5. The statements concerning hereditarily paracompact spaces are proved in 30 E.8, 9.

By 30 C.5 the hereditarily normal locally metrizable space of countable ordinals is not metrizable. For the sake of completness we restate 30 C.5 (b):

30 E.7. Theorem. *Every paracompact feebly locally pseudometrizable space is pseudometrizable.*

Proof. Since a subspace of a pseudometrizable space is pseudometrizable, we can find an open locally finite cover $\mathcal{U}$ such that each subspace $U \in \mathcal{U}$ is pseudometrizable. If $\mathcal{B}_U$ is a σ-locally finite open base for U, then the union of $\{\mathcal{B}_U \mid U \in \mathcal{U}\}$ is a σ-locally finite open base for the whole space. Now the statement follows from Metrization Theorem 30 B.1.

It turns out that the collections of 30 E.1 are not relatively feebly locally determined in a paracompact space. We restrict ourselves to $\mathbf{F}_\sigma$-sets and $\mathbf{G}_\delta$-sets.

30 E.8. *In a hereditarily paracompact space the collection of all $\mathbf{G}_\delta$-sets is relatively feebly locally determined.*

Proof. Suppose that a subset X of a hereditarily paracompact space $\mathcal{P}$ is relatively feebly locally a $\mathbf{G}_\delta$-set in $\mathcal{P}$. Choose a family $\{U_x \mid x \in X\}$ such that U_x is a neighborhood of x and $X \cap U_x$ is a $\mathbf{G}_\delta$. Without a loss of generality we may assume that all the U_x are open. If U is the union of $\{U_x \mid x \in \mathcal{P}\}$, then clearly X is feebly locally a $\mathbf{G}_\delta$-set in the paracompact subspace U of $\mathcal{P}$ and hence, by 30 E.1, X is a $\mathbf{G}_\delta$ in U. Since U is open in $\mathcal{P}$, X is a $\mathbf{G}_\delta$ in $\mathcal{P}$.

30 E.9. *In a hereditarily paracompact space $\mathscr{P}$ the collection $\mathscr{X}$ of all sets of the form $G \cap Y$, where G is open and Y is an $\mathbf{F}_\sigma$-set, is relatively feebly locally determined.*

Proof. Suppose that X relatively feebly locally belongs to $\mathscr{X}$, and take a family $\{U_x \mid x \in X\}$ such that U_x is a neighborhood of x and $U_x \cap X \in \mathscr{X}$ for each x; choose open sets G_x and $\mathbf{F}_\sigma$-sets Y_x such that $U_x \cap X = G_x \cap Y_x$ for each x in X. Since $\mathscr{P}$ is uniformizable, we can choose a family $\{V_x \mid x \in X\}$ such that V_x is an exact open neighborhood of x (thus V_x is an $\mathbf{F}_\sigma$) contained in $U_x \cap G_x$. Since $V_x \cap X = V_x \cap Y_x$, the set $V_x \cap X$ is an $\mathbf{F}_\sigma$ in $\mathscr{P}$ and hence in the subspace $V = \bigcup\{V_x \mid x \in X\}$ of $\mathscr{P}$. Since clearly X is feebly locally an $\mathbf{F}_\sigma$ in U and U is paracompact, by **30 E.1** X is an $\mathbf{F}_\sigma$ in U. Thus $X = X \cap U$ belongs to $\mathscr{X}$.