CHAPTER III

TOPOLOGICAL SPACES

(Sections 14 – 22)

The general theory of topological spaces had its origin at the beginning of the 20th century. Previously topological problems had usually been investigated for only those individual spaces and their subsets for which the concepts of a limit, a cluster point, the closure of a set, etc., had a clear intuitive meaning. In the thirties, a new type of structure was introduced (partly to meet the needs of functional analysis), namely uniformities, and in the fifties proximity spaces were investigated.

Various other types of what may be called "continuity structures" have been examined in recent years. Pending further developments, however, it seems that three kinds of these "continuity structures" constitute a substantial part of general topology, a part which, on the one hand, can be given a systematic development and, on the other hand, can serve as a sufficiently broad basis for most investigations in which we are confronted with an underlying "continuity structure". Some other "continuity structures" are mentioned in the Notes at the end of this book where we also indicate a possible unified approach to various continuity structures.

We shall now try to explain in a brief and informal manner some general ideas which concern the continuity structures under consideration (that is, closure structures, uniformities, proximities) and lead to concepts playing an important role in the subsequent developments. Disregarding historical questions, we shall concentrate on basic ideas from the standpoint of the present state of knowledge, giving special attention to their connection with mathematical analysis.

Two basic concepts appear in almost every problem of mathematical analysis, namely the concept of an operation and the concept of approximation. In application of numerical analysis, certain operations, constituting an "algorithm", are effectively performed; the degree of approximation (required or actually achieved) is also actually given. In theoretical questions of mathematical analysis, properties of various operations are investigated; instead of the actual degree of approximation, we are really interested in the possibility of approximation; we ask whether a certain series converges to an element, whether a function admits arbitrarily "good" approximation by functions of a certain kind, or whether two approximative procedures give, "in the limit", the same solution of a certain equation.

This observation, although almost trivial, leads to a more precise (but still vague) description of the "continuity structures" to be investigated.

Namely, if we put aside questions concerning operations, we are confronted with a pure theory of approximation, or rather of the possibility of approximation of elements of a set P by elements of P. Now, the structure of the "possibility of approximation" of elements of P may be conceived in various ways. We may consider this structure as given as soon as it is known, for any $x \in P$ and any $X \subset P$, whether x admits of an arbitrarily close approximation by elements of X; this approach leads to topological spaces. On the other hand, we may require a more detailed description of the "possibilities of approximation", considering the structure as given if, for any set M of pairs $\langle x, y\rangle \in P \times P$, it is known whether or not M contains pairs $\langle x, y\rangle$ with y arbitrarily close to x; this approach leads to uniform spaces. Finally, there is an "intermediate" approach under which the structure in question is given as soon as we know, for any $A \subset P$, $B \subset P$ whether or not there are $x \in A$, $y \in B$ with y arbitrarily close to x (or, which will be shown to be equivalent, if we know whether A and B are "proximal"); this approach leads to proximity spaces.

To be more concrete, consider a set P endowed with a "distance", i.e. with a real-valued function ϱ on $P \times P$ such that $0 \leqq \varrho\langle x, y\rangle = \varrho\langle y, x\rangle \leqq \varrho\langle x, z\rangle + \varrho\langle z, y\rangle$. If $x \in P$, $X \subset P$ and X contains elements y with $\varrho\langle x, y\rangle$ arbitrarily small, we say that x may be approximated by elements of X or that "x is close to X". If we restrict our attention to the fact that, for any $X \subset P$ and $x \in P$, it is known whether or not x may be approximated by elements of X, and disregard the metric ϱ, then we consider, in fact, a topological space. If we say that a set X is close to a set Y if there are elements $x \in X$, $y \in Y$ with $\varrho\langle x, y\rangle$ arbitrarily small, and confine our attention to this relation disregarding other properties of ϱ, then we investigate a proximity space, and so on. Of course, a topological, proximity or uniform structure need not be determined by a metric, and may be described in any manner sufficient for a determination of the "structure of approximation" concerned.

Of course, whether or not the investigation of the structures indicated above (and described exactly in 14, 23 and 25) is useful and has its place in mathematics can hardly be answered on the ground of any *a priori* considerations (although this may give an important heuristic lead); this can be settled only by the future development of mathematics. In the same sense, only a small part of the main results of the theory of these structures follows from general considerations only; the core of the theory constitutes, in the last instance, an answer to problems raised by the development of general analysis.

In the present chapter we shall consider basic ideas concerning closure spaces which include topological ones. A topological space is currently defined to be a struct $\langle P, \mathscr{U}\rangle$ such that P is a set and $\mathscr{U}$ is a collection of subsets of P satisfying certain conditions; the elements of $\mathscr{U}$ are called open sets. Given such a space $\langle P, \mathscr{U}\rangle$ we can define the closure operation u associated with $\mathscr{U}$ as follows: u is a single-valued relation, $\mathbf{D}u = \exp P \supset \mathbf{E}u$, and uX is the smallest set containing X such that $(P - uX) \in \mathscr{U}$. The relation u has the following properties: $u\emptyset = \emptyset$, $X \subset uX$, $u(X \cup Y) = uX \cup uY$, and $uuX = uX$. We shall examine a more general kind of spaces, the so-called closure spaces. A closure space is defined to be a struct $\mathscr{P} = \langle P, u \rangle$ where P is a set and u is a closure operation for P, i.e., a relation with the properties mentioned above except for the last property; if u also has the last property then u is termed a topological closure operation and $\mathscr{P}$ is termed a topological closure space, or merely a topological space. A classical example of a closure space which is not topological is the following: Let P be the set of all functions on the interval $I = [\![0, 1]\!]$ of real numbers and let uX, where $X \subset P$, be the set of all functions f such that some sequence $\{f_n\}$ in X converges pointwise to f. Clearly $\langle P, u \rangle$ is a closure space. If C is the set of all continuous functions on I then uC consists of all functions of the 1st Baire class, uuC is the set of all functions of the 2nd Baire class, etc., and hence u is not topological.

It may be in place to explain the reasons which led to an examination of more general spaces than topological ones, namely closure spaces, although currently topological spaces form the adopted background for topological investigations. It turns out that there are some important closure spaces which are not topological; e.g. spaces of mappings with pointwise convergence of sequences, quotient spaces, and the "sequential continuity" of mappings of topological linear spaces can be regarded as the usual continuity with respect to some closure spaces which are not topological. Furthermore, a great deal of basic definitions and theorems for topological spaces carries over to closure spaces. Finally, one can set up a general background for various continuous structures, such as uniform spaces, proximity spaces, etc. Discussing this problem we can conclude that the condition $uuX = uX$ is rather special in character. Some investigations of the present chapter are motivated by general considerations (e.g. the examination of subspaces, sums, products, etc.); to a certain degree, this also applies to the examination of pseudometrics, etc. For some properties, the motivation comes from analysis. Thus, the examination of meager and non-meager sets has its origin in the fact that some properties (by the way, unpleasant ones, as a rule) are possessed by "almost all continuous functions"; the exact definition of "almost all" involves topological properties. Naturally, the properties of being meager or non-meager, investigated in general topology, have lost their connection with this motivation and are investigated for their own sake. Nevertheless, pertinent facts from analysis remain, at least, a valuable heuristic lead.

In Sections 14 and 15 we shall describe closure spaces and characterize topological spaces among closure spaces by means of neighborhoods, cluster points and convergent nets. Section 16, which contains elementary facts concerning continuous mappings, is followed by a closely related section (17) in which some constructions of new spaces from older ones will be examined, namely subspaces, sums, products and inductive products. In Section 18 pseudometrics (more generaly, semi-pseudometrics) and some special closures for ordered sets will be examined. Subsection 18 **B** may serve as an introduction to uniform spaces. The results of 14–18 are applied to topologized algebraic structs in Section 19; the basic concepts are a topologized internal composition $\langle \sigma, u \rangle$ and a topologized external composition $\langle u, \varrho, v \rangle$ for which we define continuity and inductive continuity. Special attention is given to topological groups, which are treated as continuous topologized groups with continuous inversion.

In Section 20 separation and semi-separation in a closure space are examined and applied to connectedness. In Section 21 a general discussion of the localization of properties is given and applied to locally connected spaces.

The last section contains basic facts concerning dense, nowhere dense, meager and non-meager sets, and also Baire sets, Borel sets and Baire or Borel measurable mappings.

14. CLOSURE SPACES

All definitions, examples and propositions in this section are fundamental and they appear frequently in later developments. Most of the results of this section will be used without any reference, and therefore the reader is asked to read it carefully even though all proofs are almost evident.

In the first subsection we shall introduce the notions of a closure operation, closure space, open set, closed set, interior of a set and a closure-preserving family. In addition, we shall introduce an order in the class of all closure operations, which will be studied throughout this book.

The second part is concerned with the description of a closure operation in terms of neighborhoods. Theorems of this subsection are illustrated by two rather general examples, namely we shall introduce the order closure and the notion of a generalized order closure for a monotone ordered set; in particular, we shall define the space R of reals as the ordered set of reals endowed with the order closure, and we shall introduce a closure operation for the set of all ultrafilters of a given set X. (It should be noted that the resulting space, called the ultrafilter space associated with the set X and denoted by βX, is the Čech-Stone compactification of X endowed with the discrete closure operation.) Next, the notion of a locally finite family is introduced and its relation to closure-preserving families is clarified.

In the last subsection notions such as a cluster point, an isolated point, a regular closed and a regular open set are introduced and studied.

A. CLOSURE OPERATIONS

14 A.1. Definition. If P is a set and u is a single-valued relation on exp P ranging in exp P, then we shall say that u is a *closure operation* (or simply a *closure*) for P provided that the following conditions (also called axioms) are satisfied:

(cl 1) $u\emptyset = \emptyset$,

(cl 2) $X \subset uX$ for each $X \subset P$,

(cl 3) $u(X \cup Y) = uX \cup uY$ for each $X \subset P$ and $Y \subset P$.

A struct $\langle P, u \rangle$, where P is a set and u is a closure operation for P, will be called a *closure space*. If $\langle P, u \rangle$ is a closure space and $X \subset P$, then the set uX will be called the *closure of* X in $\langle P, u\rangle$ or under u. The closure of a set X in a closure space $\mathscr{P}$ will also be denoted by $\overline{X}^{\mathscr{P}}$; if there is no danger of misunderstanding, then we shall write simply $\overline{X}$ instead of $\overline{X}^{\mathscr{P}}$.

Conventions. In accordance with the conventions introduced in 7 A.2 we shall often write P instead of $\langle P, u\rangle$ and we shall speak of subsets of a closure space instead of subsets of the underlying set of the space in question. Moreover, we shall rarely speak of elements of a space (cf. 7 A.2); for the most part, if x belongs to the underlying set of a closure space, we shall say that x is a *point of the space* in question.

Let us notice that the underlying set of a space $\langle P, u\rangle$ is uniquely determined by u; indeed, P is the union of the domain of u. It follows that the relation $\{u \to \langle \bigcup \mathbf{D}u, u\rangle \mid u \text{ is a closure}\}$ is a one-to-one relation on the class of all closure operations ranging on the class of all closure spaces.

14 A.2. Definition. The class of all closure operations as well as the class of all closure spaces will be denoted by $\mathbf{C}$. According to the above remark this ambiguity cannot lead to a confusion. A closure u is said to be *coarser* than a closure v, and v to be *finer* than u, if $\mathbf{D}u = \mathbf{D}v$ and $uX \supset vX$ for each X (in the common domain of both u and v). Evidently the relation

$$< \; = \mathbf{E}\{\langle v, u\rangle \mid v, u \in \mathbf{C}, u \text{ is coarser than } v\}$$

is an order on the class $\mathbf{C}$. If P is a set then the set of all closures for P is denoted by $\mathbf{C}(P)$ and the restriction of $<$ to $\mathbf{C}(P)$ is also denoted by $<$. Usually $\mathbf{C}$ also denotes the class $\mathbf{C}$ ordered by $<$, and $\mathbf{C}(P)$, P being a set, the set $\mathbf{C}(P)$ ordered by $<$.

14 A.3. Let P be a set. The identity relation on $\exp P$ is clearly a closure operation for P which is finer than any closure for P; in other words, the identity relation on $\exp P$ is the finest closure for P. It will be called the *discrete closure* for P. A *discrete space* is a set endowed with the discrete closure. Setting $u\emptyset = \emptyset$ and $uX = P$ for each non-void subset of P, we obtain a closure operation for P which is obviously coarser than any other closure for P, in other words, which is the coarsest closure operation for P. This closure will be called the *accrete closure* for P, and a set endowed with the accrete closure will be called an accrete space. It is to be noted that some authors employ the word *indiscrete* instead of accrete.

We have just seen that the ordered set of all closure operations for a given set P possesses the least and the greatest elements. Rather extensive investigations of this ordered set are given in Chapter VI. For example it will be shown that the set of all closures for a given set is order-complete. These investigations become the starting point for many general constructions. Nevertheless, even in Chapters III and IV we shall make use of the notation and we shall prove some preliminary results.

14 A.4. Definition. A subset X of a closure space $\langle P, u\rangle$ will be called *closed* if $uX = X$, *open* if its complement (relative to P) is closed, i.e. if $u(P - X) = P - X$. If u is a closure operation, then a set X is called u-closed (u-open) if X is closed (open) in the space $\langle \bigcup \mathbf{D}u, u\rangle$.

Thus closed sets of a space $\langle P, u\rangle$ are exactly the "fixed elements" of u. Since always $X \subset uX$ (by axiom (cl 2)) we obtain that a subset X of a space $\langle P, u\rangle$ is closed if and only if $X \supset uX$.

14 A.5. Examples. (a) Let P be a set. Setting $uX = X$ if X is a finite subset of P and $uX = P$ if X is an infinite subset, we obtain a closure operation $u = \{X \to uX\}$ for P (prove!). Observe that $X \subset P$ is closed in $\langle P, u\rangle$ if and only if X is finite or $X = P$. The closure of any subset is a closed set. The space $\langle P, u\rangle$ is discrete if and only if P is finite.

(b) The construction in (a) admits the following generalization. Let P be a set and let m be an infinite cardinal. Define a closure operation u for P by setting $uX = X$ if the cardinal of X is less than m, and $uX = P$ otherwise. The space of (a) is obtained for $m = \aleph_0$. A set $X \subset P$ is closed in $\langle P, u\rangle$ if and only if the cardinal of X is less than m or $X = P$. The space $\langle P, u\rangle$ is discrete if and only if the cardinal of P is less than m.

(c) Let $\langle P, \leqq\rangle$ be a well-ordered set. For each $X \subset P$ let uX be the subset of P consisting of all points of X and the successors of all $x \in X$. The relation $u = \{X \to uX\}$ on exp P is a closure operation. If P contains at least three elements, then there exists a subset X of P such that $uuX \neq uX$. Indeed, if x is the least element, then $u(x) \neq uu(x)$.

(d) Let P be a set. Fix a point x of P. Let us define a closure operation u for P by setting $uX = X$ if X is finite and $uX = X \cup (x)$ otherwise. A subset X of P is closed in P if and only if X is finite or $x \in X$. The closure of any subset is a closed set.

(e) Let P be a set. Fix a point x of P. Put $u\emptyset = \emptyset$ and $uX = X \cup (x)$ for $X \neq \emptyset$. Clearly $u = \{X \to uX\}$ is a closure operation for P and $X \subset P$ is closed in $\langle P, u\rangle$ if and only if $X = \emptyset$ or $x \in X$.

(f) Let ϱ be a reflexive relation for a set P, i.e. $J_P \subset \varrho \subset P \times P$. The expansion $\{X \to \varrho[X]\}$ of ϱ (see **1 E.14**) is a closure operation for P which will sometimes be called the closure operation associated with ϱ. Indeed, $\varrho[\emptyset] = \emptyset$, $X \subset \varrho[X]$ because $J_P \subset \varrho$ and obviously $\varrho[X \cup Y] = \varrho[X] \cup \varrho[Y]$. If ϱ is a reflexive quasi-order (that is, if ϱ is also transitive), then the closure of any set is a closed set. A closure u for a set P is associated with a reflexive relation if and only if

$(*)$ $uX = \bigcup\{u(x) \mid x \in X\}$

for each $X \subset P$, i.e. the closure of a set X is the union of closures of all one-point sets contained in X. Indeed, if $u = \{X \to \varrho[X]\}$, then clearly $(*)$ is fulfilled for each $X \subset P$. Conversely, if u fulfils $(*)$ and $\varrho = \Sigma\{u(x) \mid x \in P\}$ $(= \mathbf{E}\{\langle x, y\rangle \mid y \in u(x)\})$, then clearly ϱ is reflexive and $uX = \varrho[X]$ for each $X \subset P$, i.e., u is the closure associated with ϱ. The closures associated with reflexive relations will be studied in Section 26 under the name quasi-discrete closures.

By induction we obtain at once from condition (cl 3) that

$(**)$ $\bigcup\{\overline{X_a}\} = \overline{\bigcup\{X_a\}}$

for every finite family of subsets of a space P. Formula $(**)$ need not be true if the family is infinite. For example, if $\langle P, u \rangle$ is the space in the example 14 A.5 (d), then the family $\{X_a \mid a \in A\}$, where $X_a = (a)$ and $A \subset (P - (x))$, fulfils (**) if and only if A is finite. For convenience we shall introduce the following important notion.

14 A.6. Definition. A family $\{X_a \mid a \in A\}$ of subsets of a closure space P will be called *closure-preserving* if, for each $B \subset A$, $\bigcup\{\overline{X}_a \mid a \in B\} = \overline{\bigcup\{X_a \mid a \in B\}}$.

14 A.7. *Every subfamily of a closure-preserving family is a closure-preserving family. Every finite family of subsets of space a is a closure-preserving family.*

Proof. The first statement is obvious and the second one has already been proved.

Using the fact that $X \cup Y = Y$ if and only if $X \subset Y$, we obtain from condition (cl 3) the following proposition, which asserts that if u is a closure operation for a set P, then the mapping $u : \langle \exp P, \subset \rangle \to \langle \exp P, \subset \rangle$ is order-preserving.

14 A.8. *If X and Y are subsets of a closure space P such that $X \subset Y$, then $\overline{X} \subset \overline{Y}$.*

14 A.9. Theorem. *The collection of all closed subsets of a space P is closed under finite unions and arbitrary intersections, i.e. additive and completely multiplicative. The collection of all open sets of a space P is closed under arbitrary unions and finite intersections, i.e. completely additive and multiplicative. The sets $\emptyset$ and P are simultaneously closed and open.*

Proof. If X is the union of a finite family $\{X_a\}$, then $\overline{X} = \bigcup\{\overline{X}_a\}$, because any finite family is closure-preserving by 14 A.7, and if, moreover, all X_a are closed, i.e. $\overline{X}_a = X_a$, we obtain $\overline{X} \subset \bigcup\{X_a\} = X$ which means that X is closed. Now let X be the intersection of a family $\{X_a\}$ of closed sets. Since $X \subset X_a$ for each a, by 14 A.8 we obtain $\overline{X} \subset \overline{X}_a = X_a$ for each a, i.e. $\overline{X} \subset \bigcap\{X_a\} = X$, which implies that X is closed. The set $\emptyset$ is closed by (cl 1) and P is closed by (cl 2). The proof of all assertions concerning closed sets is complete. The statements concerning open sets follow from the statements concerning closed sets and de Morgan formulas. Indeed, for any family $\{X_a\}$ in $\exp P$ the following de Morgan formulas are valid (see 2.16):

$$P - \bigcap\{X_a\} = \bigcup\{P - X_a\},$$
$$P - \bigcup\{X_a\} = \bigcap\{P - X_a\}.$$

Now, if X is the intersection of a finite family of open sets, then X is open because, by the first formula, the complement of X is closed as a finite union of closed sets. Similarly, the union of any family of open sets is open because, by the second formula, its complement is closed (as the intersection of a family $\{P - X_a\}$ of closed sets). Finally, since $\emptyset$ and P are closed, their complements P and $\emptyset$ are necessarily open.

14 A.10. Definition. With any closure u for a set P there is associated the *interior operation* int_u, usually denoted briefly by int, which is a single-valued relation on $\exp P$ ranging in $\exp P$ such that, for each $X \subset P$,

$$\operatorname{int}_u X = P - u(P - X)$$

The set $\operatorname{int}_u X$ is called the *interior of X in $\langle P, u \rangle$* or the u-interior of X.

From the definitions of a closure operation and an interior operation one immediately obtains the following assertion:

14 A.11. *In any space P the following three conditions are fulfilled:*

(int 1) $\operatorname{int} P = P$

(int 2) *For each* $X \subset P$, $\operatorname{int} X \subset X$

(int 3) *For each* $X \subset P$ *and* $Y \subset P$, $\operatorname{int}(X \cap Y) = \operatorname{int} X \cap \operatorname{int} Y$.

It is worth noticing that the conditions (int 1), (int 2) and (int 3) are characteristic for the interior operation. More precisely, one can prove the following assertion: If int is a relation on exp P ranging in exp P satisfying the conditions (int k), $k = 1, 2, 3$, and if we define

$$uX = P - \operatorname{int}(P - X)$$

for each $X \subset P$, then $u = \{X \to uX\}$ is a closure operation for P and $\operatorname{int}_u = \operatorname{int}$. Next, it is to be observed that open sets are exactly the "fixed elements" of the interior operation; stated in other words:

14 A.12. *A subset X is open if and only if* $\operatorname{int} X = X$.

If u is a closure for P and int_u is the corresponding interior operation, then $\mathbf{D}u = \mathbf{D} \operatorname{int}_u$ and $\operatorname{int}_u X = P - u(P - X)$, $uX = P - \operatorname{int}_u (P - X)$. Thus the closure of a space is uniquely determined by the interior operation of the space (and of course, the interior operation is uniquely determined by the closure). In consequence, the relation

$$\{u \to \operatorname{int}_u \mid u \in \mathbf{C}\}$$

is a one-to-one relation ranging on the class of all interior operations, and every notion based upon the notion of a closure operation can be described in terms of interior operations. Closely related to the interior operation of a space P is the notion of a neighborhood of a subset of P. The next subsection is devoted to an examination of neigborhoods and the description of a space in terms of neighborhoods.

B. NEIGHBORHOODS

14 B.1. Definition. A *neighborhood of a subset* X of a space P is any subset U of P containing X in its interior. Thus U is a neighborhood of X if and only if $X \subset P - \overline{P - U}$. By a *neighborhood of a point* x of P we mean a neighborhood of the one-point set (x). The *neighborhood system of a set* $X \subset P$ (a point $x \in P$) in the space P is the collection of all neighborhoods of the set X (the point x).

14 B.2. *Let P be a space. A subset U of P is a neighborhood of a subset X of P if and only if U is a neighborhood of each point of X. A subset U of P is open if and only if it is a neighborhood of all of its points, or equivalently, it is a neighborhood of itself.*

Proof. The first assertion is obvious and the second one is a restatement of 14 A.12 in view of the first statement.

From 14 A.11 and the definition of neighborhood one obtains at once the following result:

14 B.3. Theorem. *Let $\mathscr{U}$ be the neighborhood system of a subset X of a space P. Then $\mathscr{U}$ is a filter on P (see* 12 B.2*) the intersection of which contains X, i.e. every element of $\mathscr{U}$ contains X, finite intersections of members of $\mathscr{U}$ belong to $\mathscr{U}$ and if $P \supset V \supset U \in \mathscr{U}$ then $V \in \mathscr{U}$.*

14 B.4. Definition. Let P be a space. By 14 B.3 the neighborhood system of a set X (a point x) in P is a filter. A base or a sub-base of this filter is called, respectively, a *base* or a *sub-base of the neighborhood system of X in P*. Thus a collection $\mathscr{V}$ of subsets of P is a base of the neighborhood system of a set X (point x) if and only if each $V \in \mathscr{V}$ is a neighborhood of X (of x) and every neighborhood of X (of x) contains a $V \in \mathscr{V}$. A collection $\mathscr{W}$ of subsets of P is a sub-base of the neighborhood system of a set X (a point x) if and only if the collection of all finite intersections of elements of $\mathscr{W}$ is a base of the neighborhood system of X (of x). The terms a *local base at x* and a *local sub-base at x* will often be used instead of a base and a sub-base of the neighborhood system of the point x.

By 14 B.3 a local base at a point x is a base of a filter whose intersection contains x. There follows:

14 B.5. *If $\mathscr{U}(x)$ is a local base at a point x, then the following assertions are true:*

(nbd 1) $\mathscr{U}(x) \neq \emptyset$.

(nbd 2) *For each $U \in \mathscr{U}(x)$, $x \in U$.*

(nbd 3) *For each U_1 and U_2 in $\mathscr{U}(x)$ there exists a U in $\mathscr{U}(x)$ with $U \subset U_1 \cap U_2$.*

The following simple but very important theorem shows that the closure of a set is completely determined by neighborhoods of points of the space.

14 B.6. Theorem. *A point $x \in P$ belongs to the closure of a subset X of a space $\langle P, u \rangle$ if and only if each neighborhood of x in $\langle P, u \rangle$ intersects X.*

Proof. If a neighborhood U of x does not meet X, then $x \in \operatorname{int} U \subset \operatorname{int}(P - X) = (P - uX)$, which shows that $x \notin uX$. Conversely, if $x \notin uX$, then $P - X$ is a neighborhood of x which does not meet X.

14 B.7. Corollary. *If $\mathscr{U}$ is a local base at a point x in a space P, then $x \in \overline{X}$ if and only if $X \subset P$ and each $U \in \mathscr{U}$ intersects X.*

Proof. If $x \in \overline{X}$, then by 14 B.6 each neighborhood of x, and hence each member of $\mathscr{U}$ intersects X. Conversely, if each member of $\mathscr{U}$ intersects X, then obviously each neighborhood of x intersects X, and by 14 B.6 we obtain $x \in \overline{X}$.

14 B.8. *Let u and v be two closure operations for a set P. In order that u should be coarser than v it is necessary and sufficient that, for each $x \in P$, every u-neighborhood of x be a v-neighborhood of x.*

Proof. If u is coarser than v, then $x \in \operatorname{int}_u U = (P - u(P - U))$ implies $x \in \in (P - v(P - U)) = \operatorname{int}_v U$ because $u(P - U) \supset v(P - U)$. It follows that every u-neighborhood of a point x is a v-neighborhood of the point x. Conversely, if every u-neighborhood of a point x is a v-neighborhood of the point x, then by Theorem 14 B.6, $x \in vX$ implies $x \in uX$ for each $X \subset P$, that is, $uX \supset vX$ for each $X \subset P$, which means that u is coarser than v.

We must keep in mind that, in general, there exist many local bases at a point x. Indeed, if $\mathscr{U}$ is a local base at x and U is any element of $\mathscr{U}$, then the collection of all $V \in \mathscr{U}$, $V \subset U$, is also a local base at x. Therefore it will be useful to state the following corollary of the foregoing result.

14 B.9. Corollary. *Let u and v be two closure operations for a set P. For each x in P let $\mathscr{U}(x)$ and $\mathscr{V}(x)$ be local bases at x in $\langle P, u\rangle$ and $\langle P, v\rangle$ respectively. Then u is coarser than v if and only if, for each $x \in P$, every element of $\mathscr{U}(x)$ contains an element of $\mathscr{V}(x)$. In particular, $u = v$ if and only if, for each x in P, every $U \in \mathscr{U}(x)$ contains a $V \in \mathscr{V}(x)$ and every $V \in \mathscr{V}(x)$ contains a $U \in \mathscr{U}(x)$, or equivalently, $\mathscr{U}(x)$ and $\mathscr{V}(x)$ are bases of the same filter on P.*

Remark. A collection $\mathscr{B}$ of subsets of a space P is sometimes said to contain arbitrarily small neighborhoods of a point x if $\mathscr{B}$ contains a local base at x.

The foregoing corollary 14 B.9 asserts that every closure operation is completely determined by local bases at all points. It is sometimes convenient to define a closure operation for a set P by specifying which subsets of P are neighborhoods of what points, or, stated in other words, which filters on P are neighborhood systems of points, or, which filter bases are local bases at points. Now we shall prove that if $\{\mathscr{U}(x) \mid x \in P\}$ is any family of filter bases on P such that $x \in \bigcap \mathscr{U}(x)$ for each x, then there exists a closure operation u for P such that, for each $x \in P$, $\mathscr{U}(x)$ is a local base at x in $\langle P, u\rangle$.

14 B.10. Theorem. *For each element x of a set P let $\mathscr{U}(x)$ be a collection of subsets of P satisfying conditions* (nbd 1), (nbd 2) *and* (nbd 3) *of proposition* 14 B.5. *Then there exists exactly one closure operation u for P such that, for each $x \in P$, $\mathscr{U}(x)$ is a local base at x in $\langle P, u\rangle$.*

Proof. I. If there exists such a closure operation u for P, then by 14 B.7

$$(*)\ uX = \mathsf{E}\{x \mid x \in P, U \in \mathscr{U}(x) \Rightarrow U \cap X \neq \emptyset\}$$

for each $X \subset P$. Thus there exists at most one closure operation u on P such that the $\mathscr{U}(x)$ are local bases. We must prove that the relation u defined by $(*)$ is a closure operation on P and $\mathscr{U}(x)$ are u-local bases. – II. Clearly (cl 1) is fulfilled. The axiom (cl 2) follows from (nbd 2). To prove (cl 3), let X and Y be subsets of P. If $x \in (uX \cup uY)$, then by $(*)$ each set from $\mathscr{U}(x)$ meets X or Y, and consequently, $X \cup Y$. By $(*)$ we have $x \in u(X \cup Y)$ which proves $(uX \cup uY) \subset u(X \cup Y)$. To prove the converse inclusion, let $x \notin (uX \cup uY)$. By $(*)$ there exist U and V in $\mathscr{U}(x)$ with $U \cap X = \emptyset = V \cap Y$. By (nbd 3) there exists a W in $\mathscr{U}(x)$ contained in $V \cap U$. Clearly

$W \cap (X \cup Y) = \emptyset$. By $(*)$ $x \notin u(X \cup Y)$ which proves $u(X \cup Y) \subset (uX \cup uY)$. We have proved that u is a closure operation. – III. If $x \in P$ and $U \in \mathcal{U}(x)$, then U is a u-neighborhood of the point x for, otherwise, $x \in u(P - U)$ and by $(*)$ $V \cap \cap (P - U) \neq \emptyset$ for each $V \in \mathcal{U}(x)$, which is not correct for $V = U$. It remains to prove that every u-neighborhood W of x contains a $U \in \mathcal{U}(x)$. Let us suppose $U - W = U \cap (P - W) \neq \emptyset$ for each $U \in \mathcal{U}(x)$. Then by $(*)$ $x \in u(P - W)$, which shows that W is not a neighborhood of x.

14 B.11. Corollaries. (a) *For each element x of a set P let $\mathcal{U}(x)$ be a filter on P such that $x \in \bigcap \mathcal{U}(x)$. Then there exists exactly one closure u for P such that $\mathcal{U}(x)$ is the neighborhood system at x in $\langle P, u \rangle$ for each x in P.*

(b) *For each element x of a set P let $\mathcal{U}(x)$ be a non-void family of subsets of P such that $x \in \bigcap \mathcal{U}(x)$. Then there exists exactly one closure u for P such that $\mathcal{U}(x)$ is a local sub-base at x in $\langle P, u \rangle$ for each x in P.*

Theorem 14 B.10 and its corollaries 14 B.11 (a) and (b) will be used in the definition of the space of ultrafilters of a given set P and the generalized order closures for ordered sets, especially for the ordered set of reals. The set of all ultrafilters on a given set was considered in 12.

14 B.12. Ultrafilter space (associated with a given set X). Suppose that X is a set and βX is the set consisting of all elements of X and all free ultrafilters on X. We shall define a family $\{\mathcal{U}_x \mid x \in \beta X\}$ such that $\mathcal{U}_x$ is a filter base on βX and $x \in \bigcap \mathcal{U}_x$ for each x. If $x \in X$, then we put $\mathcal{U}_x = ((x))$. If x is a free ultrafilter on X, then

$(*)$ $\mathcal{U}_x = \mathbf{E}\{Y \cup \mathbf{E}\{y \mid y \in (\beta X - X), Y \in y\} \mid Y \in x\}$.

Obviously $x \in \bigcap \mathcal{U}_x$ for each x. By virtue of 14 B.10 there exists exactly one closure for βX such that $\mathcal{U}_x$ is a local base at x for each x in βX. The set βX endowed with this closure operation will be called the *ultrafilter space associated with* X and it will also be denoted by βX. If X is finite, then evidently βX is defined to be the discrete space with underlying set X. In the following development let X be a fixed infinite set and $\mathcal{U}_x$ be the collection given by $(*)$.

(a) If $x \in (\beta X - X)$, then $\mathcal{U}_x = \mathbf{E}\{\overline{Y} \mid Y \in x\}$ and $[\mathcal{U}_x] \cap X = x$.

Proof. Fix an x in $\beta X - X$. Obviously, if $Y \in x$ then

$$\overline{Y} = Y \cup \mathbf{E}\{y \mid y \in (\beta X - X), Y \in y\}$$

Comparing this formula with $(*)$ we obtain the first formula. The second one is self-evident.

(b) If $\{Y_i \mid i \leqq n\}$ is a disjoint finite cover of X, then $\{\overline{Y}_i\}$ is a disjoint cover of βX.

Proof. I. If $x \in \beta X - X$, then $Y_i \in x$ for some i because x is an ultrafilter on X and $\{Y_i\}$ is a finite cover of X (see 12 C.8). It follows that $\{\overline{Y}_i\}$ is a cover of βX. – II. If $x \in \overline{Y}_i \cap \overline{Y}_j$ with $i \neq j$, then $x \in (\beta X - X)$ because $X \cap \overline{Y}_i = Y_i$, $X \cap \overline{Y}_j = Y_j$

and $Y_i \cap Y_j = \emptyset$. Thus x is an ultrafilter on X and, by definition, Y_i as well as Y_j belong to x. But this is also impossible because $Y_i \cap Y_j = \emptyset$.

(c) If $Y \subset X$, then $\overline{Y}$ is simultaneously open and closed in βX.

Proof. It is obvious that $\overline{Y}$ is a neighborhood of each of its points and hence $\overline{Y}$ is open. The collection $(Y, X - Y)$ is a disjoint cover of X and consequently, $(\overline{Y}, \overline{X - Y})$ is a disjoint cover of βX by (b). Since $\overline{X - Y}$ is open, as was just proved, its complement $\overline{Y}$ in βX is closed.

14 B.13. Generalized order closures for monotone ordered sets. Suppose that $\langle P, \leqq \rangle$ is a monotone ordered set. Intervals of the form $]\!]\, \alpha, \beta \,[\![$, $]\!]\, \leftarrow, \beta \,[\![$, $]\!]\, \alpha, \rightarrow \,[\![$ and $]\!]\, \leftarrow, \rightarrow \,[\![$ $(= P)$ will be called *order-open*. Evidently, the collection of all order-open intervals containing a given point x is multiplicative and non-void, and thus it is a filter base. By virtue of **14 B.10** there exists exactly one closure operation u for P such that, for each x in P, the collection of all order-open intervals containing x is a local base at x in $\langle P, u \rangle$. This closure will be called the *order closure for* $\langle P, \leqq \rangle$. More generally, a closure v for P will be called a *generalized order closure for* $\langle P, \leqq \rangle$ if, for each x in P, there exists a local base at x consisting of intervals and containing all order-open intervals containing x.

(a) If v is a generalized order closure for $\langle P, \leqq \rangle$, then every order-open interval is open in $\langle P, v \rangle$. Indeed, any order-open interval is a neighborhood in $\langle P, v \rangle$ of all of its points (**14 B.2**).

(b) The order closure u for $\langle P, \leqq \rangle$ is the coarsest generalized order closure for $\langle P, \leqq \rangle$. By definition the order closure u is a generalized order closure for P. If v is any generalized order closure for $\langle P, \leqq \rangle$, then v is finer than u by **14 B.9**, because, for each x, the collection $\mathscr{U}_x$ of all order-open intervals containing x is a local base at x in $\langle P, u \rangle$ and simultaneously $\mathscr{U}_x$ is contained in the neighborhood system at x in $\langle P, v \rangle$.

(c) Let v be a generalized order closure for $\langle P, \leqq \rangle$ and let $x \in P$. Only the following four cases are possible:

(α) Order-open intervals containing x form a local base at x in $\langle P, v \rangle$;

(β) intervals of the form $[\![\, x, \beta \,[\![$, $x < \beta$, form a local base at x in $\langle P, v \rangle$;

(γ) intervals of the form $]\!]\, \alpha, x \,]\!]$, $\alpha < x$, form a local base at x in $\langle P, v \rangle$; and

(δ) the collection $([\![\, x, x \,]\!])$ is a local base at x in $\langle P, v \rangle$.

Of course none of these possibilities excludes any of the other ones. For example, if x is a point of P such that $]\!]\, \alpha, \beta \,[\![= (x)$ for some α and β, and u is a generalized order closure for $\langle P, \leqq \rangle$, then all cases (α) – (δ) actually appear. On the other hand if $]\!]\, \alpha, x \,]\!] \neq (x)$ for each $\alpha < x$ and $[\![\, x, \beta \,[\![\neq (x)$ for each $x < \beta$, then each case excludes each of the other ones. The simple proof of these statements is left to the reader. If (α) is fulfilled for each x, then the closure u is the order closure for $\langle P, \leqq \rangle$. If condition (β) (condition (γ)) is fulfilled for each x, then u is said to be the *closure of the approximation from the right* (*from the left*). In accordance with this terminology the order closure is sometimes called the *closure of two-sided approximation*.

If condition (δ) is fulfilled for each x in P, then evidently u is the discrete closure for P. It is to be noted that all foregoing definitions are meaningful, with slight modifications, for ordered sets which are not necessarily monotone. Nevertheless, the resulting closures are not too significant for general ordered sets and therefore we want to employ the term "order closure" for a more significant situation.

(d) A generalized order closure v for an ordered set $\langle P, \leqq\rangle$ is the order closure for $\langle P, \leqq\rangle$ if and only if every open interval (i.e. every interval in $\langle P, \leqq\rangle$which is open in $\langle P, v\rangle$) is order-open.

14 B.14. Definition. The set R of reals endowed with the order closure will be called the *space of reals* and will be denoted by R.

By 14 A.7 every finite family in a closure space is closure-preserving. Now we will introduce a wide class of closure-preserving families containing all finite families.

14 B.15. Definition. A family $\{X_a \mid a \in A\}$ of subsets of a space P is called *locally finite* if each point x of P possesses a neighborhood intersecting a finite number of X_a only, i.e. for each x in P there exists a neighborhood U of x such that the set $\mathsf{E}\{a \mid a \in A, X_a \cap U \neq \emptyset\}$ is finite.

As a straightforward consequence of the definition we obtain:

14 B.16. *If $\{X_a \mid a \in A\}$ is a locally finite family in a space P, $B \subset A$ and $Y_a \subset X_a$ for each a in B, then the family $\{Y_a \mid a \in B\}$ is also locally finite. In particular, every subfamily of a locally finite family is locally finite.*

14 B.17. Theorem. *Every locally finite family is closure-preserving.*

Proof. Since any subfamily of a locally finite family is locally finite, it suffices to show that if $\{X_a \mid a \in A\}$ is a locally finite family in a space P, then $\overline{\bigcup\{X_a \mid a \in A\}} = \bigcup\{\overline{X}_a \mid a \in A\}$. The inclusion $\supset$ follows from proposition 14 A.8 asserting that the closure considered as a mapping of $\langle \exp P, \subset\rangle$ into itself is order-preserving. To prove the converse inclusion, let us choose a point x in the set on the left side. The family being locally finite, we can choose a neighborhood U of x such that the set A_1 of all $a \in A$ for which $U \cap X_a \neq \emptyset$ is finite. In consequence, x belongs to the closure of the family $\{X_a \mid a \in A_1\}$. But $\{X_a \mid a \in A_1\}$ is finite and hence closure-preserving. Thus $x \in \overline{X}_a$ for some $a \in A_1$, which establishes that x belongs to the right side of the above equality and completes the proof.

Of course, a closure-preserving family need not be locally finite. For example, if P is a space and A is any set, then $\{X_a \mid a \in A\}$ is closure-preserving, when $X_a = P$ for each a in A. This family is not locally finite, not even point-finite, provided that A is infinite. By 14 B.16, if $\{X_a\}$ is locally finite and $Y_a \subset X_a$ for each a, then also $\{Y_a\}$ is locally finite. The similar assertion for closure-preserving families is not true. This follows from the following observation: if $\{Y_a\}$ is any family in a space P and if $X_a = P$ for each a, then $\{X_a\}$ is closure-preserving as we have noted above, and obviously $Y_a \subset X_a$ for each a. To clarify the relationship between closure-preserving and locally finite families we shall prove the following theorem.

14 B.18. Theorem. *The following two conditions are necessary and sufficient for a family $\{X_a\}$ of subsets of a space P to be locally finite:*

(a) *$\{X_a\}$ is closure-preserving;*

(b) *$\{\overline{X}_a\}$ is point-finite.*

Proof. By the foregoing theorem 14 B.17 every locally finite family is closure-preserving. Evidently, if $\{X_a\}$ is locally finite then $\{\overline{X}_a\}$ is point-finite. Thus conditions (a) and (b) are necessary. Conversely, let $\{X_a \mid a \in A\}$ be a family of subsets of a space P such that (a) and (b) are fulfilled. Choose a point $x \in P$. We must find a neighborhood U of x intersecting only a finite number of X_a. Put

$$Y = \bigcup\{X_a \mid a \in A, x \notin \overline{X}_a\}.$$

According to (a) the point x does not belong to $\overline{Y}$ and hence $U = P - Y$ is a neighborhood of x. According to (b) the set $A_1 = \mathbf{E}\{a \mid a \in A, x \in \overline{X}_a\}$ is finite. It follows that U intersects only a finite number of X_a, namely those X_a for which $a \in A_1$.

It is to be observed that condition (b) cannot be replaced by the weaker condition "$\{X_a\}$ is point-finite". Indeed, if $\langle P, u\rangle$ is the space from 14 A.5 (e) such that P is infinite, then the family $\{(y) \mid y \in P - (x)\}$ is point-finite as well as closure-preserving, but not locally finite.

14 B.19. Corollary of 14 B.18. *The union of any locally finite family of closed sets is a closed set. In other words, the collection of all closed subsets of a space is closed under locally finite unions.*

In conclusion we shall prove a property of open sets, the usefulness of which will become clear later.

14 B.20. Theorem. *If X and Y are subsets of a space P and U is a neighborhood of X, then*

(*) $X \cap \overline{Y} = X \cap \overline{Y \cap U}$.

In particular, if U is an open subset of P, then

(**) $U \cap \overline{Y} = U \cap \overline{Y \cap U}$.

Proof. By proposition 14 A.8 asserting that every closure operation is order-preserving, the inclusion $\supset$ (in (*)) holds. Let us suppose that there exists a point x in $(X \cap \overline{Y}) - (X \cap \overline{Y \cap U})$. Since $Y \subset (Y \cap U) \cup (P - U)$, we have $\overline{Y} \subset \overline{Y \cap U} \cup \overline{P - U}$. It follows that $x \in \overline{P - U}$, which contradicts the fact that U is a neighborhood of X. The second assertion is a consequence of the first one. Indeed, if U is open, then U is a neighborhood of itself (by 14 B.2) and we may put $X = U$ in (*).

C. CLUSTER POINTS

14 C.1. Definition. A *cluster point* or an *accumulation point of a set X* in a space P is a point x belonging to the closure of $X - (x)$. A *cluster point* or *an accumulation point of a space P* is defined to be a cluster point of the underlying set of P. The set of all cluster points of a set X is denoted by X' and called the *derivative of* X in P.

14 C.2. In any space P the following assertions are true:
(a) $X \subset P$ implies $\overline{X} = X \cup X'$, $(\overline{X} - X) \subset X'$;
(b) $\emptyset' = \emptyset$;
(c) $X \subset P$, $Y \subset P$ imply $(X \cup Y)' = X' \cup Y'$;
(d) $X \subset Y \subset P$ implies $X' \subset Y'$.

Proof. I. Clearly $X' \subset \overline{X}$, and if $x \in (\overline{X} - X)$ then $x \in X'$. Thus (a) is true. — II. The statements (b) and (d) are evident. — III. Since $X \subset (X \cup Y)$, $Y \subset (X \cup Y)$, we obtain from (d) that $X' \subset (X \cup Y)'$, $Y' \subset (X \cup Y)'$ and hence $(X' \cup Y') \subset \subset (X \cup Y)'$. If $x \in (X \cup Y)'$, i.e. $x \in \overline{(X \cup Y) - (x)}$, then $x \in \overline{X - (x)} \cup \overline{Y - (x)}$ by (cl 3), and consequently, $x \in X'$ or $x \in Y'$, i.e. $x \in (X' \cup Y')$ which completes the proof.

It follows from 14 C.2 (a) that the closure of a space is a uniquely determined by the relation $\{X \to X' \mid X \subset P\}$. In consequence, every notion based upon the closure operation can be described in terms of cluster points. One can show that conditions (b) and (c) are almost characteristic for the relation $\{X \to X'\}$ of a closure space. More precisely, if $\{X \to X'\}$ is a single-valued relation on $\exp P$ ranging in $\exp P$, satisfying conditions (b) and (c), and such that $x \notin (x)'$ for each $x \in P$, then there exists exactly one closure u for P such that X' is the derivative of X in the space $\langle P, u \rangle$.

14 C.3. Definition. An *isolated point of a set* X in a space P is a point x of X which is not a cluster point of X. *Isolated points of a space* P are defined to be the isolated points of the underlying set of P.

14 C.4. The closure of a subset X of a space P is the disjoint union of the set of all cluster points of X and the set of all isolated points of X. — Obvious.

14 C.5. A point x is an isolated point of a subset X of a space P if and only if there exists a neighborhood U of x in P so that $U \cap X = (x)$. A point x is an isolated point of a space P if and only if (x) is an open subset of P.

Proof. The second statement is a corollary of the first one. We shall prove the first statement. If $U \cap X = (x)$ for some neighborhood of x, then $U \cap (X - (x)) = = \emptyset$ and hence $x \notin \overline{X - (x)}$, which means that x is not a cluster point of X, i.e. x is an isolated point of X. Conversely, if $x \notin \overline{X - (x)}$, then the complement U of the set $\overline{X - (x)}$ is a neighborhood of x and clearly $U \cap X = (x)$.

Corollary. *A space P is discrete if and only if each of its points is isolated.*

14 C.6. Cluster points in ordered spaces. Suppose that $\langle P, \leqq \rangle$ is a monotone ordered set and u is a generalized order closure for $\langle P, \leqq \rangle$. By 14 C.5 $x \in P$ is an isolated point of $\langle P, u \rangle$ if and only if $[\![x, x]\!] = (x)$ is a neighborhood of x. A point x of P will be called *isolated from the left* (*from the right*) or simply *left-*

isolated (*right-isolated*), if the interval $[\![x, \rightarrow [\![$ ($]\!] \leftarrow, x]\!]$, respectively) is a neighborhood of x in $\langle P, u \rangle$. Obviously:

x is isolated from the left in $\langle P, u \rangle$ if and only if $x \notin u\ \mathbf{E}\{y \mid y < x\}$, and x is isolated from the right if and only if $x \notin u\ \mathbf{E}\{y \mid x < y\}$.

It is to be noted that left-isolated and right-isolated points were defined for a generalized order closure for a monotone ordered set $\langle P, \leqq \rangle$, and they depend essentially upon both u and $\leqq$. For example, if u is a generalized order-closure for $\langle P, \leqq \rangle$ then u is also a generalized order closure for $\langle P, \leqq^{-1} \rangle$, and clearly x is left-isolated in $\langle P, u \rangle$ relative to $\leqq$ if and only if x is right-isolated in $\langle P, u \rangle$ relative to $\leqq^{-1}$. There are many notions depending on both $\leqq$ and u. Therefore it is convenient to introduce the following definition.

14 C.7. Definition. An *ordered space* (a *generalized ordered space*) is a struct $\langle P, \leqq, u \rangle$ such that $\langle P, \leqq \rangle$ is a monotone ordered set and u is the order closure (a generalized order closure) for $\langle P, \leqq \rangle$. It will be convenient to employ the term generalized ordered space also for an underlying space of a generalized ordered space.

14 C.8. The ordered space of all ordinals $\beta < \xi$ will be denoted by T_ξ. The spaces T_ξ have several special properties, in particular if ξ is an initial ordinal, and they will often be used as counter-examples. For any ordinal α we denote by ω_α the initial ordinal η (see 11 B.6, remark 2) such that card $\mathrm{Ord}_\eta = \aleph_\alpha$.

14 C.9. Definition. A *regular open set* of a closure space P is an open subset U of P such that $U = \mathrm{int}\,\overline{U}$. A *regular closed set* is a closed set X such that $X = \overline{\mathrm{int}\,X}$.

14 C.10. *Let P be a space. A subset U is regular open if and only if its complement $P - U$ is regular closed. The collection of all regular open (regular closed) subsets of P is multiplicative (additive) and contains P and $\emptyset$.*

Proof. It is easily seen that $U = \mathrm{int}\,\overline{U}$ if and only if $(P - U) = \overline{\mathrm{int}\,(P - U)}$ which proves the first statement. According to the de Morgan formulae it remains to prove, for instance, the statements concerning regular closed sets. Obviously P and $\emptyset$ are closed, and $\overline{\mathrm{int}\,P} = \overline{P} = P$, $\overline{\mathrm{int}\,\emptyset} = \overline{\emptyset} = \emptyset$, which show that both P and $\emptyset$ are regular closed. If X and Y are regular closed, then $X \cup Y$ is closed by 14 A.9 and

$$\overline{\mathrm{int}\,(X \cup Y)} \supset \overline{\mathrm{int}\,X \cup \mathrm{int}\,Y} = \overline{\mathrm{int}\,X} \cup \overline{\mathrm{int}\,Y} = X \cup Y$$

which shows that $\overline{\mathrm{int}\,(X \cup Y)} = X \cup Y$ and completes the proof.

The union of two regular open sets need not be regular open. Indeed, in every generalized ordered space every order-open interval is regular open, but the union of two order-open intervals need not be regular open. For example, $X = (]\!] 0, 1 [\![\cup]\!] 1, 2 [\![)$ is not regular open in the space R of reals, because the closure of X is $[\![0, 2]\!]$ and the interior of $[\![0, 2]\!]$ is $]\!] 0, 2 [\![\neq X$.

15. TOPOLOGICAL SPACES

In this section an extensive and important class of closure spaces, called topological spaces (Definition 15 A.1), will be introduced and studied. A closure operation is not uniquely determined by the collection of all open sets while two topological closures coincide provided that the corresponding collections of open sets coincide. Thus a topological closure for a set P is uniquely determined by a subset of exp P. Here we shall show that generalized ordered spaces are topological and later we shall show that also metrizable spaces (18 A.12) and uniformizable spaces (= completely regular spaces) (24 A.2) are topological and that every topological group is a topological space (19 B.4).

The second part is devoted to a preliminary exposition of the convergence of nets. It will be shown that topological closures are characterized (among closure spaces) by the condition on iterated limits (15 B.13).

A. TOPOLOGICAL CLOSURES

15 A.1. Definition. A *topological closure operation* (or simply a *topological closure* or merely a T-closure) for a set P is a closure operation u for P satisfying the following condition:

(cl 4) For each $X \subset P$, $uuX = uX$.

A closure space P is said to be *topological* (or simply a T-space) if the closure of P is topological.

Stated in other words, a closure u is topological if and only if u is a transitive relation. Next, a closure u is topological if and only if $u \circ u = u$, i.e. u is an "idempotent element" relative to the internal composition $\circ$.

Notice that the discrete and the accrete closures for a set are topological, and so are all closures from 14 A.5 with the exception of those in (c) and (f).

15 A.2. Theorem. *Each of the following four conditions is necessary and sufficient for a closure space P to be a topological space.*

(a) *The closure of each subset of P is closed in P;*

(b) *The interior of each subset of P is open in P;*

(c) *For each* $x \in P$ *the collection of all open neighborhoods of* x *is a local base at* x;

(d) *For each* x *in* P, *if* U *is a neighborhood of* x, *then there exists a neighborhood* V *of* x *such that* U *is a neighborhood of each point of* V. *Stated in other words, every neighborhood of any point* $x \in P$ *is a neighborhood of a neighborhood of* x.

Proof. The pattern of the proof is (a) ⇒ (b) ⇒ (c) ⇒ (d), (a) is necessary and (d) is sufficient. Condition (a) is a restatement of condition (cl 4). In particular, (a) is necessary. If the closure of every subset of P is closed, then the interior of any set X is open as the complement of the closed set $\overline{P - X}$ (by definition 14 A.10, int $X = P - \overline{P - X}$). Thus (a) ⇒ (b). Now assume (b) and let us consider a neighborhood U of a point x of P. By (b) int U is open and by 14 B.2 int U is a neighborhood of each of its points, in particular, of x. Since obviously int $U \subset U$, condition (c) is fulfilled. Thus (b) implies (c). The implication (c) ⇒ (d) is obvious because an open set is a neighborhood of itself (pick an open V such that $x \in V \subset U$). It remains to prove that (d) is a sufficient condition. Suppose $x \in uuX$. We must prove $x \in uX$. By 14 B.6 it is sufficient to show that every neighborhood U of x intersects X. By condition (d) we can choose a neighborhood V of x such that U is a neighborhood of each point of V. Since $x \in uuX$, by 14 B.6 V intersects uX, and therefore we can choose a point y in $V \cap uX$. Since $y \in uX$ and U is a neighborhood of y (U is a neighborhood of each $z \in V$), again by 14 B.6 U intersects X.

15 A.3. Corollary. *Each of the following two conditions is necessary and sufficient for a closure space* P *to be a topological space*:

(a) *the closure of any subset* X *of* P *is the intersection of all closed sets containing* X;

(b) *the interior of any subset* X *of* P *is the union of all open subsets contained in* X.

For further characterizations of topological spaces see the exercises.

According to 15 A.3 a topological closure operation is uniquely determined by the collection of all open sets and also by the collection of all closed sets. This fact enables us to define a topological closure operation on a set P by declaring a suitable subcollection of exp P to be the collection of all open or closed sets (Theorem 15 A.6, 15 A.10). It will be seen that it is sufficient to specify suitable subcollections of open or closed sets, the so-called bases and sub-bases (Theorems 15 A.9, 15 A.13).

We begin with an adaptation of Theorem 14 B.10 to topological closures.

15 A.4. Theorem. *Let* P *be a set and for each* x *in* P *let* $\mathcal{U}(x)$ *be a collection of subsets of* P *satisfying the conditions* (nbd 1), (nbd 2) *and* (nbd 3) *of* 14 B.5 *and also the following condition*:

(nbd 4) *For each* x *in* P *and each* U *in* $\mathcal{U}(x)$ *there exists a* V *in* $\mathcal{U}(x)$ *such that for each* y *in* V, *some* $W \in \mathcal{U}(y)$ *is contained in* U.

Then there exists exactly one closure operation u *on* P *such that* $\mathcal{U}(x)$ *is a local base at* x *in* $\langle P, u \rangle$ *for each* $x \in P$. *The closure operation* u *is topological.*

Proof. According to Theorem 14 B.10 there exists exactly one closure operation u on P such that the collections $\mathscr{U}(x)$ are local bases (condition (nbd 4) is not used). It remains to prove that u is a topological closure; this follows from Theorem 15 A.2 because condition (nbd 4) implies condition (d).

15 A.5. Theorem. *Let $\mathscr{B}$ be a collection of subsets of a set P such that the following two conditions are fulfilled:*

(ob 1) *The union of $\mathscr{B}$ is P;*

(ob 2) *For each B_1 and B_2 in $\mathscr{B}$ and for each x in $B_1 \cap B_2$ there exists a B in $\mathscr{B}$ with $x \in B \subset B_1 \cap B_2$.*

Then there exists exactly one closure operation u on P such that, for each x in P, the collection $\mathscr{B}(x) = \mathbf{E}\{B \mid x \in B \in \mathscr{B}\}$ is a local base at x in $\langle P, u\rangle$. This closure is a topological closure operation, and a subset U of $\langle P, u\rangle$ is open if and only if U is the union of a subcollection of $\mathscr{B}$. In particular, all sets of $\mathscr{B}$ are open in $\langle P, u\rangle$.

Proof. Obviously the collections $\mathscr{B}(x)$ fulfil conditions (nbd 1) – (nbd 4). By Theorem 15 A.4 there exists exactly one closure u on P such that the collections $\mathscr{B}(x)$ are local bases and this closure is topological. Next, every $B \in \mathscr{B}$ is open in $\langle P, u\rangle$ because it is a neighborhood of each of its points. Indeed, if $x \in B$ then $B \in \mathscr{B}(x)$. Since any union of open sets is an open set, we have proved that unions of sets from $\mathscr{B}$ are open. Finally, if U is an open subset of $\langle P, u\rangle$, U is a neighborhood of each of its points, and consequently we can choose a family $\{B_x \mid x \in U\}$ so that $x \in B_x \in \mathscr{B}(x)$ and $B_x \subset U$. Obviously the union of $\{B_x\}$ is U. Thus every open subset of $\langle P, u\rangle$ is the union of a subcollection of $\mathscr{B}$, concluding the proof.

By 14 A.9 the collection $\mathscr{O}$ of all open subsets of a closure spaces $\langle P, u\rangle$ fulfils the following three conditions:

(o 1) The set P belongs to $\mathscr{O}$;

(o 2) The union of any subcollection of $\mathscr{O}$ belongs to $\mathscr{O}$;

(o 3) The intersection of any two members of $\mathscr{O}$ belongs to $\mathscr{O}$.

It is to be observed that by (o 2) the empty set belongs to $\mathscr{O}$, since the union of an empty collection is, by definition, the empty set. In the converse direction we shall prove the following result.

15 A.6. Theorem. *Let $\mathscr{O}$ be a collection of subsets of a set P fulfilling conditions* (o 1), (o 2) *and* (o 3). *Let C be the set of all closure operations u for P such that $\mathscr{O}$ is the collection of all open subsets of $\langle P, u\rangle$. There exists exactly one topological closure operation in C and this closure is a coarsest element in C.*

Proof. Clearly, the collection $\mathscr{O}$ fulfils conditions (ob 1) and (ob 2) of 15 A.5. By Theorem 15 A.5 there exists exactly one closure operation u on P such that, for each $x \in P$, the collection of all $U \in \mathscr{O}$ containing x is a local base at x in $\langle P, u\rangle$; again by 15 A.5 a set V is open in $\langle P, u\rangle$ if and only if V is a union of a subcollection of $\mathscr{O}$. But $\mathscr{O}$ is closed under unions by (o 2) and consequently $\mathscr{O}$ is the collection of all u-open sets. Thus $u \in C$. Moreover, Theorem 15 A.5 asserts that u is a topological closure. It remains to show that u is the coarsest closure from C; this follows from the following result which will often be needed in the sequel.

15 A.7. *A topological closure* $u \in \mathbf{C}(P)$ *is coarser than a closure* $v \in \mathbf{C}(P)$ *if and only if every u-open set is v-open, or equivalently, if every u-closed set is v-closed. In particular, two topological closures u and v for a set P are identical if and only if, for each subset X of P, the following two equivalent conditions are fulfilled:*

(a) *X is u-open* $\Leftrightarrow$ *X is v-open;*

(b) *X is u-closed* $\Leftrightarrow$ *X is v-closed.*

Proof. First suppose that a closure u is coarser than a closure v. By 14 B.8, if U is a u-neighborhood of a point x, then U is a v-neighborhood of x. Since a set is open if and only if it is a neighborhood of each of its points (by 14 B.2), it follows that every u-open set is v-open (observe that the assumption that u is topological was not needed). Conversely, let u be topological and let every u-open set be v-open. By 14 B.9, to prove that u is coarser than v it is sufficient to show that, for each point x, there exists a local base $\mathscr{U}(x)$ at x in $\langle P, u\rangle$ such that any member U of $\mathscr{U}(x)$ is a v-neighborhood of x. Since u is topological, by 15 A.2 the collection $\mathscr{U}(x)$ of all u-open sets containing x is a local base at x in $\langle P, u\rangle$. Now, if $U \in \mathscr{U}(x)$, then U is u-open, by our assumption U is also v-open, and since an open set is a neighborhood of each of its points, U is a v-neighborhood of x.

15 A.8. Definition. An *open base of a topological space* $\langle P, u\rangle$ is a collection $\mathscr{B}$ of subsets of P such that a subset U of $\langle P, u\rangle$ is open if and only if it is the union of subcollection of $\mathscr{B}$. An *open sub-base of a topological space* $\langle P, u\rangle$ is a collection $\mathscr{B}_1$ of subsets of P such that the collection $\mathscr{B}$ of all finite intersections of sets from $\mathscr{B}_1$ is an open base of $\langle P, u\rangle$.

Stated in other words, an open base of a topological space is a collection $\mathscr{B}$ of sets such that the smallest completely additive collection of sets containing $\mathscr{B}$ is identical with the collection of all open sets. It is easy to find necessary and sufficient conditions for a given family of sets to be an open base or an open sub-base of a topological space.

15 A.9. Theorem. *Let* $\mathscr{B}$ *be a collection of subsets of a set P. There exists a topological closure operation u on P such that* $\mathscr{B}$ *is an open base of* $\langle P, u\rangle$ *if and only if the family* $\mathscr{B}$ *fulfils conditions* (ob 1) *and* (ob 2) *of* 15 A.5. *In order that* $\mathscr{B}$ *be an open sub-base for a topological space* $\langle P, u\rangle$ *it is necessary and sufficient that the union of* $\mathscr{B}$ *be P.*

Proof. The assertion concerning sub-bases is a straightforward consequence of that concerning open bases which will now be proved. According to Theorem 15 A.5 the conditions (ob 1) and (ob 2) are sufficient for the collection $\mathscr{B}$ to be an open base of a topological space $\langle P, u\rangle$. Conversely, suppose that $\mathscr{B}$ is an open base of a topological space $\langle P, u\rangle$. The union of $\mathscr{B}$ is P because P is open in $\langle P, u\rangle$ (by 14 A.9). Next, suppose $x \in B_1 \cap B_2$ with $B_i \in \mathscr{B}$. Since B_1 and B_2 are open, their intersection $B_1 \cap B_2$ is also open (by 14 A.9). Since $\mathscr{B}$ is an open base, $B_1 \cap B_2$ is the union of a subcollection $\mathscr{B}_1$ of $\mathscr{B}$. Clearly some $B \in \mathscr{B}_1$ contains x.

Up to now we have restricted ourselves to open sets. Since a set is open if and only if its complement is closed, the collection of all open sets is uniquely determined by the collection of all closed sets. By 14 A.9, the collection $\mathscr{C}$ of all closed subsets of a space $\langle P, u\rangle$ fulfils the following three conditions:

(c 1) The empty set belongs to $\mathscr{C}$;

(c 2) The intersection of any subcollection of $\mathscr{C}$ belongs to $\mathscr{C}$;

(c 3) The union of any two members of $\mathscr{C}$ belongs to $\mathscr{C}$.

It is to be noted that condition (c 2) implies $P \in \mathscr{C}$. Indeed, the intersection of an empty family of subsets of P is P by our Definition 2.11.

15 A.10. Theorem. *If $\mathscr{C}$ is a collection of subsets of a set P satisfying the conditions* (c 1), (c 2) *and* (c 3), *then there exists exactly one topological closure operation u for P such that $\mathscr{C}$ is the collection of all closed sets of $\langle P, u\rangle$. For any $X \subset P$ the closure of X is the intersection of all $C \in \mathscr{C}$ containing X.*

15 A.11. Definition. A *closed base of a topological space* P is a collection $\mathscr{B}$ of subsets of P such that $X \subset P$ is closed if and only if X is the intersection of a subcollection of $\mathscr{B}$. A *closed sub-base of a topological space* P is a collection $\mathscr{B}_1$ of subsets of P such that the collection $\mathscr{B}$ of all finite unions of members of $\mathscr{B}_1$ is a closed base of P.

From the de Morgan formulae 2.16 the following result follows at once:

15 A.12. *$\mathscr{B}$ is a closed base or a closed sub-base of a topological space P if and only if the collection consisting of the complements* (*in P*) *of all $B \in \mathscr{B}$ is an open base or an open sub-base, respectively, of P.*

Combining 15 A.12 and 15 A.9 we obtain at once the following characterization of collections which are closed bases or sub-bases of a topological space (applying the de Morgan formulae).

15 A.13. Theorem. *A collection $\mathscr{B}_1$ of subsets of a set P is a closed sub-base of a topological space $\langle P, u\rangle$ if and only if the intersection of $\mathscr{B}_1$ is empty. A collection $\mathscr{B}_1$ of subsets of a set P is a closed base of a topological space $\langle P, u\rangle$ if and only if the following two conditions* (cb 1) *and* (cb 2) *are fulfilled*:

(cb 1) *The intersection of $\mathscr{B}_1$ is empty*;

(cb 2) *For each B_1 and B_2 in $\mathscr{B}_1$ and $x \notin (B_1 \cup B_2)$ there exists a B in $\mathscr{B}_1$ such that $x \notin B \supset B_1 \cup B_2$.*

The introduced notions will be illustrated upon generalized ordered spaces. We recall that generalized order closures for monotone ordered sets were introduced in 14 B.13.

15 A.14. Theorem. *Let $\langle P, \leqq\rangle$ be a monotone ordered set. The order closure u for $\langle P, \leqq\rangle$ is topological and the collection of all order-open intervals is an open base for $\langle P, u\rangle$. A closure v for P is a generalized order closure for $\langle P, \leqq\rangle$ if and only if v is topological, every order-open interval is v-open* (*i.e. v is finer* *than the order closure) and a collection of intervals forms an open base for* $\langle P, v\rangle$.

Proof. Let $\mathscr{U}$ by the collection of all order-open intervals in $\langle P, \leqq\rangle$. – I. The first statement is almost evident. By definition, $\mathscr{U}_x = \mathbf{E}\{U \mid U \in \mathscr{U}, x \in U\}$ is a local base at x in $\langle P, u\rangle$ for each x. By virtue of 15 A.5 the closure u is topological and $\mathscr{U}$ is an open base for $\langle P, u\rangle$. – II. Now let v be a generalized order closure for $\langle P, \leqq\rangle$. For each x in P let $\mathscr{V}_x$ be the set of all neighborhoods of x in $\langle P, v\rangle$ of the following form: $]\!]\, \alpha, \beta \,[\![$ or $]\!]\, \leftarrow, \rightarrow [\![$, $]\!]\, \alpha, x \,]\!]$, $[\![\, x, \beta \,[\![$ or $[\![\, x, x \,]\!] = (x)$ (see 14 B.13 (c)). Each element I of each $\mathscr{V}_x$ is open in $\langle P, v\rangle$. Indeed, it is easily seen that I is a neighborhood of each of its points. This is evident in the last case ($I = (x)$) and this has already been shown for the first case ($I = \,]\!]\, \alpha, \beta \,[\![$ or $I = \,]\!]\, \leftarrow, \rightarrow [\![$); in the remaining two cases, $I = \,]\!]\, \alpha, x \,]\!]$ or $I = [\![\, x, \beta \,[\![$, the interval I is, by definition, a neighborhood of x, and $I - (x)$ is open as an order-open interval. Now according to 15 A.2 the space $\langle P, v\rangle$ is topological (there exists a local base at x consisting of open sets for each point x). Finally, for instance the union of $\{\mathscr{V}_x \mid x \in P\}$ is an open base for $\langle P, v\rangle$. – III. Now let v be a topological closure for P such that some collection $\mathscr{V}$ of intervals containing the collection $\mathscr{U}$ of all order-open intervals is an open base for $\langle P, v\rangle$. For each x in P the collection $\mathscr{V}_x = \mathbf{E}\{V \mid x \in V \in \mathscr{V}\}$ is a local base at x in $\langle P, v\rangle$ and clearly $\mathscr{V}_x$ contains all order-open intervals containing x. By definition 14 B.13 the closure v is a generalized order closure for $\langle P, \leqq\rangle$.

Let $\langle P, \leqq, v\rangle$ be a generalized ordered space and let $\mathscr{V}$ be the collection of all open intervals of $\langle P, \leqq, v\rangle$. Since $\mathscr{V}$ is an open base of $\langle P, v\rangle$, every open subset of $\langle P, v\rangle$ is the union of a subcollection of $\mathscr{V}$. It will be shown immediately, that an open subset of $\langle P, v\rangle$ need not be the union of a disjoint subcollection of $\mathscr{V}$. On the other hand, we shall show that every open subset of $\langle P, v\rangle$ is the union of a disjoint collection of open (in $\langle P, v\rangle$) interval-like sets (in $\langle P, \leqq\rangle$). Recall that an interval-like set is a set I such that $x \in I$, $y \in I$ implies $[\![\, x, y \,]\!] \subset I$.

15 A.15. Theorem. *Let $\mathscr{W}$ be the collection of all open interval-like subsets of a generalized ordered space $\langle P, \leqq, v\rangle$. For each open subset U of $\langle P, v\rangle$ there exists (exactly one) disjoint subcollection $\mathscr{W}_0$ of $\mathscr{W}$ such that*

(a) *U is the union of $\mathscr{W}_0$; and*

(b) *if $\mathscr{W}'$ is a disjoint subcollection of $\mathscr{W}$ with union U, then $\mathscr{W}'$ is a refinement of $\mathscr{W}_0$.*

Proof. Let U be an open subset of $\langle P, v\rangle$. If U is empty then $\mathscr{W}' = (\emptyset)$ possesses the required properties. Suppose $U \neq \emptyset$. For each x in U let V_x be the union of all $W \in \mathscr{W}$ containing x and let

$$\mathscr{W}_0 = \mathbf{E}\{V_x \mid x \in U\}.$$

It is easily seen that $\mathscr{W}_0$ possesses all the required properties.

It can be easily shown that an ordered set $\langle P, \leqq\rangle$ is boundedly order-complete (i.e., every non-void bounded subset possesses a least upper bound as well as a greatest lower bound) if and only if every interval-like subset of $\langle P, \leqq\rangle$ is an interval (ex. 11).

Combining the "only if" part of this result with the foregoing Theorem 15 B.15 we obtain at once the following important result.

15 A.16. Theorem. *If v is a generalized order closure for a boundedly order-complete ordered set $\langle P, \leqq \rangle$, then every open subset of $\langle P, v \rangle$ is the union of a disjoint collection of open (in $\langle P, v \rangle$) intervals (in $\langle P, \leqq \rangle$).*

Corollary. *If u is the order closure for a boundedly order-complete ordered set $\langle P, \leqq \rangle$, then every open subset of $\langle P, u \rangle$ is the union of a disjoint collection of order-open intervals.*

Proof. By 14 B.13 every open interval in $\langle P, u \rangle$ is an order-open interval in $\langle P, \leqq \rangle$.

B. CONVERGENCE

In this subsection we begin the study of convergence of nets. Here we restrict ourselves to the definition and some elementary properties; an advanced theory will be developed in Section 35. The main results are the description of the closure-structure of a space in terms of convergent nets (15 B.4), and the very suggestive characterization of topological spaces by the condition on iterated limits (15 B.13). The reader will see that the concept of a net converging to a point is a generalization of the notion of a sequence of real numbers converging to a real number, with which the reader is surely familiar. It is to be noted that the concept of a net converging to a point is a fundamental notion in analysis.

Recall that a subset B of an ordered set $\langle A, \leqq \rangle$ is said to be right (left) cofinal if each element of A is followed (preceded) by an element of B. The intersection of two right cofinal subsets need not be a right cofinal set. We shall work with sets which intersect any right (left) cofinal set in a right (left) cofinal set, and ordered sets such that if a subset does not have the property just mentioned then its complement is right (left) cofinal.

15 B.1. Definition. A subset B of an ordered set $\langle A, \leqq \rangle$ is said to be *right (left) residual* if each element of A is followed (preceded) by an element b of B such that the fibre (inverse fibre) of $\leqq$ at b is contained in B, i.e. $\leqq [(b)] \subset B$ ($\leqq^{-1} [(b)] \subset B$, respectively). Recall that a *right (left) directed set* (cf. Definition 10 E.1) is a non-void quasi-ordered set such that all its finite subsets are right (left) bounded, i.e. a quasi-ordered set $\langle A, \leqq \rangle$ such that $A \neq \emptyset$ and $\leqq [(a)] \cap \leqq [(b)] \neq \emptyset$ ($\leqq^{-1} [(a)] \cap \leqq^{-1} [(b)] \neq \emptyset$) for each a and b in A. For brevity we shall use *directed, cofinal, residual* instead of right directed, right cofinal and right residual. Finally, we shall say that a relation ϱ *directs a set* X if $\langle X, \varrho \rangle$ is a directed set.

In order that a set B be residual in an ordered set $\langle A, \leqq \rangle$ it is necessary and sufficient that $B \cap C$ be cofinal in $\langle A, \leqq \rangle$ whenever C is cofinal in $\langle A, \leqq \rangle$. Indeed, if B is residual and C cofinal and if a is any element of A, then $\leqq [(b)] \subset B$ for some b, $a \leqq b$ because B is residual, and there exists a c in C such that $b \leqq c$ because C

is cofinal; thus $a \leqq c$ and $c \in B \cap C$ which proves that $B \cap C$ is cofinal. Conversely, if B is not residual then there exists an a in A such that $\leqq [(b)] \cap (P - B) \neq \emptyset$ for each b, $a \leqq b$, and hence the set $C = (P - \leqq [(a)]) \cup (\leqq [(a)] \cap (P - B))$ is cofinal but $B \cap C \cap (\leqq [(a)]) = \emptyset$ which shows that $B \cap C$ is not cofinal.

It is evident (and follows from the preceding statement) that each residual set is cofinal and it follows from the preceding statement that the intersection of two residual sets is a residual set. Clearly a subset containing a residual set is residual, and therefore the *collection of all residual subsets of an ordered set* $\langle A, \leqq \rangle$ *is a filter on* A *which is proper whenever* $A \neq \emptyset$.

A cofinal set need not be residual; a cofinal fibre is, however, residual.

Each of the following equivalent conditions characterize directed sets among all non-void ordered sets $\langle A, \leqq \rangle$:

(1) *Each fibre is cofinal (and hence residual).*

(2) *If* $B \subset A$ *intersects each residual subset of* $\langle A, \leqq \rangle$, *then* B *is cofinal (i.e. if* B *is not cofinal then* $A - B$ *is residual).*

(3) *If* $B \subset A$ *intersects each cofinal subset of* $\langle A, \leqq \rangle$, *then* B *is residual (i.e., if* B *is not residual then* $A - B$ *is cofinal).*

(4) *If* $\{B_i\}$ *is a finite cover of* A, *then some* B_i *is cofinal.*

Proof. Condition (1) is equivalent to the following statement: $\leqq [a] \cap \leqq [b] \neq \emptyset$ for each $a, b \in A$ (i.e. any two elements are followed by an element). Consequently, if $\langle A, \leqq \rangle$ is directed then (1) is true, and if $A \neq \emptyset$ and (1) is true then $\langle A, \leqq \rangle$ is directed. We shall prove that conditions (1)–(4) are equivalent. Clearly conditions (2) and (3) are equivalent. Assuming (1), if B intersects each residual set then B intersects each fibre (because each fibre is residual), and hence B is cofinal. Thus (1) $\Rightarrow$ (2). If (2) is true and $\{B_i\}$ is a finite cover of A such that no B_i is cofinal, then each $A - B_i$ is residual, and hence $C = \bigcap\{A - B_i\}$ is also residual. Next, clearly $C = \emptyset$ and $A \neq \emptyset$, which contradicts the fact that C is residual. Hence (2) $\Rightarrow$ (4). Finally, assuming (4) let us consider the cover $(\leqq [a], A - \leqq [a])$ of A. The set $B = (A - \leqq [a])$ is not cofinal (because $B \cap \leqq [a] = \emptyset$); by (4) the fibre $\leqq [a]$ is cofinal. Hence (4) $\Rightarrow$ (1).

We shall need the following simple results whose proof is easy and therefore left to the reader. If B is a cofinal subset of $\langle A, \leqq \rangle$ and C is residual in $\langle A, \leqq \rangle$, then $B \cap C$ is residual in $\langle B, \leqq_B \rangle$. The product of a non-void family of directed sets is a directed set, and if B is cofinal in a directed set $\langle A, \leqq \rangle$ then $\langle B, \leqq_B \rangle$ is a directed set. It is sclf-evident that the assumption "B is cofinal" is essential.

All the results just stated will be used without any reference. Now we are prepared to give fundamental definitions.

15 B.2. Definition. A *net* is a pair $\langle N, \leqq \rangle$ such that N is a non-void family and $\leqq$ is an order for the domain of N. Thus a net is a domain-structured single-valued relation and in accordance with the general convention all terminology and all conventions concerning relations apply to nets, e.g. a *net* $\langle N, \leqq \rangle$ *ranges in a struct* ξ

if N ranges in the underlying class of ξ. A net $\langle N, \leqq \rangle$ will be called *monotone* (*directed*) if the ordered set $\langle \mathbf{D}N, \leqq \rangle$ is monotone (directed). A point x is said to be a *limit point* (an *accumulation point*) of a net $\langle N, \leqq \rangle$ in a closure space $\langle P, u \rangle$ if $x \in P$, N ranges in P and the set $N^{-1}[U]$ is right residual (right cofinal) in $\langle \mathbf{D}N, \leqq \rangle$ for each neighborhood U of x in $\langle P, u \rangle$.

One can go to examples 15 B.6. We prefer to begin with the proof of fundamental results which show the significance of the concepts just introduced.

15 B.3. *Every limit point of a net (in a closure space) is an accumulation point. Let $\mathcal{N} = \langle N, \leqq \rangle$ be a net in a closure space $\langle P, u \rangle$. Consider the set X of all accumulation points of $\mathcal{N}$ in $\langle P, u \rangle$ and the intersection Y of the closures of sets $N[A]$, where A varies over all residual subsets of $\langle \mathbf{D}N, \leqq \rangle$. Then $X \subset Y$; particularly, if $\mathcal{N}$ ranges in a subset Z of P then $X \subset uZ$. If $\mathcal{N}$ is directed then $X = Y$.*

Proof. The first statement is evident. If $x \in (P - Y)$ then $x \in (P - u\,N[A])$ for some residual set A and hence $U = P - N[A]$ is a neighborhood of x. The set $B = N^{-1}[U]$ is not cofinal in $\langle \mathbf{D}N, \leqq \rangle$ because $B \cap A = \emptyset$ and A is residual. By definition x is not an accumulation point of $\mathcal{N}$. Conversely, assuming that $\mathcal{N}$ is directed and U is a neighborhood of a point x of Y, we must show that $B = N^{-1}[U]$ is cofinal in $\langle \mathbf{D}N, \leqq \rangle$. As noted above it is sufficient to prove that B intersects each residual set. But this is almost evident: if A is residual then $x \in u\,N[A]$ and hence the neighborhood U of x intersects $N[A]$ which shows that $A \cap B = N^{-1}[U \cap N[A]]$ is non-void.

Remark. Show that in general $X \neq Y$.

15 B.4. Theorem. *Each of the following conditions* (a) – (d) *is necessary and sufficient for a point x to be an element of the closure of a subset X of a space $\mathcal{P}$*:

(a) *x is a limit point of a directed net ranging in X.*
(b) *x is a limit point of a net ranging in X.*
(c) *x is an accumulation point of a directed net ranging in X.*
(d) *x is an accumulation point of a net ranging in X.*

Proof. By 15 B.3. each of the conditions is sufficient. Since condition (a) implies each of the conditions (b), (c) and (d) (obviously (a) implies (b), (c) implies (d), and (a) implies (c) by 15 B.3), it will suffice to prove that condition (a) is necessary. Assume that x belongs to the closure of X and consider the neighborhood system $\mathcal{U}$ at x. Since $\mathcal{U}$ is a proper filter on $\mathcal{P}$, $\mathcal{U}$ is directed by the inverse inclusion $\supset$. Next, since each $U \in \mathcal{U}$ intersects X, there exists a family $\{x_U \mid U \in \mathcal{U}\}$ such that $x_U \in X \cap U$ for each U. Thus $\langle \{x_U \mid U \in \mathcal{U}\}, \supset \rangle$ is a directed net ranging in X. It is almost self-evident that this net converges to x in $\mathcal{P}$. Indeed, if U is a neighborhood of x, then $U \in \mathcal{U}$, and if $V \in \mathcal{U}$, $V \subset U$, then $x_V \in V \subset U$ and hence $x_V \in U$; since the set $\langle \mathcal{U}, \supset \rangle$ is directed, the set $\mathbf{E}\{V \mid V \in \mathcal{U}, V \subset U\}$ is residual.

The foregoing theorem shows that the closure structure of a space is uniquely determined by the convergence of directed nets; more precisely, if u and v are closures and **Lim** u and **Lim** v denotes respectively the class of all pairs $\langle \mathcal{N}, x \rangle$ such that $\mathcal{N}$ is a directed net converging to x relative to u or v, then **Lim** u = **Lim** v implies $u = v$. It follows that every notion based upon closures can be described in terms of convergence of directed nets. As an example of a description of a notion based upon the notion of a closure operation we shall state the following corollary of 15 B.4. Later we shall give a description of topological closures.

15 B.5. *A point x is a cluster point of a subset Y in a closure space P if and only if x is a limit point in P of a net (directed net) ranging in $Y - (x)$. Similarly with "limit point" replaced by "accumulation point".*

Proof. We know that x is a cluster point of Y if and only if $x \in \overline{X}$, where $X = Y - (x)$. Apply 15 B.4.

As a further example one can observe that U is a neighborhood of a point x if and only if x is a limit point of no net (directed net) ranging in the complement of U. Similarly with "limit point" replaced by " accumulation point".

15 B.6. Examples. (a) If $\langle N, \leqq \rangle$ is a constant net with the (unique) value x, then $\langle N, \leqq \rangle$ converges to x in any space containing its range.

(b) If P is a discrete space, then a net $\langle N, \leqq \rangle$ in P converges to a point x of P if and only if $N_a = x$ for all a in a right residual subset of $\langle \mathbf{D}N, \leqq \rangle$.

(c) A space P is an accrete space if and only if every net in P converges to each point of P. Thus a net may possess many limit points.

(d) The notion of a convergent net appears frequently in elementary calculus. For example, the sum of a family $\{x_a \mid a \in A\}$ of real numbers is defined as the limit point, if it exists, of the net

$$\langle \{\sum \{x_a \mid a \in F\} \mid F \text{ is a finite subset of } A\}, \subset \rangle .$$

We shall return to this example in Section 19, where the sum of a family in a commutative topological group will be considered. Next, the upper and the lower Riemann integral of a function on an interval $[\![a, b]\!]$ of real numbers can be defined as the limit point of a net $\langle N, \leqq \rangle$ whose domain is the set of all subdivisions of $[\![a, b]\!]$, the values of N are upper and lower Darboux sums and $\leqq$ is an appropriate order.

15 B.7. Convergence of sequences. Throughout the following, the word sequence will be used also for the net $\langle S, \leqq \rangle$ where S is a sequence, that is a single-valued relation on N, and $\leqq$ is the natural order for $\mathbf{D}S$ = N. It is interesting to notice that the convergence of sequences can be described without mentioning the order for N. Indeed, a subset A of N is right residual in $\langle$N, $\leqq \rangle$ if and only if its complement in N is finite and a subset A of N is right cofinal if and only if A is infinite. Thus a sequence $\langle S, \leqq \rangle$ in a space P converges to a point x of P if and only if each neighborhood of x contains all S_n except for a finite number of n's. Similarly, x is an accumulation point of $\langle S, \leqq \rangle$ if and only if every neighborhood of x contains S_n for an infinite number of n's. It is to be noted that one could define limit points and accumulation points of a family $\{x_a \mid a \in A\}$ in a space P as follows: x is a limit point if every neighborhood of x contains all x_a except for a finite number of a's, and x is a cluster point if every neighborhood of x contains x_a for an infinite number of a's, i.e., if $\{x_a\}$ is not "locally finite at x". Nevertheless these notions do not seem to be sufficiently important.

In general the closure of a subset of a space cannot be described in terms of convergence of sequences, that is, a point x may belong to the closure of a set X in a space P whereas no sequence ranging in X converges to x. This will be shown by the following two examples:

(a) Let T_{ω_1+1} be the ordered space of all ordinals $\leqq \omega_1$. It is easily seen that no sequence S ranging in T_{ω_1} converges to ω_1. Indeed, since the range $\mathbf{E}S$ of S is countable, there exists a right bound α of $\mathbf{E}S$. Now clearly $]\!] \alpha, \rightarrow]\!]$ is a neighborhood of ω_1 in T_{ω_1+1} which is disjoint with $\mathbf{E}S$, and hence S does not converge to ω_1.

(b) Let us consider the ultrafilter space βX of an infinite set X. The closure of X in βX is βX but no sequence ranging in X converges to any point of $\beta X - X$. Assuming that a sequence S ranging in X converges to a point $y \in (\beta X - X)$ we shall derive a contradiction. If $\mathbf{E}S$ is finite, then $\beta X - \mathbf{E}S$ is a neighborhood of y and hence S does not converge to y, which contradicts our assumption. If $\mathbf{E}S$ is infinite then $\mathbf{E}S$ can be written as the union of two disjoint infinite sets, say X_1 and X_2, and hence $X_1 \notin y$ or $X_2 \notin y$. If $X_i \notin y$, where $i = 1$ or $i = 2$, then $X - X_i$ is a neighborhood of y in βX (see 14 B.12) and hence an infinite number of S_n lies in the complement of a neighborhood of y, which implies that S does not converge to y. It is interesting to notice that any net $\langle N, \leqq \rangle$ ranging in X has at least one accumulation point in βX. For a proof let us consider the collection η of all sets $Y \subset X$ such that $N[B] \subset Y$ for some residual set B in $\langle \mathbf{D}N, \leqq \rangle$. Clearly η is a proper filter of sets on X and therefore we can choose an ultrafilter y on X containing η. If y is fixed and $\bigcap y = (x)$, then clearly $N_a = x$ for all a in a cofinal subset of $\langle \mathbf{D}N, \leqq \rangle$, and therefore x is an accumulation point of $\langle N, \leqq \rangle$ in βX. If y is free, then $y \in (\beta X - X)$ and if U is any neighborhood of y then $Y \subset U$ for some Y in y; we shall show that $N[B] \subset Y$ for some cofinal set B in $\langle \mathbf{D}N, \leqq \rangle$. If $a \in \mathbf{D}N$ then $N[\leqq [(a)]] \in \eta$ and hence $N[\leqq [(a)]] \in y$, which implies that $N[\leqq [(a)]] \cap Y \neq \emptyset$ and hence $N_b \in Y$ for some b following a. It is to be noted that it can be shown that each net in βX has an accumulation point in βX.

The class of all spaces which can be described in terms of the convergence of sequences will be investigated in Section 35. Here we will consider a rather extensive and important subclass, consisting of spaces with countable local characters.

15 B.8. Definition. Let P be a closure space. If $x \in P$ then the smallest cardinal of a local base at x is called the *local character of P at x* or the *local character of x in the space P*. The least upper bound of local characters of P at x, $x \in P$, is called *the local character of P*. It is to be noted that a space of a countable local character is often said to satisfy the first axiom of countability.

If P is topological then the *total character* of P is defined to be the smallest cardinal of an open base for P.

For elementary properties of local characters consult the exercises. Here we only observe that the space R of reals is of a countable infinite local character. Obviously the family $\{]\!] x - 1/n, x + 1/n [\![\mid n = 1, 2, \ldots\}$ is a local base at x in R for each x and clearly no $x \in$ R is of a finite local character.

15 B.9. Theorem. *Let us suppose that a space P is of a countable local character at x. Then x belongs to the closure of a subset X of P if and only if x is a limit point in P of a sequence ranging in X.*

Proof. The "if" part is a particular case of 15 B.4 because a sequence is a net. Conversely, suppose that $x \in \overline{X}$ and $\{U_n \mid n \in \mathrm{N}\}$ is a local base of P at x. Choose a sequence $\{x_n\}$ so that $x_n \in \bigcap\{U_k \mid k \leqq n\} \cap X$ for each n. Clearly the sequence $\{x_n\}$ ranges in X and converges to x in the space P.

Sometimes it is convenient to know whether or not the closure structure of a space can be described in terms of the convergence of nets whose ordered domains belong to a given class of ordered sets; more precisely, given a space P and a class K of ordered sets, we ask whether $x \in \overline{X}$ implies that x is a limit point of a net $\langle N, \leqq\rangle$ ranging in X such that $\langle \mathbf{D}N, \leqq\rangle$ belongs to K. By 15 B.9 every space with a countable local character can be described in terms of convergence of sequences. Some theorems of this type are given in the exercises where also the closely related theory of convergence of filters in a space will be sketched.

Before proceeding further, we shall introduce some conventions which will be useful in the more complicated situations which follow.

15 B.10. Conventions. (a) A net is a struct and therefore all conventions concerning structs apply to nets, e.g. if $\langle N, \leqq\rangle$ is a net then we often speak about N as a net, and moreover, in this case, if we say that a relation N is a net and the order for $\mathbf{D}N$ is not indicated, then automatically, in accordance with Section 12, this order is denoted by $\leqq$. Often the order for $\mathbf{D}N$ is given by the context, e.g. as we agreed, if a sequence is considered as a net then the order is the natural order for N. — (b) The following conventions are more significant: we shall say that a net $\langle N, \leqq\rangle$ is *eventually* (*frequently*) in a struct ξ if and only if the set $N^{-1}[|\xi|]$ is residual (cofinal) in $\langle \mathbf{D}N, \leqq\rangle$ (remember that $|\xi|$ denotes the underlying class of ξ). Using this terminology we can restate the definition of limit (accumulation) points as follows: a point x is a limit (an accumulation) point of a net $\mathscr{N}$ in a space P if and only if $\mathscr{N}$ is eventually (frequently) in each neighborhood of x. — (c) Finally, a limit point of a net (in a given space) is currently denoted by $\lim \mathscr{N}$, and if $\mathscr{N}$ is denoted, in accordance with (a), by $\{N_a\}$, then merely by $\lim N_a$ or $\lim \{N_a\}$. This notation is very convenient if each net converges to at most one point. Then the symbol lim can be treated as a single-valued relation which assigns to each convergent net its limit point and then the symbol $\lim \mathscr{N}$ denotes the value of lim at $\mathscr{N}$.

Now we proceed to a characterization of topological spaces in terms of convergence. By definition, a closure space $\langle P, u \rangle$ is topological if $uuX = uX$ for each $X \subset P$. Let $x \in uuX$. By 15 B.4, in order that $x \in uX$ it is sufficient that some net $\mathcal{M}$ ranging in X converge to x in $\langle P, u \rangle$. By 15 B.4 there exists a net $\mathcal{N} = \langle N, \leqq \rangle$ ranging in uX and converging to x in $\langle P, u \rangle$. Since each value N_a of N lies in uX, we can choose a family $\{\mathcal{M}^a \mid a \in \mathbf{D}N\}$ such that each $\mathcal{M}^a$ is a net $\langle M^a, \leqq_a \rangle$ which ranges in X and converges to the point N_a in the space $\langle P, u \rangle$. It is natural to ask whether there is a net $\mathcal{M}$, obtained in a certain manner from the nets $\mathcal{M}^a$, such that the range of $\mathcal{M}$ is contained in the union of ranges of nets $\mathcal{M}^a$ and that $\mathcal{M}$ converges to x, whenever the space is topological. Thus the requirement that $\mathcal{M}$ converge to x, for each choice of $\mathcal{N}$ and $\{\mathcal{M}^a\}$, will be a necessary and sufficient condition for the space P to be topological. If $\mathcal{N}$ and all $\mathcal{M}^a$ are sequences then it is natural to conjecture that some diagonal sequence can be taken as $\mathcal{M}$; more precisely, if P is a topological space and $\{S^n \mid n \in \mathbf{N}\}$ is a sequence of sequences $S^n = \{S_{nk} \mid k \in \mathbf{N}\}$ ranging in P and if $S = \{S_n\}$ is a sequence in P such that S_n is a limit point of S^n for each n and S converges to a point x, then some "diagonal sequence" $\{S_{n_i k_i} \mid i \in \mathbf{N}\}$ converges to x. This is actually true if P is of a countable local character, as stated by the following theorem; that this is not true in general will be shown in the example 15 B.12.

15 B.11. Theorem. *The following condition is necessary and sufficient for a closure space $\langle P, u \rangle$ with a countable local character to be topological.*

If $\{S^n\}$ is a sequence of sequences $S^n = \{S_{nk} \mid k \in \mathbf{N}\}$ of points of P and $S = \{S_n\}$ is a sequence of points of P such that S^n converges to S_n for each n and S converges to a point x, then some diagonal sequence $\{S_{nk_n} \mid n \in \mathbf{N}\}$ converges to x.

Proof. I. The sufficiency will be proved although, in fact, it has been proved in the remarks preceding the Theorem. Suppose that $x \in uuX$. Assuming the condition we must prove $x \in uX$. Since the space is of a countable local character, by virtue of 15 B.9 we can choose a sequence $S = \{S_n\}$ ranging in uX which converges to x in $\langle P, u \rangle$, and then, for the same reason, a sequence $\{S^n\}$ of sequences $S^n = \{S_{nk} \mid k \in \mathbf{N}\}$ ranging in X so that S^n converges to the point S_n for each n in $\mathbf{N}$. By the condition some diagonal sequence $S' = \{S_{n_i k_i} \mid i \in \mathbf{N}\}$ converges to x. Since S' ranges in X, by 15 B.4 we obtain $x \in uX$ which completes the proof. — II. Now suppose that $\langle P, u \rangle$ is topological and S^n, S and x fulfil the assumptions of the condition. Choose a local base $\{U_i \mid i \in \mathbf{N}\}$ at x consisting of open sets (this is possible since P is topological and of a countable local character at x). Clearly we may assume that $U_i \supset U_l$ if $i \leqq l$, and since $S = \{S_n\}$ converges to x, that $S_i \in U_i$ for each i (note that exercise 5 asserts precisely what is needed). Now, U_i being a neighborhood of S_i and the sequence $\{S_{in} \mid n \in \mathbf{N}\}$ being convergent to S_i, we can choose a S_{in_i}, in U_i. It is easily seen that $\{S_{in_i} \mid i \in \mathbf{N}\}$ converges to x; this completes the proof.

15 B.12. Example to diagonal sequences. We shall show that the condition of the foregoing theorem need not be fulfilled in a topological space which is not of a countable local character. Let us consider a set P consisting of all points of $(\mathsf{N} \times \mathsf{N}) \cup \mathsf{N}$ (here and in what follows we shall assume that the sets $\mathsf{N} \times \mathsf{N}$ and N are disjoint, see Notes) and of a further point x. Evidently the following three conditions determine exactly one closure u for P:

(a) each element of $\mathsf{N} \times \mathsf{N}$ is isolated in $\langle P, u\rangle$, that is, (y) is open in $\langle P, u\rangle$ for each $y \in \mathsf{N} \times \mathsf{N}$;

(b) a set $U \subset P$ is a neighborhood of an $n \in \mathsf{N}$ if and only if $n \in U$ and $((n) \times \mathsf{N})) - - U$ is finite;

(c) a set U is a neighborhood of the point x if and only if $x \in U$ and, except for a finite number of $n \in \mathsf{N}$, U is also a neighborhood of each $n \in \mathsf{N}$.

It is almost evident that $\langle P, u\rangle$ is a topological space (use, for instance 15 A.2). Let S^n, $n \in \mathsf{N}$ be the sequence $\{\langle n, i\rangle \mid i \in \mathsf{N}\}$. Clearly each sequence S^n converges to n and the sequence $\{n \mid n \in \mathsf{N}\}$ converges to x. It will be shown that no sequence $\{\langle n_i, m_i\rangle \mid i \in \mathsf{N}\}$ converges to x. Suppose that some $S = \{\langle n_i, m_i\rangle\}$ converges to x. Since $P - ((n) \times \mathsf{N})$ is a neighborhood of x for each n, S is frequently in no $((n) \times \times \mathsf{N})$, particularly $\mathbf{E}S \cap ((n) \times \mathsf{N})$ is finite for each n. It follows that $P - \mathbf{E}S$ is a neighborhood of x which contradicts our assumption that S converges to x.

Now we proceed to the general case. Example 15 B.12 shows that we must look for more complicated "diagonal nets" than the most natural ones, considered for sequences in 15 B.11.

15 B.13. Theorem. *A closure space P is topological if and only if the following condition, called the condition on iterated limits, is fulfilled:*

Let $\mathscr{A} = \langle A, \leqq\rangle$ be a directed set and let $\{\mathscr{B}_a \mid a \in A\}$ be a family of directed sets, $\mathscr{B}_a = \langle B_a, \leqq_a\rangle$. Put

$$\mathscr{C} = \langle C, \prec\rangle = \mathscr{A} \times \prod\{\mathscr{B}_a \mid a \in A\}$$

and let ϱ be the single-valued relation which assigns to each $\langle \alpha, \{b_a\}\rangle \in C$ the element $\langle \alpha, b_\alpha\rangle$ of $\sum\{B_a\}$, i.e.,

$$\varrho = \{\langle \alpha, \{b_a\}\rangle \to \langle \alpha, b_\alpha\rangle \mid \langle \alpha, \{b_a\}\rangle \in C\}\,.$$

(Thus $\mathbf{D}\varrho = C$, $\mathbf{E}\varrho = \sum\{B_a \mid a \in A\}$.) Let N be a relation on $B = \sum\{B_a \mid a \in A\}$ ranging in P, M be a relation on A ranging in P and x be a point of P such that the net $\langle M, \leqq\rangle$ converges to x in P, and, for each a in A, the net $\langle\{N_{\langle a,b\rangle} \mid b \in B_a\}, \leqq_a\rangle$ converges to M_a. Then the directed net $\langle N \circ \varrho, \prec\rangle$ converges to x.

Proof. I. The proof of sufficiency was given in the remark preceding 15 B.11. – II. To prove necessity, first let us observe that if A' is residual in A and $\{B'_a \mid a \in A\}$ is a family such that B'_a is residual in $\mathscr{B}_a$ for each a, then the product set $A' \times \times \prod\{B'_a \mid a \in A\}$ is residual in $\langle C, \prec\rangle$. – III. Now let P be a topological space and N, M and x fulfil the requirements of the condition. Suppose that U is any neighborhood of x. We must show that $\langle N \circ \varrho, \prec\rangle$ is eventually in U. Choose an open neighborhood V of x contained in U. Since $\langle M, \leqq\rangle$ converges to x, $\langle M, \leqq\rangle$ is eventually in V, and hence we can choose a residual subset A' of A so that $M[A'] \subset$

$\subset V$. If $a \in A'$, then V is a neighborhood of M_a, and $\langle \{N_{\langle a,b \rangle} \mid b \in B_a\}, \leqq_a \rangle$ being convergent to M_a, we can choose a residual subset B'_a of $\mathscr{B}_a$ so that $N_{\langle a,b \rangle} \in V$ for each b in B'_a. If $a \in (A - A')$, then put $B'_a = B_a$. Consider the set

$$C' = A' \times \prod\{B'_a \mid a \in A\}.$$

As we remarked in II, the set C' is residual in $\langle C, \prec \rangle$. By choice of $\{B'_a\}$ we have $(N \circ \varrho)[C'] = N[\varrho[C']] \subset V$ which shows that $\langle N \circ \varrho, \prec \rangle$ is eventually in V and hence in U.

15 B.14. Remarks. (a) In **15 B.11** the "diagonal sequence" depends essentially upon the closure of the space. In **15 B.13** the net $N \circ \varrho$ does not depend upon the closure of the space because ϱ depends upon the ordered sets $\mathscr{A}$ and $\mathscr{B}_a$ only.

(b) In accordance with the conventions **15 B.10** the conclusion of the condition in **15 B.13** can be stated more suggestively as follows:

$$\lim \langle N \circ \varrho, \prec \rangle = \lim_a \lim_b \{N_{\langle a,b \rangle}\}$$

provided that the iterated limit (on the right side) exists.

15 B.15. Convergence in ordered spaces. Let u be a generalized order closure for a monotone ordered set $\langle P, \leqq \rangle$. If A is a non-void subset of P, then clearly $\langle \mathsf{J}_A, \leqq_A \rangle$ as well as $\langle \mathsf{J}_A, \leqq_A^{-1} \rangle$ are directed nets. It turns out that the closure u can be described in terms of the convergence of such nets.

(a) *If x is a limit point of a net $\langle \mathsf{J}_A, \leqq_A \rangle$, then $x = \sup A$.*

Proof. Let x be a limit point of a net $\langle \mathsf{J}_A, \leqq_A \rangle$. If $x \leqq \alpha$, $x \neq \alpha$ for some $\alpha \in A$, then $]\!]\leftarrow, \alpha[\![$ is a neighborhood of x in which the net J_A is not frequently. Thus x is an upper bound of A. If y is another upper bound and $y \leqq x$, $y \neq x$, then $]\!]\, y, \rightarrow [\![$ is a neighborhood of x which does not intersect $A = \mathbf{E}\mathsf{J}_A$ and hence x is not a limit point of J_A. Thus x is the least upper bound.

(b) *If $A \neq \emptyset$, $\sup A$ exists and the point $\sup A$ is not isolated from the left in $\langle P, u \rangle$, then the net $\langle \mathsf{J}_A, \leqq_A \rangle$ converges to $\sup A$.*

Proof. Obviously $\langle \mathsf{J}_A, \leqq_A \rangle$ is eventually in each set $]\!]\, \alpha, \sup A\,]\!]$, $\alpha < \sup A$. Since $\sup A$ is not isolated from the left, each neighborhood of the point $\sup A$ contains an interval $]\!]\, \alpha, \sup A\,]\!]$, $\alpha < \sup A$, and consequently J_A is eventually in each neighborhood of $\sup A$.

(c) *$x \in uX$ if and only if x is a limit point of either the net $\langle \mathsf{J}_A, \leqq_A \rangle$ or the net $\langle \mathsf{J}_A, \leqq_A^{-1} \rangle$ for some $A \subset X$.*

15 B.16. Order convergence. Let $\langle N, \prec \rangle$ be a net in an ordered set $\langle P, \leqq \rangle$ (not necessarily monotone). The order upper limit (lower limit) of $\langle N, \prec \rangle$ in $\langle P, \leqq \rangle$ denoted by $\limsup \langle N, \prec \rangle$ ($\liminf \langle N, \prec \rangle$), is defined to be the greatest lower bound (least upper bound) of the set of all $x \in P$ such that x is an upper (lower) bound of $N[A]$ for some residual subset A of $\langle \mathbf{D}N, \prec \rangle$. Obviously,

$$\liminf \langle N, \prec \rangle \leqq \limsup \langle N, \prec \rangle$$

whenever the limits exist. If $\limsup \langle N, \prec \rangle = \liminf \langle N, \prec \rangle$, then $\langle N, \prec \rangle$ is said to be *ordered-convergent* and the point $\limsup \langle N, \prec \rangle$ is denoted by $\lim \langle N, \prec \rangle$ and called the *order-limit* of $\langle N, \prec \rangle$ in $\langle P, \leqq \rangle$.

(a) If $\leqq$ is monotone and u is the order closure for $\langle P, \leqq \rangle$, then x is a limit point of a net $\langle N, \prec \rangle$ in $\langle P, u \rangle$ if and only if the point x is the order-limit of $\langle N, \prec \rangle$ in $\langle P, \leqq \rangle$. Roughly speaking, *if $\leqq$ is monotone and u is the order closure, then the convergence in $\langle P, u \rangle$ coincides with the order-convergence.*

Proof. We restrict ourselves to the case when x is neither the greatest nor the least element. First suppose that x is a limit point of a net $\langle N, \prec \rangle$ in $\langle P, u \rangle$. Let $\alpha < x < \beta$. The interval $]\!] \alpha, \beta [\![$ is a neighborhood of x and consequently $\langle N, \prec \rangle$ is eventually in it, i.e. $N[A] \subset]\!] \alpha, \beta [\![$ for some residual subset A of $\langle \mathbf{D}N, \prec \rangle$. Since α and β were chosen arbitrarily, we obtain from the definition that $\limsup \langle N, \prec \rangle = x = \liminf \langle N, \prec \rangle$. Conversely, suppose that the order-limit $\lim \langle N, \prec \rangle$ exists. By definition, if $\alpha < \lim \langle N, \prec \rangle < \beta$, then there exist points $\alpha' \in P$, $\beta' \in P$ and residual subsets A and B of $\langle \mathbf{D}N, \prec \rangle$ so that $\alpha < \alpha' \leqq \leqq \lim \langle N, \prec \rangle \leqq \beta' < \beta$, α' is a lower bound of $N[A]$ and β' is an upper bound of $N[B]$. Evidently the set $C = A \cap B$ is residual in $\langle \mathbf{D}N, \prec \rangle$ and clearly $N[C] \subset [\![\alpha', \beta']\!] \subset \subset]\!] \alpha, \beta [\![$. Thus $\langle N, \prec \rangle$ is eventually in the interval $]\!] \alpha, \beta [\![$. But such intervals form a local base at x in $\langle P, u \rangle$ and hence, $\langle N, \prec \rangle$ is eventually in each neighborhood of $\lim \langle N, \prec \rangle$ which means that $\lim \langle N, \prec \rangle$ is a limit point of $\langle N, \prec \rangle$ in $\langle P, u \rangle$.

(b) *Let $\leqq$ be monotone and let u be the order closure for $\langle P, \leqq \rangle$. If $\langle N, \prec \rangle$ is a directed net in $\langle P, \leqq \rangle$ such that $x = \limsup \langle N, \prec \rangle$ exists, then x is an accumulation point of $\langle N, \prec \rangle$ in $\langle P, u \rangle$.*

Proof. Let U be any neighborhood of x. Choose a neighborhood $]\!] \alpha, \beta [\![\subset U$ of x. Thus $\alpha < x < \beta$. Since $x < \beta$, $\langle N, \prec \rangle$ is eventually in $]\!] \leftarrow, \beta [\![$ and since $\alpha < x$, $\langle N, \prec \rangle$ is not eventually in $]\!] \leftarrow, \alpha]\!]$, it follows that $\langle N, \prec \rangle$ is frequently in $]\!] \leftarrow, \beta]\!] -]\!] \leftarrow, \alpha]\!] =]\!] \alpha, \beta [\![$ and hence in U, which shows that x is an accumulation point of $\langle N, \prec \rangle$ in $\langle P, u \rangle$.

(c) *If $\langle P, v \rangle$ is order-complete (not necessarily monotone), then* $\limsup \mathcal{N}$ *and* $\liminf \mathcal{N}$ *exist for each net $\mathcal{N} = \langle N, \prec \rangle$, and*

$$\limsup \mathcal{N} = \inf \{\sup N[A] \mid A \textit{ is residual in } \langle \mathbf{D}N, \prec \rangle\},$$
$$\liminf \mathcal{N} = \sup \{\inf N[A] \mid A \textit{ is residual in } \langle \mathbf{D}N, \prec \rangle\}.$$

Proof. Obvious.

From (c) it follows immediately that

(d) *if $\langle P, \leqq \rangle$ is boundedly order-complete then* $\limsup \mathcal{N}$ *and* $\liminf \mathcal{N}$ *exists for each eventually bounded net $\mathcal{N}$ in $\langle P, \leqq \rangle$. If $\mathcal{N}$ is bounded then the formulae of* (c) *hold.*

(e) *Every directed net in an order-complete monotone ordered space has an accumulation point, and every bounded directed net in a boundedly order-complete* *ordered space has an accumulation point.* In particular, if $\mathscr{P}$ is the ordered space $[\![a, b]\!]$ of reals or a closed segment of ordinals with the order closure operation, then each net in $\mathscr{P}$ has an accumulation point. Every bounded net in $\mathbf{R}$ has an accumulation point.

The remaining part is devoted to the concept of a subnet and generalized subnet of net.

15 B.17. Definition. A net $\langle M, \prec\rangle$ is a *subnet of a net* $\langle N, \leqq\rangle$ *under the relation* ϱ if $\mathbf{D}\varrho = \mathbf{D}M$ and $\varrho : \langle \mathbf{D}M, \prec\rangle \to \langle \mathbf{D}N, \leqq\rangle$ is an order-embedding (in particular, ϱ is one-to-one and $a \prec b$ implies $\varrho a \leqq \varrho b$) such that $\mathbf{E}\varrho$ is cofinal in $\langle \mathbf{D}N, \leqq\rangle$ and $M = N \circ \varrho$. If, in addition, $\mathbf{E}\varrho$ is right residual, then $\langle M, \prec\rangle$ is called a *residual subnet* of $\langle N, \leqq\rangle$ under ϱ. A *subsequence of a sequence* S is a sequence which is a subnet of S. A net $\langle M, \prec\rangle$ is a *generalized subnet of a net* $\langle N, \leqq\rangle$ *under* ϱ if ϱ is a single-valued relation on $\mathbf{D}M$ ranging in $\mathbf{D}N$ such that $M = N \circ \varrho$, and $\varrho^{-1}[B]$ is residual in $\langle \mathbf{D}M, \prec\rangle$ provided that B is residual in $\langle \mathbf{D}N, \leqq\rangle$.

15 B.18. Remarks to the definition. (a) Our definition of a subsequence coincides with the usual one, that is, $\{R_i \mid i \in \mathbf{N}\}$ is a subsequence of a sequence $\{S_n \mid n \in \mathbf{N}\}$ if and only if there exists an increasing sequence $\{n_i \mid i \in \mathbf{N}\}$ of natural numbers such that $R_i = S_{n_i}$ for each i in $\mathbf{N}$. In this case $\{R_i\}$ is a subsequence of $\{S_n\}$ under $\{n_i\}$.

(b) If $\langle N, \leqq\rangle$ is a net and A is a right cofinal subset of $\langle \mathbf{D}N, \leqq\rangle$, then obviously $\langle N_A, \leqq_A\rangle$ is a subnet of $\langle N, \leqq\rangle$ under the identity relation J_A (where N_A is the restriction of N to A and $\leqq_A$ is the restriction of the order $\leqq$ to A). Moreover, $\langle N_A, \leqq_A\rangle$ is a subnet of $\langle N, \leqq\rangle$ under the identity relation if and only if A is a right cofinal subset of $\langle \mathbf{D}N, \leqq\rangle$.

(c) The notion of a subnet is a slight generalization of that of a net restricted to a right cofinal subnet (see (b)). Actually, it is almost self-evident that $\langle M, \prec\rangle$ is a subnet of a net $\langle N, \leqq\rangle$ under a relation ϱ if and only if $\langle N_{\mathbf{E}\varrho}, \leqq_{\mathbf{E}\varrho}\rangle$ is a subnet of $\langle N, \leqq\rangle$ under the identity relation. (Notice that $\varrho : \langle \mathbf{D}M, \prec\rangle \to \langle \mathbf{E}\varrho, \leqq_{\mathbf{E}\varrho}\rangle$ is an order-isomorphism.)

(d) In the definition of a generalized subnet the requirement "B residual $\Rightarrow \varrho^{-1}[B]$ residual" can be restated as follows: for each residual subset B of $\langle \mathbf{D}N, \leqq\rangle$ there exists a residual subset A of $\langle \mathbf{D}M, \prec\rangle$ such that $\varrho[A] \subset B$. In particular, if $\langle \mathbf{D}M, \prec\rangle$ and $\langle \mathbf{D}N, \leqq\rangle$ are directed then the requirement is equivalent to each of the following conditions: for each residual subset B of $\langle \mathbf{D}N, \leqq\rangle$ there exists an element α of $\mathbf{D}M$ such that $\alpha \prec a$ implies $\varrho a \in B$; for each β in $\mathbf{D}N$ there exists an α in $\mathbf{D}M$ such that $\alpha \prec a$ implies $\beta \leqq \varrho a$.

15 B.19. *If* $\langle M, \prec\rangle$ *is a subnet of a net* $\langle N, \leqq\rangle$ *under* ϱ, *then* $\langle M, \prec\rangle$ *is a generalized subnet of* $\langle N, \leqq\rangle$ *under* ϱ.

Proof. We must show that $\varrho^{-1}[B]$ is residual in $\langle \mathbf{D}M, \prec\rangle$ whenever B is residual in $\langle \mathbf{D}N, \leqq\rangle$. But this is almost evident. Indeed, since $\mathbf{E}\varrho$ is cofinal in $\langle \mathbf{D}N, \leqq\rangle$,

$B \cap \mathbf{E}\varrho$ is necessarily residual in the ordered subset $\mathbf{E}\varrho$ of $\langle \mathbf{D}N, \leqq \rangle$; since the mapping $\varrho : \langle \mathbf{D}M, \prec \rangle \to \mathbf{E}\varrho$ is an isomorphism of the ordered set $\langle \mathbf{D}M, \prec \rangle$ onto the ordered subset $\mathbf{E}\varrho$ of $\langle \mathbf{D}N, \leqq \rangle$, it follows that $\varrho^{-1}[B \cap \mathbf{E}\varrho]$ $(= \varrho^{-1}[B])$ is residual in $\langle \mathbf{D}M, \prec \rangle$, which completes the proof.

15 B.20. Theorem. *If a net $\langle N, \leqq \rangle$ converges to a point x in a space P, then each generalized subnet of $\langle N, \leqq \rangle$ converges to x in P.*

Proof. Let $\langle M, \prec \rangle$ be a generalized subnet of $\langle N, \leqq \rangle$ under ϱ and U a neighborhood of x; then clearly

$$M^{-1}[U] = \varrho^{-1}[N^{-1}[U]],$$

and consequently, if $N^{-1}[U]$ is residual in $\langle \mathbf{D}N, \leqq \rangle$ then $M^{-1}[U]$ is residual in $\langle \mathbf{D}M, \prec \rangle$; this establishes the theorem.

15 B.21. Theorem. *If a directed net $\langle N, \leqq \rangle$ in a space P does not converge to a point x of P, then there exists a subnet $\langle M, \prec \rangle$ of $\langle N, \leqq \rangle$ such that no generalized subnet of $\langle M, \prec \rangle$ converges to x.*

Proof. Suppose that $\langle N, \leqq \rangle$ does not converge to x. By definition there exists a neighborhood U of x such that $\langle N, \leqq \rangle$ is not eventually in U, i.e., $\langle N, \leqq \rangle$ is frequently in $P - U$ ($\leqq$ is directed). Put $A = N^{-1}[P - U]$ and consider the subnet $\mathscr{M} = \langle N_A, \leqq_A \rangle$ of $\langle N, \leqq \rangle$. Since $\mathscr{M}$ ranges in $P - U$, each generalized subnet ranges in $P - U$ as well, and consequently, no generalized subnet of $\mathscr{M}$ converges to x.

Remark. The assumption that the net is directed is essential.

15 B.22. Theorem. *If x is an accumulation point of a (directed) net $\mathscr{N}$ in a space P, then x is a limit point of some (directed) generalized subnet $\mathscr{M}$ of $\mathscr{N}$.*

Proof. I. Suppose that x is an accumulation point of a net $\mathscr{N} = \langle N, \leqq \rangle$ in a space P. Let $\mathscr{U}$ be a local base of the neighborhood system at x, and consider the ordered subset $\langle A, \prec \rangle$ of the product-ordered set

$$\langle \mathscr{U}, \supset \rangle \times \langle \mathbf{D}N, \leqq \rangle$$

where

$$A = \mathbf{E}\{\langle U, b \rangle \mid N_b \in U\},$$

and the following single-valued relation

$$\varrho = \{\langle U, b \rangle \to b \mid \langle U, b \rangle \in A\},$$

which is a restriction of the projection of the product $\mathscr{U} \times \mathbf{D}N$ onto $\mathbf{D}N$. We shall prove that $\mathscr{M} = \langle N \circ \varrho, \prec \rangle$ is a generalized subnet of $\langle N, \leqq \rangle$ under ϱ which possesses the required properties. — II. First we shall prove that $\mathscr{M}$ is directed provided that $\mathscr{N}$ is directed. Assuming $a_i = \langle U_i, b_i \rangle \in A$, $i = 1, 2$, choose a U in $\mathscr{U}$ so that $U \subset U_1 \cap U_2$, and then choose a b in $\mathbf{D}N$ such that $N_b \in U$ and b follows both b_1 and b_2 in $\mathbf{D}N$. Then $\langle U, b \rangle \in A$ and clearly $\langle U, b \rangle$ follows both a_1 and a_2 in $\langle A, \prec \rangle$. — III. Auxiliary assertion: A is cofinal in $\langle \mathscr{U}, \supset \rangle \times \langle \mathbf{D}N, \leqq \rangle$. Let $\langle U_0, b_0 \rangle$ be any element of $\mathcal{U} \times \mathbf{D}N$. Since x is an accumulation point of $\mathcal{N}$ and U_0 is a neighborhood of x, we can choose a b in $\mathbf{D}N$ such that $b_0 \leqq b$ and $N_b \in U_0$. Thus $\langle U_0, b\rangle \in A$ and $\langle U_0, b\rangle$ follows $\langle U_0, b_0\rangle$ in $\langle \mathcal{U}, \supset\rangle \times \langle \mathbf{D}N, \leqq\rangle$. – IV. We shall prove that $\mathcal{M}$ is a generalized subnet of $\mathcal{N}$ and that $\mathcal{M}$ converges to x. Let B be residual in $\mathbf{D}N$ and let U_0 be a neighborhood of x. Choose a U in $\mathcal{U}$ with $U \subset U_0$ and consider the set $\mathcal{V} \times B$ where $\mathcal{V} = \mathbf{E}\{V \mid V \in \mathcal{U}, V \subset U\}$. Since $\langle \mathcal{U}, \supset\rangle$ is directed, $\mathcal{V}$ is residual in $\mathcal{U}$. Hence $\mathcal{V} \times B$ is residual in $\langle \mathcal{U}, \supset\rangle \times \langle \mathbf{D}N, \leqq\rangle$; by III the set $A_1 = A \cap (\mathcal{V} \times B)$ is residual in $\langle A, \prec\rangle$.

It is easily seen that $\varrho[A_1] \subset B$ and $(N \circ \varrho)[A_1] \subset U \subset U_0$. Thus $\mathcal{M}$ is a generalized subnet of $\mathcal{N}$, and $\mathcal{M}$ converges to x. The proof is complete.

Corollary. *If u is the order closure for a monotone order-complete ordered set $\langle P, \leqq\rangle$ then each directed net in $\langle P, u\rangle$ possesses a convergent generalized subnet.*

Proof. By 15 B.16 (e), every net in $\langle P, \leqq\rangle$ has an accumulation point.

15 B.23. *If x is an accumulation point of a net $\mathcal{N}$, then x need not be a limit point of a subnet of $\mathcal{N}$.* For example, in the ultrafilter space of an infinite set X, every sequence in X has a cluster point but a countable net in X is convergent in βX if and only if it is constant on a residual set. It follows that a cluster point of a sequence S may be a limit point of no subsequence of S. It is to be noted that generalized subnets were introduced to obtain 15 B.22.

15 B.24. Theorem. *In a closure space with a countable local character each accumulation point of a sequence S is a limit point of a subsequence of S.*

Proof. We shall prove somewhat more. Suppose that x is an accumulation point of a sequence $\{x_n\}$ in a space P and P is of a countable local character at x. Choose a local base $\{U_n\}$ at x. By induction we can construct an increasing sequence $\{n_i\}$ in N so that

$$x_{n_i} \in \bigcap\{U_n \mid n \leqq i\}.$$

Clearly $\{x_{n_i} \mid i \in \mathrm{N}\}$ is a subsequence of $\{x_n\}$ converging to x.

Corollary. *Every bounded sequence in R possesses a convergent subsequence.*

Proof. By 15 B.16 (e) every bounded sequence in R has an accumulation point and R is of a countable local character by the remark following Definition 15 B.8.

15 B.25. Convention. For conciseness of formulation, in the sequel by a net we shall mean a directed net (except for some cases in which the distinction is maintained expressly).

16. CONTINUOUS MAPPINGS

In this section we begin the study of the notion of a continuous mapping of one closure space into another. We shall describe the continuity of a mapping by means of neighborhoods and nets. Particular attention will be given to the case in which the range carrier of the mapping is topological; in this case the continuity admits a description by means of open sets or closed sets. In this connection in subsection B the notion of the topological modification of a closure operation will be introduced and studied. We shall see later that many theorems about topological spaces can be reduced to corresponding results for closure spaces by the appropriate application of the properties of topological modification. The section concludes with some remarks concerning homeomorphisms, i.e. bijective mappings f of closure spaces such that f as well as its inverse f^{-1} is continuous.

A closure operation for a set P was defined to be a single-valued relation u on $\exp P$ and ranging in $\exp P$ satisfying certain conditions. A closure u for a set P is entirely determined by the relation $\varrho = \{x \to X \mid x \in uX\}$ which occurs in proofs more frequently than u; of course $uX = \varrho^{-1}[(X)]$. This relation is more intuitive than u; it shows that a closure for a set determines, roughly speaking, what points are proximal to which sets, and this is precisely the intuitive sense of the notion of a closure operation. Now the definition of a continuous mapping as a mapping preserving the relation $\{x \text{ is proximal to } X\}$ is evident: a mapping f of a space $\langle P, u\rangle$ into a space $\langle Q, v\rangle$ is continuous if $x \in uX$ implies $fx \in vf[X]$. Without doubt, the reader is familiar with the notion of a continuous mapping, at least in the case of special closure spaces, e.g. functions on R, and therefore, perhaps, this motivation will enable the reader to unterstand the intuitive meaning of theorems which follow.

A. GENERALITIES

16 A.1. Definition. Let f be a mapping of a closure space $\mathscr{P}$ into a closure space $\mathscr{Q}$. The mapping f is said to be *continuous at a point* x of $\mathscr{P}$ if

(*) $X \subset |\mathscr{P}|$, $x \in \overline{X}$ imply $fx \in \overline{f[X]}$.

The mapping f is said to be *continuous* if it is continuous at each point x of $\mathcal{P}$, or equivalently, if

(**) $X \subset |\mathcal{P}|$ implies $f[\overline{X}] \subset \overline{f[X]}$.

The set of all continuous mappings of a space $\mathcal{P}$ into a space $\mathcal{Q}$ will be denoted by $\mathbf{C}(\mathcal{P}, \mathcal{Q})$.

Since the closure operation for a set P is an order-preserving relation under $\subset$, condition (**) is equivalent to the following

(***) $Y \subset |\mathcal{Q}| \Rightarrow \overline{f^{-1}[Y]} \subset f^{-1}[\overline{Y}]$.

Indeed, assuming (**), if $X = f^{-1}[Y]$, then $f[\overline{X}] \subset \overline{Y}$, which obviously implies $\overline{X} \subset f^{-1}[\overline{Y}]$ and this is the right side of the condition (***). Conversely, assuming (***), choose an arbitrary $X \subset P$, and consider the sets $Y = f[X]$ and $X_1 = f^{-1}[Y]$; by (***) we have $\overline{X}_1 = \overline{f^{-1}[Y]} \subset f^{-1}[\overline{Y}]$ which implies $f[\overline{X}_1] \subset \overline{Y} = \overline{f[X_1]} = \overline{f[X]}$. Since $X \subset X_1$, we have $\overline{X} \subset \overline{X}_1$ and hence $f[\overline{X}] \subset f[\overline{X}_1] \subset \overline{f[X]}$.

From the definition it follows at once that every mapping of a discrete space into any space is continuous, and every mapping of any space into an accrete space is continuous. From the definitions one obtains the following description of the relation $\{u$ is coarser than $v\}$ in terms of continuity.

16 A.2. *A closure operation u for a set P is coarser than a closure operation v for P if and only if the identity mapping of $\langle P, v\rangle$ onto $\langle P, u\rangle$ is continuous.*

The following result is almost evident but very important.

16 A.3. *Let $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ be closure spaces, f a mapping of $\mathcal{P}$ into $\mathcal{Q}$ and g a mapping of $\mathcal{Q}$ into $\mathcal{R}$. If f is continuous at a point $x \in \mathcal{P}$ and g is continuous at the point fx, then the composition $g \circ f : \mathcal{P} \to \mathcal{R}$ is continuous at the point x. In particular, if $f : \mathcal{P} \to \mathcal{Q}$ and $g : \mathcal{Q} \to \mathcal{R}$ are continuous, then their composite $g \circ f : \mathcal{P} \to \mathcal{R}$ is continuous.*

Proof. If $x \in \overline{X}$, then $fx \in \overline{f[X]}$ by the continuity of f at x, and $gfx \in \overline{g[f[X]]}$ by the continuity of g at fx. But $gfx = (g \circ f)\, x$ and $g \circ f[X] = g[f[X]]$. The continuity of $g \circ f$ at x follows.

Now we proceed to various characterizations of continuity. We begin with the characterization in terms of neighborhoods.

16 A.4. Theorem. *In order that a mapping f of a closure space $\mathcal{P}$ into another one $\mathcal{Q}$ be continuous at a point $x \in \mathcal{P}$ it is necessary and sufficient that the inverse image $f^{-1}[V]$ of each neighborhood of fx be a neighborhood of x, or equivalently, that for each neighborhood V of fx there exist a neighborhood U of x such that $f[U] \subset V$.*

Proof. I. Necessity: If $U = f^{-1}[V]$, where $V \subset Q = |\mathcal{Q}|$, is not a neighborhood of x, then by definition $x \in \overline{P - U}$, and finally by the continuity of f at x, $fx \in \overline{f[P - U]} = \overline{f[P] - V} \subset \overline{Q - V}$ which means that V is not a neighborhood of fx in Q. Consequently, if V is a neighborhood of fx then $f^{-1}[V]$ is a neighborhood of x. II. – Sufficiency: If $x \in P$, $X \subset P$ and $fx \notin \overline{f[X]}$, then $V = Q - \overline{f[X]}$ is a neighborhood of fx, and by hypothesis $f^{-1}[V]$ is a neighborhood of x. Clearly, $f^{-1}[V] \cap X = \emptyset$ which implies $x \notin \overline{X}$. It follows that $x \in \overline{X}$ implies $fx \in \overline{f[X]}$. Thus (*) of 16 A.1 holds and f is continuous at x.

16 A.5. Corollary. *A mapping of a space $\mathscr{P}$ into a space $\mathscr{Q}$ is continuous if and only if, for each x in $\mathscr{P}$, the inverse image of every neighborhood of fx is a neighborhood of x, or equivalently, every neighborhood of fx contains the image of a neighborhood of x.*

16 A.6. *If f is a continuous mapping of a space $\mathscr{P}$ into a space $\mathscr{Q}$, then the inverse image of each open (closed) subset of $\mathscr{Q}$ is an open (closed) subset of $\mathscr{P}$.*

Proof. If U is open in $\mathscr{Q}$, then U is a neighborhood of each of its points (by 14 B.2), and by 16 A.5, $f^{-1}[U]$ is a neighborhood of each of its points which means that $f^{-1}[U]$ is open (by 14 B.2).

If Y is closed in $\mathscr{P}$, then $U = \mathscr{Q} - Y$ is open; it has just been proved that the set $f^{-1}[U]$ is open. But $f^{-1}[Y] = P - f^{-1}[U]$. Thus $f^{-1}[Y]$ is closed.

16 A.7. *Let f be a continuous mapping of a space $\mathscr{P}$ into a space $\mathscr{Q}$. If $\{Y_a\}$ is a locally finite family in $\mathscr{Q}$, then $\{f^{-1}[Y_a]\}$ is a locally finite family in $\mathscr{P}$.*

Proof. If V is a neighborhood of fx intersecting only a finite number of Y_a then $f^{-1}[V]$ is a neighborhood of x (by 16 A.4) intersecting only a finite number of $f^{-1}[Y_a]$.

16 A.8. Theorem. *Each of the following conditions is necessary and sufficient for a mapping of a space $\mathscr{P}$ into a space $\mathscr{Q}$ to be continuous at a point $x \in \mathscr{P}$:*

(a) *If x is an accumulation point of a (directed) net N in $\mathscr{P}$, then fx is an accumulation point of the net $f \circ N$ in $\mathscr{Q}$.*

(b) *If x is a limit point of a (directed) net N in $\mathscr{P}$, then fx is a limit point of the net $f \circ N$ in $\mathscr{Q}$.*

Proof. I. Necessity of both conditions. Suppose that f is continuous at x, N is a net in $\mathscr{P}$, x is an accumulation (a limit) point of N, and finally, V is a neighborhood of fx. We must show that $f \circ N$ is frequently (eventually) in V. But V contains the image $f[U]$ of some neighborhood of x (by continuity of f at x and 16 A.4); since N is frequently (eventually) in U, $f \circ N$ is frequently (eventually) in $f[U] \subset V$.

II. Sufficiency of both conditions. Suppose that the condition (a) (the condition (b)) obtains and $x \in \overline{X}$. By 15 B.4 there exists a directed net N ranging in X such that x is an accumulation (a limit) point N in $\mathscr{P}$. By our assumption, fx is an accumulation (a limit) point of $f \circ N$. Since obviously $f \circ N$ ranges in $f[X]$, again by 15 B.4 we obtain $fx \in \overline{f[X]}$ which establishes the continuity of f at x.

16 A.9. Corollary. *Each of the following two conditions is necessary and sufficient for a mapping f of a space $\mathscr{P}$ into a space $\mathscr{Q}$ to be continuous:*

(a) *If a point x of $\mathscr{P}$ is an accumulation point of a (directed) net N in $\mathscr{P}$, then fx is an accumulation point of $f \circ N$ in $\mathscr{Q}$.*

(b) *If a point x of $\mathscr{P}$ is a limit point of a (directed) net N in $\mathscr{P}$, then fx is a limit point of $f \circ N$ in $\mathscr{Q}$.*

By 16 A.6, if $f : \mathscr{P} \to \mathscr{Q}$ is continuous, then for any X open (closed) in $\mathscr{Q}$, $f^{-1}[X]$ is open (closed) in $\mathscr{P}$. Now we shall prove that if $\mathscr{Q}$ is a topological space, then each of these conditions is also sufficient for f to be continuous.

16 A.10. Theorem. *Each of the following conditions is necessary and sufficient for a mapping of a closure space $\mathscr{P}$ into a topological space $\mathscr{Q}$ to be continuous:*

(a) *The inverse image under f of every open subset of $\mathscr{Q}$ is an open subset of $\mathscr{P}$ (i.e. if X is open in $\mathscr{Q}$ then $f^{-1}[X]$ is open in $\mathscr{P}$).*

(b) *The inverse image under f of every closed subset of $\mathscr{Q}$ is a closed subset of $\mathscr{P}$ (i.e. if X is closed in $\mathscr{Q}$ then $f^{-1}[X]$ is closed in $\mathscr{P}$).*

Proof. Both conditions are necessary by 16 A.6 (without assuming that $\mathscr{Q}$ is topological). Both conditions are equivalent because, with $P = |\mathscr{P}|$, $Q = |\mathscr{Q}|$, $f^{-1}[X] = P - f^{-1}[Q - X]$ for each $X \subset Q$, and a set is open if and only if its complement is closed (by definition). We shall prove the sufficiency of (a). Let $x \in P$ and let V be a neighborhood of fx. Since $\mathscr{Q}$ is topological, we can choose an open neighborhood U of fx with $U \subset V$ (by 15 A.2 (c)). By condition (a) the set $f^{-1}[U]$ is open in $\mathscr{P}$, and obviously $x \in f^{-1}[U]$. Thus $f^{-1}[U]$ is a neighborhood of x in $\mathscr{P}$. Since V and x were chosen arbitrarily, f is continuous by 16 A.5.

It is to be noted that 16 A.10 is a generalization of 15 A.7 which asserts that a topological closure operation u for P is coarser than a closure v for P if and only if any u-open set is v-open.

B. TOPOLOGICAL MODIFICATION

Now let $\langle P, u \rangle$ be a closure space and let $\mathcal{O}$ be the collection of all open subsets of $\langle P, u \rangle$. By 14 A.9 the collection $\mathcal{O}$ fulfils the conditions (o 1), (o 2) and (o 3) of Theorem 15 A.6 according to which there exists exactly one topological closure operation v on P such that the collection $\mathcal{O}$ is the set of all v-open sets, and this closure v is the coarsest closure in the collection C_0 of all closures w on P such that $\mathcal{O}$ is the collection of all w-open sets. In particular, v is coarser than u. If $X \subset P$, then

(*) $vX = \bigcap \{F \mid X \subset F,\ F \text{ is closed in } \langle P, u \rangle\}$.

Indeed, since v is a topological closure, vX is closed in $\langle P, v \rangle$, $P - vX$ is open in $\langle P, v \rangle$ and by our assumption, $P - vX$ is open in $\langle P, u \rangle$ and hence vX is closed in $\langle P, u \rangle$. In particular, (*) holds.

16 B.1. Definition. The *topological modification*, or simply the *T-modification*, of a closure u for a set P, denoted by τu, is defined to be the closure operation $v =$

$= \{X \to vX\}$ for P where vX is given by $(*)$. The topological modification of a closure space $\langle P, u \rangle$ is defined to be the space $\langle P, v \rangle$ where v is the T-modification of u and is denoted by $\tau \langle P, u \rangle$. The relation $\tau = \{u \to \tau u\}$ on the class of all closure operations is called topological modification.

16 B.2. Each of the following conditions is necessary and sufficient for a closure operation v to be the topological modification of a closure u on a set P:

(a) the closure v is topological and the collections of all u-open sets and of all v-open sets coincide;

(b) the closure v is topological and the collections of all u-closed sets and of all v-closed sets coincide.

Proof. By the considerations preceding Definition 16 B.1 condition (a) is necessary and sufficient. Since a set is closed if and only if its complement is open, conditions (a) and (b) are equivalent.

16 B.3. *The topological modification of a closure operation u for a set P is the finest topological closure on P coarser than u.* Stated in other words, the following condition is necessary and sufficient for a closure operation v on a set P to be the topological modification of a closure operation u for P: v is a topological closure (for P) coarser than u, and if w is any topological closure (for P) coarser than u, then w is coarser than v.

Proof. Obviously τu is a topological closure coarser than u. If w is any closure on P coarser than u, then every w-open set is u-open (by 15 A.7, or 16 A.2 and 16 A.10) and hence τu-open. If w is, in addition, topological, then this implies that w is coarser than τu (by 15 A.7 or 16 A.2 and 16 A.10). Thus the condition is necessary. Conversely, let v be a closure fulfilling the condition. We must prove $\tau u = v$. Since τu is a topological closure coarser than u, τu is coarser than v. Thus we have $u \prec v \prec \tau u$. Since the open sets of $\langle P, u \rangle$ and $\langle P, \tau u \rangle$ are identical, by 15 A.7 the open sets of v and τu are identical, which implies $v = \tau u$ because of 15 A.7.

Now we proceed to the formulation and proof of the main result.

16 B.4. *Let u be a closure for a set P. In order that v be the topological modification of u it is necessary and sufficient that v be a topological closure for the set P and each mapping f of $\langle P, u \rangle$ into a topological space Q be continuous if and only if the mapping f is continuous as a mapping of $\langle P, v \rangle$ into Q (that is, $f : \langle P, v \rangle \to Q$ is continuous).*

Proof. I. First assume the condition. Since the identity mapping J of $\langle P, u \rangle$ onto $\langle P, \tau u \rangle$ is continuous, by the condition the mapping $\mathsf{J} : \langle P, v \rangle \to \langle P, \tau u \rangle$ is also continuous which implies (by 16 A.2) that v is finer than τu. Since $\mathsf{J} : \langle P, v \rangle \to \to \langle P, v \rangle$ is continuous, by the condition the mapping $\mathsf{J} : \langle P, u \rangle \to \langle P, v \rangle$ is also continuous, which implies (by 16 A.3) that v is coarser than u. Thus v is a topological closure coarser than u and finer than τu. By 16 B.3 necessarily $v = \tau u$. —

II. Conversely, let $v = \tau u$. If f is a continuous mapping of $\langle P, \tau u\rangle$ into a closure space Q (not necessarily topological), then $f: \langle P, u\rangle \to Q$ is also continuous as the composition of continuous mappings $J: \langle P, u\rangle \to \langle P, \tau u\rangle$ and $f: \langle P, \tau u\rangle \to Q$. Conversely, if $f: \langle P, u\rangle \to Q$ is continuous, then the inverse image of every open subset of Q is u-open (16 A.6), and hence τu-open (16 B.2) which implies $f: \langle P, \tau u\rangle \to Q$ is continuous provided that Q is topological (16 A.10).

Because of the importance of the topological modification we shall describe two further constructions which yield the topological modification of a closure operation u. The first construction describes neighborhoods in the topological modification in terms of neighborhoods in $\langle P, u\rangle$.

16 B.5. *Let $\langle P, u\rangle$ be a closure space and let $x \in \mathscr{P}$. Let $\mathscr{U}$ be the collection of all sets U of the form $U = \bigcup\{U_n \mid n \in \mathsf{N}\}$, where U_0 is a u-neighborhood of x and U_{n+1} is a u-neighborhood of U_n for each $n \in \mathsf{N}$. Then $\mathscr{U}$ is a local base at x in $\langle P, \tau u\rangle$.*

Proof. It is sufficient to show that $U \in \mathscr{U}$ if and only if U is a u-open set containing x. If U is open and $x \in U$, then U can be written in the form $U = \bigcup\{U_n\}$ with $U_n = U$ for each n. Indeed, an open set is a neighborhood of itself. Conversely, if $U \in \mathscr{U}$ and $U = \bigcup\{U_n\}$, where U_0 is a u-neighborhood of x and U_{n+1} is a u-neighborhood of U_n for each $n \in \mathsf{N}$, then U is u-open, because U is a neighborhood of each of its points. Indeed, if $y \in U$, then $y \in U_n$ for some n and U_{n+1} is a u-neighborhood of y; since $U_{n+1} \subset U$, U is also a u-neighborhood of y . Evidently $x \in U$.

Now the second construction will be described. We omit details but the reader is requested to complete all proofs. Let $\langle P, u\rangle$ be a closure space. Let us consider the transfinite sequence $\{u_\alpha\}$ of relations on $\exp P$ ranging in $\exp P$ such that $u_0 = u$ and

$$(*) \qquad u_\alpha X = \bigcup\{u_\beta \circ u_\beta X \mid \beta < \alpha\}$$

for each cardinal $\alpha \geqq 1$ and each $X \subset P$. It is easy to verify that each u_α is a closure operation (by induction) and $u_\alpha X \supset u_\beta X$ for $\alpha \geqq \beta$, i.e. u_α is coarser than u_β provided that $\alpha \geqq \beta$. In other words, $\{u_\alpha\}$ is an increasing sequence of closure operations on P. If F is a closed subset of $\langle P, u\rangle$ containing a set X, then obviously $u_\alpha X \subset F$ for each α. Thus every u_α is finer than the topological modification τu of u. By definition of topological closures, u_α is topological if and only if $u_\alpha = u_{\alpha+1}$. If $u_\alpha X = u_{\alpha+1} X$ for some X, then by induction we obtain at once that $u_\alpha X = u_\beta X$ for each $\beta \geqq \alpha$. In particular, if $u_\alpha = u_{\alpha+1}$, then $u_\alpha = u_\beta$ for $\alpha \leqq \beta$. But if $u_\alpha = u_{\alpha+1}$, i.e. if u_α is a topological closure, then necessarily $u_\alpha = \tau u$ because $u \prec u_\alpha \prec \tau u$. If γ is an ordinal of cardinality greater than that of P, then $u_\gamma = u_{\gamma+1}$ and hence $u_\gamma = \tau u$. Indeed, let X be any subset of P and let us consider the set

$$Y = \bigcup\{u_{\alpha+1} X - u_\alpha X \mid \alpha < \gamma\} .$$

Since $Y \subset P$, the cardinal of Y is at most that of P, that is, less than that of γ. Since the family $\{u_{\alpha+1} X - u_\alpha X\}$ is disjoint, necessarily at most one $u_{\alpha+1} X - u_\alpha X$ must be void, i.e. $u_{\alpha+1}X = u_\alpha X$ for some $\alpha < \gamma$. It follows that $u_{\gamma+1}X = u_\gamma X$. Since X was arbitrarily chosen, $u_{\gamma+1} = u_\gamma$. The results obtained are summarized in the following statement.

16 B.6. *Let u be a closure operation on a set P and let $\{u_\alpha\}$ be the transfinite sequence of mappings of* exp P *into itself defined as follows:* $u_0 = u$ and $u_\alpha X$ is *given by (*) for each $\alpha \geqq 1$ and $X \subset P$. Then $\{u_\alpha\}$ is a sequence of closure operations such that $u_\alpha = u_\beta$ for sufficiently large α and β, and each u_α is coarser than u and finer than τu. If $u_\alpha = u_{\alpha+1}$, then $u_\alpha = u_\beta$ for all $\beta \geqq \alpha$ and $u_\alpha = \tau u$.*

C. HOMEOMORPHISMS

In conclusion we shall introduce the concept of a homeomorphism.

16 C.1. Definition. A *homeomorphism* is a bijective mapping f for closure spaces such that both f and f^{-1} are continuous. We shall say that f is a homeomorphism of $\mathscr{P}$ onto $\mathscr{Q}$ if f is a homeomorphism such that $\mathscr{P} = \mathbf{D}^*f$ and $\mathscr{Q} = \mathbf{E}^*f$.

16 C.2. *The identity mapping of a space onto itself is a homeomorphism; if f is a homeomorphism then f^{-1} is also a homeomorphism; and finally the composition of two homeomorphisms is a homeomorphism. In particular, the relation $\mathbf{E}\{\langle\mathscr{P}, \mathscr{Q}\rangle \mid$ there exists a homeomorphism of $\mathscr{P}$ onto $\mathscr{Q}\}$ is an equivalence relation on the class of all closure spaces.*

Proof. The first and the second statement are self-evident and the third follows from the second and 16 A.3 (observe that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ if both f and g are bijective).

Remark. The relation of 16 C.2 is the class consisting of the structures of homeomorphisms. Indeed, a mapping $f = \langle \text{gr}\, f, \mathscr{P}, \mathscr{Q}\rangle$ is a struct, gr f is the underlying class of f (called the graph of f) and $\langle\mathscr{P}, \mathscr{Q}\rangle$ is the structure of f.

16 C.3. Definition. A space $\mathscr{Q}$ is a *homeomorph* of $\mathscr{P}$ if there exists a homeomorphism of $\mathscr{P}$ onto $\mathscr{Q}$. By 16 C.2 the relation $\mathbf{E}\{\langle\mathscr{P}, \mathscr{Q}\rangle \mid \mathscr{Q}$ is a homeomorph of $\mathscr{P}\}$ is an equivalence on the class of all closure spaces. Two spaces are said to be *homeomorphic* if one is a homeomorph of the other.

For example, two discrete spaces $\mathscr{P}$ and $\mathscr{Q}$ (i.e. sets endowed with the finest closure operations) are homeomorphic if and only if there exists a one-to-one mapping of $\mathscr{P}$ onto $\mathscr{Q}$, that is, if and only if the cardinals of $\mathscr{P}$ and $\mathscr{Q}$ are equal. This follows from the fact that every mapping of a discrete space into any space is continuous. Also two accrete spaces (sets endowed with the coarsest closure operations) are homeomorphic if and only if their cardinals are equal. This is so because every mapping into an accrete space is continuous, regardless of the domain space.

The notion of a homeomorphism is fundamental and therefore we list a sequence of necessary and sufficient conditions for a mapping to be a homeomorphism. The proof follows immediately from the definition and corresponding results for continuous mappings.

16 C.4. *Let f be a one-to-one mapping of a space $\mathscr{P}$ into a space $\mathscr{Q}$. If f is a homeomorphism, then all the following conditions are fulfilled. Each of the conditions* (a)–(d) *implies that f is a homeomorphism. If $\mathscr{P}$ and $\mathscr{Q}$ are topological spaces, then also each of the conditions* (e) *and* (f) *implies that f is a homeomorphism.*

(a) $X \subset |\mathscr{P}| \Rightarrow f[\overline{X}] = \overline{f[X]}$;

(b) *for each $x \in |\mathscr{P}|$, X is a neighborhood of x if and only if $f[X]$ is a neighborhood of fx;*

(c) *x is a limit point of a (directed) net N in $\mathscr{P}$ if and only if fx is a limit point of $f \circ N$ in $\mathscr{Q}$;*

(d) *x is an accumulation point of a (directed) net N in $\mathscr{P}$ if and only if fx is an accumulation point of $f \circ N$ in $\mathscr{Q}$;*

(e) *$X \subset |\mathscr{P}|$ is open in $\mathscr{P}$ if and only if $f[X]$ is open in $\mathscr{Q}$;*

(f) *$X \subset |\mathscr{P}|$ is closed in $\mathscr{P}$ if and only if $f[X]$ is closed in $\mathscr{Q}$.*

Remark. A more formal statement of conditions (a)–(f) may be in place. Let f be a bijective mapping of $\langle P, u \rangle$ onto $\langle Q, v \rangle$. Denoting by ϱ the relation $\{X \to f[X] \mid X \subset P\}$, we can state condition (a) as follows: $(\varrho \times \varrho) \circ u \subset v$. Of course, this inclusion is equivalent with the equality $(\varrho \times \varrho) \circ u = v$. Denoting by $\mathscr{M}$ the relation consisting of all pairs $\langle X, x \rangle$ such that X is a neighborhood of x in $\langle P, u \rangle$ and denoting by $\mathscr{N}$ the similar relation for $\langle Q, v \rangle$, we find at once that condition (b) can be stated as follows: $\varrho \times f$ is a bijective relation for $\mathscr{M}$ and $\mathscr{N}$. Denoting by σ the single-valued relation which assigns to each net N in $\mathscr{P}$ its transform $f \circ N$ under f, condition (c) can be stated as follows: the image of **Lim** u (see 15 B.4) under the relation $\sigma \times f$ is **Lim** v (remember that σ is bijective). A similar restatement of remaining conditions is left to the reader.

In general it is often difficult to discover whether two spaces are homeomorphic or not. Usually, to show that two spaces are not homeomorphic, we try to find a property of spaces which is possessed by one of the spaces but not by the other and such that, if a space possesses this property, then all its homeomorphs also possess this property. Such a property is called a topological property.

16 C.5. Definition. A *topological property* is a property such that if a closure space possesses this property, then all homeomorphs of P also possess this property.

For example, "the cardinal of the underlying set of $\mathscr{P}$ is m" and "$\mathscr{P}$ is discrete" are topological properties. A space exhibiting both these properties (for fixed m) is determined up to a homeomorphism. Next, "$\mathscr{P}$ is topological" is evidently a topological property, i.e. no topological space is a homeomorph of a space which is not topological. In general, every property described by means of closure operations, neighborhoods or convergence of nets is a topological property. Later on, many examples will be given in which the fact that two spaces are not homeomorphic is established by exhibiting a topological property which is possessed by one of them but not by the other.

In conclusion we shall show that the following condition is not sufficient for two spaces $\langle P, u\rangle$ and $\langle Q, v\rangle$ to be homeomorphic: there exists a one-to-one continuous mapping f of $\langle P, u\rangle$ onto $\langle Q, v\rangle$ and a one-to-one continuous mapping g of $\langle Q, v\rangle$ onto $\langle P, u\rangle$.

Of course this condition is necessary. Stated more formally, the relation consisting of all pairs $\langle\langle P, u\rangle, \langle Q, v\rangle\rangle$ with the property mentioned above is strictly larger than the class of all pairs of homeomorphic spaces.

16 C.6. Example. Let A_1 and A_2 be disjoint countable infinite sets, and let us consider the closure u for $A = A_1 \cup A_2$ such that $uX = u_1(X \cap A_1) \cup u_2(X \cap A_2)$ where u_1 is the discrete closure for A_1 and u_2 is the accrete closure for A_2. Let $\langle A_3, u_3\rangle$ be any countable space disjoint with A and let v be the closure for $B = A \cup A_3$ such that $vX = \bigcup\{u_i(X \cap A_i) \mid i = 1, 2, 3\}$. It is easily seen that there exists a one-to-one continuous mapping f of $\langle A, u\rangle$ onto $\langle B, v\rangle$ and a one-to-one continuous mapping g of $\langle B, v\rangle$ onto $\langle A, u\rangle$. E.g. let f be a one-to-one mapping of $\langle A, u\rangle$ onto $\langle B, v\rangle$ such that $f[A_1] = A_1 \cup A_3$ and let g be a one-to-one mapping of $\langle B, v\rangle$ onto $\langle A, u\rangle$ such that $g[A_2 \cup A_3] = A_2$. It is easily seen that both f and g are continuous. It is easy to find a countable $\langle A_3, u_3\rangle$ such that the spaces $\langle A, u\rangle$ and $\langle B, v\rangle$ are not homeomorphic. E.g. let A_3 consists of two points, say x and y, and let $u_3(x) = (x, y)$, $u_3(y) = (y)$; clearly $v(x) = (x, y)$, $v(y) = (y)$. On the other hand there exists no pair $\langle x_1, y_1\rangle$ such that $u(x_1) = (x_1, y_1)$, $u(y_1) = (y_1)$. Thus $\langle A, u\rangle$ and $\langle B, v\rangle$ are not homeomorphic. In 20 ex. 7 we shall give another example.

17. SUBSPACES, SUMS AND PRODUCTS

This section is devoted to fundamental constructions of new spaces from given spaces. We begin with the motivation. It is to be noted that all results stated here will be proved in this section.

If Q is a subset of the underlying set of a space $\langle P, u\rangle$, then clearly the relation $v = \{X \to Q \cap uX \mid X \subset Q\}$ is a closure operation for Q which is called the relativization of u to Q; the space $\langle Q, v\rangle$ is said to be a subspace of $\langle P, u\rangle$. It turns out that the relativization v of u to Q is the unique closure for Q such that a mapping f of a space $\langle R, w\rangle$ into $\langle Q, v\rangle$ is continuous if and only if the mapping $\operatorname{gr} f : \langle R, w\rangle \to \langle P, u\rangle$ is continuous. Stated in other words, a subspace of a space is defined so that a mapping f for closure spaces will be continuous if and only if the mapping $\operatorname{gr} f : \mathbf{D}^*f \to \mathbf{E}f$, where $\mathbf{E}f$ is considered as a subspace of $\mathbf{E}^*f$, is continuous. Subspaces are studied in subsection A.

Now let $\{\langle P_a, u_a\rangle\}$ be a family of spaces. It is often necessary to construct a space $\langle P, u\rangle$ such that there exists a disjoint cover $\{Q_a\}$ of $\langle P, u\rangle$ consisting of open sets (and hence, each Q_a is also closed because its complement in P is open as the union of open sets) such that each Q_a endowed with the relativization of u is a homeomorph of $\langle P_a, u_a\rangle$. Such a space can be constructed as follows: we take for P the sum of the family $\{P_a\}$ of underlying sets and define $uX = \Sigma\{u_a X_a\}$ for each $X = \Sigma\{X_a\} \subset P$. It turns out that each mapping $\operatorname{inj}_a : \langle P_a, u_a\rangle \to \operatorname{inj}_a [P_a]$, where $\operatorname{inj}_a [P_a]$ is considered as a subspace of $\langle P, u\rangle$, is a homeomorphism, and $\{\operatorname{inj}_a [P_a]\}$ is a disjoint cover of $\langle P, u\rangle$ consisting of open sets. The space $\langle P, u\rangle$ is called the sum of the family $\{\langle P_a, u_a\rangle\}$ and the closure u is called the sum closure. Next, the sum closure can be characterized as the finest closure for the sum P of underlying sets rendering all mappings $\operatorname{inj}_a : \langle P_a, u_a\rangle \to P$ continuous. Sum closures are examined in subsection B.

In the subsection C products of spaces are studied. The product of a family $\{\langle P_a, u_a\rangle\}$ of spaces, denoted by $\Pi\{\langle P_a, u_a\rangle\}$, can be defined as the product P of the family $\{P_a\}$ of underlying sets endowed with a closure operation u, called the product closure, such that the following two equivalent conditions are fulfilled:

(a) a net $\langle N, \leqq\rangle$ converges to x in $\langle P, u\rangle$ if and only if the net $\langle \operatorname{pr}_a \circ N, \leqq\rangle$ converges to the point $\operatorname{pr}_a x$ in $\langle P_a, u_a\rangle$ for each a,

(b) u is the coarsest closure for P such that all mappings $\mathrm{pr}_a : \langle P, u\rangle \to \langle P_a, u_a\rangle$ are continuous.

It is to be noticed that we shall define the product closure by specifying neighborhood of points and statements (a) and (b) will be theorems. Next, the product closure is not the only appropriate closure for P. In Section 33 we shall introduce the inductive product of a family of spaces and in Section 35 we shall introduce the sequential product of certain spaces (those which can be described by the convergence of sequences). – Here, in subsection D, the inductive product $\mathrm{ind}(\langle P, u\rangle \times \langle Q, v\rangle)$ of two spaces $\langle P, u\rangle$ and $\langle Q, v\rangle$ will be defined to be the finest closure w for $P \times Q$ such that all mappings

$$\{x \to \langle x, y\rangle\} : \langle P, u\rangle \to \langle P \times Q, w\rangle, y \in Q$$

and

$$\{y \to \langle x, y\rangle\} : \langle Q, v\rangle \to \langle P \times Q, w\rangle, x \in P$$

are continuous. The significance of the inductive products lies in the fact that a mapping $f : \mathrm{ind}(\langle P, u\rangle \times \langle Q, v\rangle) \to \langle R, w\rangle$ is continuous if and only if all mappings $\{x \to f\langle x, y\rangle\} : \langle P, u\rangle \to \langle R, w\rangle, y \in Q$, and $\{y \to f\langle x, y\rangle\} : \langle Q, v\rangle \to \langle R, w\rangle, x \in$ $\in P$, are continuous, that is, if and only if, f is "separately continuous", or continuous "separately in each variable". If $\mathscr{U}$ is a local base at x in $\langle P, u\rangle$ and $\mathscr{V}$ is a local base at y in $\langle Q, v\rangle$, then $[\mathscr{U}] \times [\mathscr{V}] (= \mathbf{E}\{U \times V \mid U \in \mathscr{U}, V \in \mathscr{V}\})$ is a local base at $\langle x, y\rangle$ in $\langle P, u\rangle \times \langle Q, v\rangle$, while the collection of all "crosses" $(U \times (y)) \cup ((x) \times V)$ is a local base at $\langle x, y\rangle$ in $\mathrm{ind}(\langle P, u\rangle \times \langle Q, v\rangle)$.

A. SUBSPACES

17 A.1. Definition. Let $\langle P, u\rangle$ be a closure space and let $Q \subset P$. It is easy to verify that the relation $v = \{X \to Q \cap uX\}$ on $\exp Q$ ranging in $\exp Q$ is a closure operation for Q. The closure v is called the *relativization of u to Q* and the space $\langle Q, v\rangle$ is called a *subspace of* $\langle P, u\rangle$. Thus a subspace of a space is uniquely determined by the underlying set. If no confusion is likely to result and if P is a space, then a subspace of P is denoted by a single letter, say Q, the closure of a set $X \subset Q$ in Q is denoted $\overline{X}^Q$ and called the *relative closure* of X. It is clear what we mean by a *relatively open* or *relatively closed set*, a *relative neighborhood*, etc.

A class of closure spaces is said to be *hereditary* if, with each space $\mathscr{P}$, it contains all subspaces of $\mathscr{P}$.

For example, every subspace of a discrete space is discrete and every subspace of an accrete space is accrete. The relativization of a closure u for a set P to a subset Q of P is the coarsest closure v for Q such that the identity mapping of $\langle Q, v\rangle$ into $\langle P, u\rangle$ is continuous. More precisely:

17 A.2. *Let $\langle P, u\rangle$ be a closure space and let $Q \subset P$. A closure v for Q is the relativization of u to Q if and only if the following two conditions are fulfilled:*

(a) *The mapping* $J_Q : \langle Q, v \rangle \to \langle P, u \rangle$ *is continuous*;

(b) *if* w *is a closure for* Q *such that* $J_Q : \langle Q, w \rangle \to \langle P, u \rangle$ *is continuous, then* w *is finer than* v.

Proof. I. First suppose that v is the relativization of u to Q, i.e., $vX = Q \cap uX$ for each $X \subset Q$. The condition (a) is fulfilled because $J_Q[vX] = vX \subset uX$ for each $X \subset Q$. If $J_Q : \langle Q, w \rangle \to \langle P, u \rangle$ is continuous, then, by definition, $J_Q[wX] \subset u\, J_Q[X]$ for each $X \subset Q$, i.e. $wX \subset uX$ for each $X \subset Q$; but $wX \subset Q$ and hence $wX \subset Q \cap$ $\cap\, uX$ for each $X \subset Q$, that is, $wX \subset vX$ for each $X \subset Q$, which shows that w is finer than v. – II. Now let v be a closure for Q satisfying conditions (a) and (b). The relativization v' of u to Q fulfils conditions (a) and (b) by I. From (b) we obtain that v is finer than v' and v' is finer than v which implies $v = v'$.

The following two straightforward consequences of the definition will usually be used without references.

17 A.3. *Let* $\mathcal{Q}$ *be a subspace of a space* $\mathcal{P}$. *Then*

(a) *if* $\mathcal{P}$ *is topological, then* $\mathcal{Q}$ *is also topological*;

(b) *if the underlying set of a space* $\mathcal{R}$ *is contained in that of* $\mathcal{Q}$, *then* $\mathcal{R}$ *is a subspace of* $\mathcal{Q}$ *if and only if it is a subspace of* $\mathcal{P}$.

17 A.4. *Let* $\mathcal{Q}$ *be a subspace of a closure space* $\mathcal{P}$. *Then*

(a) *If* Y *is closed in* $\mathcal{P}$, *then* $Y \cap |\mathcal{Q}|$ *is closed in* $\mathcal{Q}$. *If* $\mathcal{P}$ *is topological, then every closed subset of* $\mathcal{Q}$ *is of the form* $|\mathcal{Q}| \cap Y$ *with* Y *closed in* $\mathcal{P}$.

(b) *If* Y *is open in* $\mathcal{P}$ *then* $Y \cap |\mathcal{Q}|$ *is open in* $\mathcal{Q}$. *If* $\mathcal{P}$ *is topological, then every open set of* $\mathcal{Q}$ *is of the form* $|\mathcal{Q}| \cap Y$ *with* Y *open in* $\mathcal{P}$.

(c) *If* $|\mathcal{Q}|$ *is closed (open) in* $\mathcal{P}$ *and* X *is closed (open) in* $\mathcal{Q}$, *then* X *is closed (open) in* $\mathcal{P}$.

Proof. Evidently assertions (a) and (b) are equivalent. Write $P = |\mathcal{P}|$, $Q = |\mathcal{Q}|$. We shall prove (a) and (c). If Y is closed in P, that is, if $\overline{Y} = Y$, then $\overline{Y \cap Q}^Q = Q \cap \overline{Y \cap Q} \subset Q \cap Y$. Thus $Q \cap Y$ is relatively closed. If P is topological and X is closed in Q, then $\overline{X}$ is closed in P and $\overline{X} \cap Q = X$. Thus every relatively closed set is the intersection of a closed set with Q. If Q is closed in P and $X \subset Q$ is relatively closed, that is, if $\overline{Q} = Q$ and $\overline{X}^Q = \overline{X} \cap Q = X$, then $\overline{X} \subset \overline{Q} = Q$ and finally $\overline{X} = \overline{X} \cap Q = \overline{X}^Q = X$. Finally if Q is open and X is relatively open, i.e. $\overline{P - Q} \subset$ $\subset P - Q$ and $\overline{Q - X} \cap Q \subset Q - X$, then $\overline{P - X} = \overline{P - Q} \cup \overline{Q - X} \subset P - X$.

If $\mathcal{P}$ is not a topological space, then a relatively closed (open) set need not be the intersection of the subspace and a closed (open) set. Actually, the following rather general result is true.

17 A.5. *In order that a closure space* P *be topological, it is necessary and sufficient that for each subspace* Q *of* P *every relatively closed (open) set be of the form* $Q \cap Y$ *with* Y *closed (open) in* P.

Proof. The condition is necessary by 17 A.4 (a). Suppose that a space P is not topological. Then there exists a subset X of P such that $\overline{\overline{X}} \neq \overline{X}$. Consider the subspace $Q = X \cup (\overline{\overline{X}} - \overline{X})$ of P. Clearly X is closed in Q. Nevertheless, if Y is closed in P and $Y \supset X$, then $Y \supset \overline{\overline{X}}$ and hence $Y \cap Q = Q$; it follows that $Q \cap Y \neq X$. Clearly the set $Q - X$ is relatively open and $Q - X = Q \cap U$ for no open set U.

Let $\langle Q, v \rangle$ be a subspace of a closure space $\langle P, u \rangle$ and consider the topological modification $\langle Q, \tau v \rangle$ of $\langle Q, v \rangle$ and the subspace $\langle Q, v^* \rangle$ of $\langle P, \tau u \rangle$. The space $\langle Q, v^* \rangle$ is topological as a subspace of a topological space (by 17 A.3 (a)). By 17 A.4 (b) the collection $\mathscr{V}^*$ of all v^*-open sets consists of all sets of the form $Q \cap U$, U open in $\langle P, \tau u \rangle$; but τu is the topological modification of u, and hence U is open in $\langle P, \tau u \rangle$ if and only if it is open in $\langle P, u \rangle$. By 17 A.4 (b) each set from $\mathscr{V}^*$ is open in $\langle Q, v \rangle$ and hence in $\langle Q, \tau v \rangle$. Both closures v^* and τv are topological, and therefore, if $\mathscr{V}$ denotes the collection of all v-open sets, then $v^* = \tau v$ if and only if $\mathscr{V}^* = \mathscr{V}$, and v^* is coarser than τv if and only if $\mathscr{V}^* \subset \mathscr{V}$ (by 15 A.7). Thus we have proved the following theorem.

17 A.6. Theorem. *Let $\langle Q, v \rangle$ be a subspace of a closure space $\langle P, u \rangle$. The relativization v^* of τu to Q is always coarser than τv. In order that $\tau v = v^*$ it is necessary and sufficient that every open (closed) subset of $\langle Q, v \rangle$ be of the form $Q \cap U$ with U open (closed) in $\langle P, u \rangle$.*

Combining 17 A.5 and 17 A.6 we obtain the following proposition.

17 A.7. *In order that a closure space P be topological it is necessary and sufficient that for each subspace Q of P the topological modification τQ of τQ be a subspace of τP.*

It is to be noted that for some subspaces Q of P the topological modification τQ of Q may be a subspace of τP even if P is not topological. The following proposition describes a wide class of such subspaces.

17 A.8. *Let Q be a subspace of a closure space P such that $Q = U \cap C$, where U is open in P and C is closed in P. Then every closed (open) subset X of Q is of the form $Q \cap Y$ with Y closed (open) in P. By* 17 A.6 *τQ is a subspace of τP. (Notice that the condition is fulfilled if Q is open or closed in P.)*

Proof. Let X be closed in Q. It is sufficient to find a closed subset Y of C such that $Y \cap Q = X$. Evidently the set Q is open in C and hence $C - Q$ is closed in C. Put $Y = X \cup (C - Q)$. Since clearly $\overline{Y}^C = \overline{X}^C \cup (C - Q) = X \cup (C - Q) = Y$, Y is closed in C, and hence in P (17 A.4 (c)), and obviously $Y \cap Q = X$. If V is open in Q, then $X = Q - V$ is closed in Q and, as we have just proved, there exists a closed set Y in P with $Y \cap Q = X = Q - V$. Clearly $P - Y$ is open in P and $Q \cap (P - Y) = V$.

Now we proceed to various descriptions of relativized closure operations.

17 A.9. Theorem. *Let $\langle Q, v \rangle$ and $\langle P, u \rangle$ be closure spaces such that $Q \subset P$. Each of the following conditions is necessary and sufficient for $\langle Q, v \rangle$ to be a subspace of $\langle P, u \rangle$:*

(a) *for each* $X \subset Q$, $\operatorname{int}_v X = Q \cap \operatorname{int}_u (X \cup (P - Q))$;

(b) *if* $x \in Q$, *then a set* $V \subset Q$ *is a neighborhood of* x *in* $\langle Q, v \rangle$ *if and only if there exists a neighborhood* U *of* x *in* $\langle P, u \rangle$ *such that* $U \cap Q = V$;

(c) *if* N *is a net in* Q *and* $x \in Q$, *then* x *is a limit point (an accumulation point) of* N *in* $\langle P, u \rangle$ *if and only if* x *is a limit point (an accumulation point) of* N *in* $\langle Q, v \rangle$.

Proof. The pattern of the proof will be: (a) is necessary, (a) $\Rightarrow$ (b) $\Rightarrow$ (c) and (c) is sufficient. – I. Suppose that $\langle Q, v \rangle$ is a subspace and $X \subset Q$. By definition of the interior operation we have $\operatorname{int}_u (X \cup (P - Q)) = P - u(P - (X \cup (P - Q)) = P - u(Q - X)$ and consequently

$$Q \cap \operatorname{int}_u (X \cup (P - Q)) = Q - u(Q - X) = Q - (Q \cap u(Q - X)) = \\ = Q - v(Q - X) = \operatorname{int}_v X .$$

Thus (a) is necessary. – II. Now suppose (a). If V is a neighborhood of an $x \in Q$ in $\langle Q, v \rangle$, then by (a) the set $U = V \cup (P - Q)$ is a neighborhood of x in $\langle P, u \rangle$ and clearly $U \cap Q = V$. Conversely, if U is a neighborhood of x in $\langle P, u \rangle$, then also the set $U \cup (P - Q)$ is a neighborhood of x in $\langle P, u \rangle$ and by (a) the set $(U \cup (P - Q)) \cap Q = U \cap Q$ is a neighborhood of x in $\langle Q, v \rangle$. Thus (a) $\Rightarrow$ (b). – III. Since the implication (b) $\Rightarrow$ (c) is almost self-evident, it remains to show that (c) is sufficient. – IV. Suppose (c) and let $x \in Q$, $X \subset Q$. If $x \in vX$, then there exists a net N in X such that x is a limit (an accumulation) point of N in $\langle Q, v \rangle$ (by 15 B.4). By the condition, x is also a limit (an accumulation) point of N in $\langle P, u \rangle$, and hence $x \in uX$ by 15 B.4. Thus $vX \subset (Q \cap uX)$. Conversely, if $x \in uX$, then there exists a net N in X such that x is a limit (an accumulation) point of N in $\langle P, u \rangle$. By the condition x is also a limit (an accumulation) point of N in $\langle Q, v \rangle$ and hence (by 15 B.4) $x \in vX$. Thus $vX \supset (Q \cap uX)$.

17 A.10. Corollary. *Let* Q *be a subspace of a closure space* P. *If* $\mathcal{U}$ *is a local base (a local sub-base) at a point* $x \in Q$ *in* P, *then* $[\mathcal{U}] \cap Q = \mathsf{E}\{U \cap Q \mid U \in \mathcal{U}\}$ *is a local base (a local sub-base) at* x *in* Q. *As a consequence, if* $\mathcal{U}$ *is a base (a sub-base) of the neighborhood system in* P *of a set* $X \subset Q$, *then* $[\mathcal{U}] \cap Q$ *has the same property in* Q.

Let us state the following immediate consequence of 17 A.3 and 17 A.4.

17 A.11. *Let* Q *be a subspace of a topological space* P. *If* $\mathcal{U}$ *is the collection of all open subsets of* P *or an open base or an open sub-base for* P, *then* $[\mathcal{U}] \cap P$ *has the same property in* Q.

Sometimes we have defined a closure for a set by specifying neighborhoods of points. Often it will be convenient to define a closure u for a set P by specifying some subspaces of $\langle P, u \rangle$, neighborhoods of some points and closures of some sets in such a manner that the closure u will be uniquely determined. E. g., let P be a set, $x \in P$ and let $\mathcal{U}$ be a filter base in P with $x \in \bigcap \mathcal{U}$. By 14 B.10 there exists exactly one closure u for P such that $\mathcal{U}$ is a local base at x in $\langle P, u \rangle$ and, for each $y \in P - (x)$, the filter base $((y))$ is a local base at y in $\langle P, u\rangle$. Clearly $P - (x)$ is an open discrete subspace of $\langle P, u\rangle$, that is, $P - (x)$ is an open subset of $\langle P, u\rangle$ and the subspace $P - (x)$ of $\langle P, u\rangle$ is discrete. It is easily seen that the closure u can be defined as follows: u is the closure for P such that $P - (x)$ is an open discrete subspace of $\langle P, u\rangle$ and $\mathscr{U}$ is a local base at x in $\langle P, u\rangle$. For another example, let P be the union of two disjoint infinite sets P_1 and P_2 and let $x_i \in P_i$, $i = 1, 2$. Then there exists exactly one closure u for P such that P_1 and P_2 are discrete subspaces of $\langle P, u\rangle$ and an $x \in P_j$ belongs to the closure of a subset X of P_i, $i \neq j$, if and only if X is infinite and $x = x_j$.

Having defined the notion of a subspace of a closure space, in accordance with 7 B.5 we can define the restriction of a mapping for closure spaces as follows:

17 A.12. Definition. The *restriction of a mapping f for closure spaces* is a mapping g for closure spaces such that the underlying abstract mapping $|g|$ ($= \langle \mathrm{gr}\, g, |\mathbf{D}^*g|, |\mathbf{E}^*g|\rangle$) is a restriction of $|f|$ ($= \langle \mathrm{gr}\, f, |\mathbf{D}^*f|, |\mathbf{E}^*f|\rangle$) and $\mathbf{D}^*g$ and $\mathbf{E}^*g$ are subspaces of closure spaces $\mathbf{D}^*f$ and $\mathbf{E}^*f$ respectively. If $\mathbf{D}^*g = \mathbf{D}^*f$, then g is called a *range-restriction* of f, if $\mathbf{E}^*g = \mathbf{E}^*f$, then g is called a *domain-restriction* of f. If g is a restriction of f, then g is said to be the *restriction of f to a mapping of $\mathbf{D}^*g$ into $\mathbf{E}^*g$, if g* is a domain-restriction (range-restriction) of f, then g is said to be the *domain-restriction* (the *range-restriction*) *of f to $\mathbf{D}^*g$* ($\mathbf{E}^*g$). The domain-restriction of f to $\mathscr{P}$ is denoted by $f \mid \mathscr{P}$; thus $f \mid \mathscr{P} = \langle \mathrm{gr}\, f \mid |\mathscr{P}|, \mathscr{P}, \mathbf{E}^*f\rangle$. A mapping f is said to be an *extension* (*domain-extension*, *range-extension*) *of a mapping g for closure spaces* if and only if g is a restriction (domain-restriction, range-restriction) of f.

The result which follows will be used frequently in the sequel and therefore it will be proved in detail even though it is almost evident.

17 A.13. Theorem. *Every restriction of a continuous mapping is a continuous mapping. A mapping f for closure spaces is continuous if and only if the range-restriction of f to the subspace $\mathbf{E}f$ of $\mathbf{E}^*f$ is continuous.*

We shall need the following more general result:

17 A.14. *If g is the restriction of a mapping f for closure spaces and if f is a continuous at a point x of $\mathbf{D}g$, then g is also continuous at x. A mapping f for closure spaces is continuous at a point x if and only if the range-restriction g of f to the subspace $\mathbf{E}f$ of $\mathbf{E}^*f$ is continuous at x.*

Proof. I. Suppose that g is the restriction of f and f is continuous at an $x \in \mathbf{D}g$. To prove that g is continuous at x, by virtue of 16 A.4 it is sufficient to show that $g^{-1}[V]$ is a neighborhood of x in $\mathbf{D}^*g$ whenever V is a neighborhood of gx in $\mathbf{E}^*g$. Let V be a neighborhood of gx in $\mathbf{E}^*g$. Since $\mathbf{E}^*g$ is a subspace of $\mathbf{E}^*f$, by 17 A.9 we can choose a neighborhood U of gx in $\mathbf{E}^*f$ so that $V = U \cap \mathbf{E}^*g$. Since $\mathrm{gr}\, g = \mathrm{gr}\, f \cap (\mathbf{D}g \times \mathbf{E}g)$ we obtain $g^{-1}[V] = f^{-1}[U] \cap \mathbf{D}^*g$. Since f is continuous at x, $f^{-1}[U]$ is a neighborhood of x in $\mathbf{D}^*f$, and hence, $\mathbf{D}^*g$ being a subspace of $\mathbf{D}^*f$, the right side of the above equality is a neighborhood of x in $\mathbf{D}^*g$; this completes the proof. — II. The "only if" part of the second statement is a particular case of the first statement. To prove the "if" part, suppose that g is continuous at x and U is any neighborhood of $fx(= gx)$ in $\mathbf{E}^*f$. The set $U \cap \mathbf{E}f$ is a neighborhood of gx in the subspace $\mathbf{E}f$ of $\mathbf{E}^*f$ and hence $g^{-1}[U \cap \mathbf{E}f]$ is a neighborhood of x in $\mathbf{D}^*g = \mathbf{D}^*f$. But clearly $f^{-1}[U] = g^{-1}[U \cap \mathbf{E}f]$.

17 A.15. Corollary to 17 A.13. *Every range-extension of a continuous mapping is continuous.*

On the other hand, the domain-extension of a continuous mapping need not be continuous. Indeed, every mapping for closure spaces is the domain-extension of a continuous mapping (observe that every mapping of a one-point space into any space is continuous).

Let f be a mapping of a space P into a space Q. If $\{X_a \mid a \in A\}$ is a cover of P (that is, by 12 A.1, a family of subsets of $|P|$ the union of which is $|P|$) and if, for each a, the restriction of f to the subspace X_a of P is continuous, then clearly f need not be continuous, but under certain additional assumptions (on the cover $\{X_a\}$) the mapping f is necessarily continuous.

17 A.16. Theorem. *Let $\{X_a \mid a \in A\}$ be a locally finite cover of a space P. If f is a mapping of P into a space Q such that the domain-restriction of f to each subspace $\overline{X}_a$ is continuous, then f is continuous.*

Proof. Let us suppose that $X \subset P$, $x \in \overline{X}$. We must show that $fx \in \overline{f[X]}$. Since $\{X_a\}$ is a cover of P, the family $\{X \cap X_a\}$ is a cover of X, and being locally finite (14 B.16) and hence closure-preserving (14 B.18), x belongs to the closure of some set $X \cap X_a$. Since the domain-restriction g of f to the subspace $R = \overline{X \cap X_a}$ is continuous and $x \in \overline{X \cap X_a^R}$ $(= \overline{X \cap X_a^P})$, we obtain $gx \in \overline{g[X \cap X_a]}$; but $gx = fx$ and $f[X \cap X_a] = g[X \cap X_a]$, and hence $fx \in \overline{f[X \cap X_a]}$, which implies $fx \in \overline{f[X]}$.

It is to be noted that the assumption that the domain-restriction to each $\overline{X}_a$ is continuous cannot be replaced by the weaker assumption "the domain-restriction to each X_a is continuous". For example, if P is an accrete two-point space, $A = |P|$, $X_a = (a)$, then for each mapping f the domain-restrictions are continuous; however, obviously there exists a mapping f of P which is not continuous (for instance we can take the identity mapping of P onto $|P|$ endowed with the discrete closure).

Next, the assumption that $\{X_a\}$ is locally finite cannot be replaced by the weaker assumption that $\{X_a\}$ is closure-preserving. For example, let P be an infinite space with exactly one cluster point, say x, $A = |P|$, $X_a = (a, x)$ for each a. It is easily seen that P is not discrete and hence there exists a mapping of P into a space which is not continuous, $\{X_a\}$ is a closure-preserving family of closed sets and each subspace X_a of P is discrete, and consequently every mapping of X_a into any space is continuous.

17 A.17. Definition. A cover $\{X_a\}$ of a closure space P will be called *closed* (*open*) if each X_a is closed (open). An *interior cover* of a space P is a cover $\{X_a\}$ of P such that $\{\operatorname{int} X_a\}$ is a cover of P.

For example, every open cover is an interior cover. As a corollary of 17 A.16 we obtain the following extremely useful theorem.

17 A.18. *If $\{X_a\}$ is a locally finite closed cover of a space P and if f is a mapping of a space P into a space such that all restrictions $f \mid X_a$ are continuous, then f is continuous.*

17 A.19. *If $\{X_a\}$ is an interior cover of a space P and f is a mapping of P into a space Q such that all domain-restrictions $f \mid X_a$ are continuous, then f is continuous.*

Proof. Let $x \in P$ and let U be a neighborhood of fx. Choose an index a so that $x \in \operatorname{int} X_a$. Since $f \mid X_a$ is continuous, there exists a neighborhood V of x in X_a so that $(f \mid X_a)[V] \subset U$, and hence $f[V] \subset U$. It is easily seen that V is a neighborhood of x in P. By 16 A.4 f is continuous.

17 A.20. Definition. An *embedding* of a space P into a space Q is a mapping f of P into Q such that the range-restriction of f to $\mathbf{E}f$ is a homeomorphism. We shall say that a space P *admits an embedding* into a space Q if there exists an embedding of P into Q.

Thus every embedding for closure spaces is an injective mapping. If $Q \subset P$ then the identity mapping of a space $\langle Q, v\rangle$ into a space $\langle P, u\rangle$ is an embedding if and only if $\langle Q, v\rangle$ is a subspace of $\langle P, u\rangle$. From the corresponding definitions one can derive the following direct description of embeddings which will usually be used without reference in the sequel.

17 A.21. *A mapping f of a space $\langle P, u\rangle$ into a space $\langle Q, v\rangle$ is an embedding if and only if the following two conditions are fulfilled:*

(a) *f is injective; and*
(b) *for each $X \subset P$, $f[uX] = vf[X] \cap \mathbf{E}f$.*

In conclusion we shall examine subspaces of ordered spaces. Let u be a generalized ordered closure for a monotone ordered set $\langle P, \leqq\rangle$, Q a subset of P, v the relativization of u to Q and $\leqq_Q$ the restriction of $\leqq$ to Q. We shall prove that v is a generalized order closure for $\langle \mathcal{Q}, \leqq_Q\rangle$, and if u is the order closure, then v need not be the order closure. Finally, if v is a generalized order closure for an ordered set $\langle Q, \leqq_Q\rangle$, then there exists an ordered set $\langle P, \leqq\rangle$ such that $\langle Q, \leqq_Q\rangle$ is an ordered subset of $\langle P, \leqq\rangle$ and v is the relativization of the order closure for $\langle P, \leqq\rangle$.

17 A.22. Theorem. *A subspace $\langle Q, v\rangle$ of a generalized ordered space $\langle P, u\rangle$ is a generalized ordered space; if $\leqq$ is a monotone order for P such that u is a generalized order closure for $\langle P, \leqq\rangle$, then v is a generalized order closure for $\langle Q, \leqq_Q\rangle$ where $\langle Q, \leqq_Q\rangle$ is an ordered subset of $\langle P, \leqq\rangle$.*

Proof. For each x in Q we must find a local base at x in $\langle Q, v \rangle$ consisting of intervals in $\langle Q, \leqq_Q \rangle$. Fix an x in Q and let us consider the set $\mathscr{U}$ of all intervals in $\langle P, \leqq \rangle$ which are neighborhoods of x in $\langle P, u \rangle$. If $U \cap Q = (x)$ for some U in $\mathscr{U}$, then clearly $((x))$ is a local base at x in $\langle Q, v \rangle$, and evidently (x) is an interval in $\langle Q, \leqq_Q \rangle$, namely the closed interval with end-points x. In the remaining cases we shall need the evident fact that $U \cap Q$ is an interval in $\langle Q, \leqq_Q \rangle$ whenever U is an interval in $\langle P, \leqq \rangle$ with end-points belonging to Q. Suppose that $U \cap Q \neq (x)$ for each U in $\mathscr{U}$. If there exists a U in $\mathscr{U}$ such that $U \cap Q \subset [\![x, \rightarrow [\![$, then for each $V \in \mathscr{U}$, $V \subset U$, there exists an interval $W \in \mathscr{U}$ such that $W \cap Q$ is an interval in $\langle Q, \leqq_Q \rangle$ with end-points x and $y \in Q$; indeed we can choose any $y \in Q$, such that $x < y \in V$. Similarly in the case $U \cap Q \subset]\!] \leftarrow, x]\!]$ for some U in $\mathscr{U}$. In the remaining case, for each U in $\mathscr{U}$ we can choose $z \in (Q \cap U)$, $y \in (Q \cap U)$ such that $z < < x < y$; clearly the interval $]\!] z, y [\![$ belongs to $\mathscr{U}$ and its intersection with Q is contained in $U \cap Q$.

It is to be noted that the statement becomes false if the expressions "generalized order closure" are replaced by expressions "order closure"; e.g. if $\langle P, \leqq \rangle$ is the ordered set of reals and $Q =]\!] \leftarrow, 0]\!] \cup]\!] 1, \rightarrow [\![$, then the order closure w for $\langle Q, \leqq_Q \rangle$ is strictly coarser than the relativization v of the order closure for $\langle P, \leqq \rangle$; actually, $0 \in w\,]\!] 1, \rightarrow [\![$ but $0 \notin v\,]\!] 1, \rightarrow [\![$.

17 A.23. Theorem. *Every generalized order closure v is a relativization of an order closure u; in addition, if v is a generalized order closure for an ordered set $\langle Q, \leqq_Q \rangle$ then $\langle Q, \leqq_Q \rangle$ is an ordered subset of an ordered set $\langle P, \leqq \rangle$ such that v is a relativization of the order closure for $\langle P, \leqq \rangle$.*

Proof. Let $\langle R, \prec \rangle$ be the lexicographic product of the ordered set $\langle Q, \leqq_Q \rangle$ and the ordered set S of integers (the proof remains true with $S = (-1, 0, 1)$) (the order is not indicated), that is, $R = Q \times S$ and $\langle x, n \rangle \prec \langle y, m \rangle$ if and only if either $x \leqq_Q y$, $x \neq y$, or $x = y$ and m follows n in S. Let T be a subset of R (thus T is a relation) such that $T[(x)] = S$ if x is isolated in $\langle Q, \leqq_Q, v \rangle$, $T[(x)] = \mathsf{N}$ if x is right-isolated but not left-isolated in $\langle Q, \leqq_Q, v \rangle$, $T[(x)] = (S - \mathsf{N}) \cup (0)$ if x is left-isolated but not right-isolated in $\langle Q, \leqq_Q, v \rangle$ and $T[(x)] = (0)$ in the remaining case, i.e. if x is neither left-isolated nor right-isolated. Let $\langle T, \prec_T \rangle$ be an ordered subset of $\langle R, \prec \rangle$, u' be the order closure for $\langle T, \prec_T \rangle$ and v' be the relativisation of v to the subset $Q \times (0)$ of T. It is easily seen that the mapping $\{x \rightarrow \langle x, 0 \rangle\} : \langle Q, \leqq_Q \rangle \rightarrow \langle T, \prec_T \rangle$ is order-preserving and the mapping $\{x \rightarrow \langle x, 0 \rangle\} : \langle Q, v \rangle \rightarrow \langle Q, v' \rangle$ is a homeomorphism. Now the reader will have no difficulties in constructing $\langle P, \leqq \rangle$; it is enough to find a set $P \supset Q$ and a one-to-one mapping f of P onto T such that $fx = \langle x, 0 \rangle$ for each $x \in Q$.

Remark. A generalized order closure for a set P need not be an order closure for P; more precisely, if u is a generalized order closure for $\langle P, \leqq \rangle$ then there need not exist an order $\prec$ for P such that u is an order closure for $\langle P, \prec \rangle$. In Section 20 we shall show that the closure structure of the subspace $[\![0, 1]\!] \cup]\!]2, 3[\![$ of the space R of reals has this property.

B. SUMS OF SPACES

Let us recall that the sum of a family $\{X_a \mid a \in A\}$ of sets is the set $\Sigma\{X_a\}$ of all pairs $\langle a, x\rangle$ such that $a \in A$ and $x \in X_a$. If $Y \subset \Sigma\{X_a\}$, then there exists exactly one family $\{Y_a\}$ such that $Y = \Sigma\{Y_a\}$. Next, for each element a, the injection relation at a, denoted by inj_a, is the single-valued relation which assigns to each element x of X_a the pair $\langle a, x\rangle$. Thus we can write

$$\Sigma\{X_a\} = \bigcup\{\mathrm{inj}_a [X_a]\} .$$

17 B.1. Definition. The *sum of a family* $\{\langle P_a, u_a\rangle \mid a \in A\}$ *of closure spaces*, denoted by $\Sigma\{\langle P_a, u_a\rangle \mid a \in A\}$, is defined to be the space $\langle P, u\rangle$ where P is the sum of the family of sets $\{P_a\}$ and u is the *sum closure*, denoted sometimes by $\Sigma\{u_a\}$ and defined by

$$(\Sigma\{u_a\})\, \Sigma\{X_a\} = \Sigma\{u_a X_a\}$$

for each subset $X = \Sigma\{X_a\}$ of P. Thus

$$\Sigma\{\langle P_a, u_a\rangle\} = \langle \Sigma\{P_a\}, \Sigma\{u_a\}\rangle .$$

To show that the sum closure is well-defined, we must prove that the sum closure is actually a closure operation, i.e. that it fulfils the conditions (cl i) of Definition 14 A.1. Since $\Sigma\{X_a\}$ is empty if and only if each X_a is empty, condition (cl 1) is true. Since $\Sigma\{X_a\} \subset \Sigma\{Y_a\}$ if and only if $X_a \subset Y_a$ for each a, we obtain condition (cl 2). Finally, additivity follows from the formula $\Sigma\{X_a \cup Y_a\} = (\Sigma\{X_a\}) \cup (\Sigma(Y_a))$.

17 B.2. Theorem. *Let* $\langle P, u\rangle$ *be the sum of a family* $\{\langle P_a, u_a\rangle \mid a \in A\}$ *of closure spaces. Then*

(a) *the mapping* $\mathrm{inj}_a : \langle P_a, u_a\rangle \to \langle P, u\rangle$, *called the canonical embedding of* $\langle P_a, u_a\rangle$ *into* $\langle P, u\rangle$, *is an embedding for each* a *in* A,

(b) *the set* $\mathrm{inj}_a [P_a]$ *is simultaneously open and closed in* $\langle P, u\rangle$ *for each* a *in* A,

(c) *if all* $\langle P_a, u_a\rangle$ *are topological, then so is* $\langle P, u\rangle$,

(d) *if all* $\langle P_a, u_a\rangle$ *are discrete, then so is* $\langle P, u\rangle$.

Proof. Let i_a denote the mapping $\mathrm{inj}_a : \langle P_a, u_a\rangle \to \langle P, u\rangle$. — I. Clearly i_a is injective and $i_a[u_a X] = u\, i_a[X] = \mathbf{E} i_a \cap u\, i_a[X]$, which shows that i_a is an embedding (by 17 A.21). — II. Obviously each set $\mathrm{inj}_a [P_a]$ is closed. Since clearly the family $\{\mathrm{inj}_a [P_a]\}$ is disjoint and closure-preserving, each set $\mathrm{inj}_a [P_a]$ is open because its complement is closed as the union of a closure-preserving family of closed sets (14 A.7), namely $\{\mathrm{inj}_b [P_b] \mid b \in A - (a)\}$ — III. If all $\langle P_a, u_a\rangle$ are topological then $uu\, \Sigma\{X_a\} = u\, \Sigma\{u_a X_a\} = \Sigma\{u_a u_a X_a\} = \Sigma\{u_a X_a\} = u\, \Sigma\{X_a\}$ which shows that u is topological. — IV. If all $\langle P_a, u_a\rangle$ are discrete, then $u\, \Sigma\{X_a\} = \Sigma\{u X_a\} = \Sigma\{X_a\}$ which shows that u is also discrete.

Now we shall prove that, roughly speaking, the sum closure is the finest closure for the sum of underlying sets such that all injections are continuous. This description of sum closure may be compared with the description 17 A.2 of relativized closures.

17 B.3. *Let* $\{\langle P_a, u_a\rangle \mid a \in A\}$ *be a family of spaces. A closure* u *for* $P = \Sigma\{P_a\}$ *is the sum closure if and only if the following two conditions are fulfilled*:

(a) *All mappings* $\mathrm{inj}_a : \langle P_a, u_a\rangle \to \langle P, u\rangle$ *are continuous.*

(b) *If* v *is a closure for* P *such that all mappings* $\mathrm{inj}_a : \langle P_a, u_a\rangle \to \langle P, v\rangle$ *are continuous, then* v *is coarser than* u.

Proof. It follows from (b) that there exists at most one closure u for P satisfying the conditions (a) and (b). It remains to show that the sum closure u fulfils conditions (a) and (b). Condition (a) is fulfilled by 17 B.2 (a). If v fulfils the assumptions of (b), then clearly

$$v(\Sigma\{X_a\}) \supset \bigcup\{v\ \mathrm{inj}_a X_a\} \supset \Sigma\{u_a X_a\} = u\ \Sigma\{X_a\}$$

which shows that v is coarser than u.

In conclusion we shall prove an important description of the continuity of a mapping of a sum of spaces, which is a consequence of 17 B.3 but which will be proved directly here.

17 B.4. Theorem. *A mapping* f *of a sum space* $\langle P, u\rangle = \Sigma\{\langle P_a, u_a\rangle\}$ *into a space* $\langle Q, v\rangle$ *is continuous if and only if each composition*

(*) $f \circ (\mathrm{inj}_a : \langle P_a, u_a\rangle \to \langle P, u\rangle)$

is continuous.

Proof. If f is continuous then all mappings of (*) are continuous as compositions of continuous mappings. Conversely, suppose that all mappings of (*) are continuous. If $X = \Sigma\{X_a\}$ is any subset of P, then $f[X] = \bigcup\{f \circ \mathrm{inj}_a [X_a]\}$, $f[uX] = \bigcup\{f \circ \mathrm{inj}_a [u_a X_a]\}$ and $f \circ \mathrm{inj}_a [u_a X_a] \subset v(f \circ \mathrm{inj}_a [X_a])$ (by the continuity of mappings of (*)), so that we obtain

$$v\, f[X] \supset \bigcup\{v(f \circ \mathrm{inj}_a [X_a])\} \supset \bigcup\{f \circ \mathrm{inj}_a [u_a X_a]\} = f[uX]$$

which establishes the continuity of f. An alternate proof of the continuity of f can be obtained from 17 A.19. Indeed, each set $\mathrm{inj}_a [P_a]$ is open by 17 B.2 and clearly the domain-restriction of f to each subspace $\mathrm{inj}_a [P_a]$ is continuous because $\mathrm{inj}_a : \langle P_a, u_a\rangle \to \mathrm{inj}_a [P_a]$ is a homeomorphism.

It is to be noted that all results of this subsection become corollaries of the results of Section 33 devoted to the inductive construction of closure spaces.

In conclusion we shall introduce the concepts of the sum and the reduced sum of a family of mappings.

17 B.5. Definition. The *sum of a family of mappings for closure spaces* $\{f_a\}$ is the mapping f of $\Sigma\{\mathbf{D}^* f_a\}$ into $\Sigma\{\mathbf{E}^* f_a\}$ which assigns to each $\langle a, x\rangle$ the point

$\langle a, f_a x\rangle$. Thus gr f is the relational sum of $\{\text{gr} f_a\}$ (5 C.6). If $\mathbf{E}^* f_a = \mathscr{P}$ for each index a, then we define the *reduced sum of* $\{f_a\}$ to be the mapping of $\Sigma\{\mathbf{D}^* f_a\}$ into $\mathscr{P}$ which assigns to each point $\langle a, x\rangle$ the point $f_a x$.

The proofs of the following two theorems are simple and therefore may be left to the reader.

17 B.6. Theorem. *Let f be the sum of a family of mappings for closure spaces $\{f_a\}$. Then f is continuous, an embedding or homeomorphism if and only if each f_a has the corresponding property.*

17 B.7. *The reduced sum of a family of mappings for closure spaces $\{f_a\}$ is continuous if and only if each f_a is continuous.*

C. PRODUCTS OF SPACES

Recall that, by 5 A.4, the product of a family $\{P_a \mid a \in A\}$ of sets is the set $\Pi\{P_a\}$ consisting of all families $\{x_a \mid a \in A\}$ such that $x_a \in P_a$ for each a. Next, the projection relation at an element a is the relation pr_a which assigns to each family $\{x_b \mid b \in B\}$ with $a \in B$ the member x_a. The mapping $\text{pr}_a : \Pi\{P_b \mid b \in A\} \to P_a$, where $a \in A$, will be called the *projection* of the product in question into its a-th coordinate set P_a, or simply into P_a. It is to be noted that each projection of a product is surjective whenever the product is non-void.

Now let $\{\langle P_a, u_a\rangle \mid a \in A\}$ be a family of closure spaces, P be the product of the family $\{P_a\}$ of underlying sets and π_a be the projection of P into P_a for each a. For each x in P let $\mathscr{U}_x$ be the collection of all sets of the form

$$\pi_a^{-1}[V] = \mathbf{E}\{y \mid y \in P, \text{pr}_a y \in V\}$$

where $a \in A$ and V is a neighborhood of the point $\text{pr}_a x$ in $\langle P_a, u_a\rangle$. By Theorem 14 B.11 there exists exactly one closure operation u for P such that $\mathscr{U}_x$ is a local sub-base at x in $\langle P, u\rangle$ for each x in P. Clearly, for each x in P, the smallest multiplicative collection containing $\mathscr{U}_x$ which is a local base at x in $\langle P, u\rangle$, consists of all sets of the form

$$\bigcap\{\pi_a^{-1}[V] \mid a \in F\} = \mathbf{E}\{y \mid y \in P, a \in F \Rightarrow \text{pr}_a y \in V_a\}, \tag{1}$$

where F is a finite subset of A and V_a is a neighborhood of $\text{pr}_a x$ for each a in F.

17 C.1. Definition. The *product of a family* $\{\langle P_a, u_a\rangle \mid a \in A\}$ *of closure spaces*, denoted by $\Pi\{\langle P_a, u_a\rangle\}$, is defined to be the product P of the family $\{P_a\}$ of underlying sets endowed with the closure operation u defined above. This closure u, called the *product closure*, will sometimes be denoted by $\Pi\{u_a\}$. The neighborhoods of x of the form (1) will be called *canonical neighborhoods* (of x). The mappings $\text{pr}_a : \Pi\{P_a\} \to P_a$ will be called *projections of the product* (into coordinate spaces). The product (more precisely, *pair-product*) of two spaces $\langle P, u\rangle$ and $\langle Q, v\rangle$, the definition of which is clear, will be denoted by $\langle P, u\rangle \times \langle Q, v\rangle$ and the product closure will be denoted by $u \times v$. Thus we have:

$$\Pi\{\langle P_a, u_a\rangle\} = \langle \Pi\{P_a\}, \Pi\{u_a\}\rangle, \ \langle P, u\rangle \times \langle Q, v\rangle = \langle P \times Q, u \times v\rangle .$$

A class $K \subset \mathbf{C}$ is called *completely productive* if $\Pi\{\mathscr{P}_a\} \in K$ for any $\{\mathscr{P}_a\}$ in K.

It is to be noted that the symbol $\Pi\{u_a\}$, where u_a are closures, has many quite different meanings. It can be treated as the product of sets; in this case, it consists of all families $\{\langle X_a, Y_a\rangle\}$ such that $Y_a = u_a X_a$. Next, it can be treated as the relational product; in this case it consists of all pairs $\langle\{X_a\}, \{Y_a\}\rangle$ such that $Y_a = u_a X_a$ for each a. Finally it can be treated as the product closure. In what follows, unless otherwise stated, if u_a is considered as a closure operation, then the symbol $\Pi\{u_a\}$ will denote the product closure. It is interesting and also very important that the product closure $\Pi\{u_a\}$ is a relation-extension of the relation $\{\Pi\{X_a\} \to \Pi\{u_a X_a\}\}$ for exp P ranging in exp P which is "induced" by the relational product $\Pi_{\mathrm{rel}}\{u_a\}$ and the relation $\{\{X_a\} \to \Pi\{X_a\}\}$. Stated more directly:

17 C.2. *The closure of a set* $\Pi\{X_a\}$ *in a product space* $\Pi\{\langle P_a, u_a\rangle\}$ *is* $\Pi\{u_a X_a\}$, *i.e.*,

$$(\Pi\{u_a\})(\Pi\{X_a\}) = \Pi\{u_a X_a\} .$$

Proof. It is evident that each canonical neighborhood of a point x intersects $\Pi\{X_a\}$ if and only if $\mathrm{pr}_a\, x \in u_a X_a$ for each a.

It is to be noted that if a subset of the product is not of the form treated in 17 C.2 then there exists no simple formula such as, for instance, in the case of the sum closure or relativized closure.

17 C.3. *Let* $x = \{x_a\}$ *be a point of a product space* $P = \Pi\{P_a \mid a \in A\}$ *and let* $\{\mathscr{U}_a\}$ *be a family. If* $\mathscr{U}_a$ *is a local sub-base at* x_a *in* P_a *for each* a, *then the collection of all sets of the form* $\mathbf{E}\{y \mid y \in P, \mathrm{pr}_a\, y \in U\}$, *where* $a \in A$ *and* $U \in \mathscr{U}_a$, *is a local sub-base at* x *in* P. *If each* $\mathscr{U}_a$ *is a local base at* x *in* P_a, *then the collection of all sets of the form* $\mathbf{E}\{y \mid y \in P, a \in F \Rightarrow \mathrm{pr}_a\, y \in U_a\}$, *where* $F \subset A$ *is finite and* $U_a \in \mathscr{U}_a$ *for each* a *in* F, *is a local base at* x *in* P. — The simple proof is left to the reader.

17 C.4. Theorem. *The product of a family of topological spaces is a topological space.*

Proof. Let P be the product of a family $\{P_a\}$ of topological spaces. It is sufficient to show that, for each x in P, the open canonical neighborhoods of x form a local base at x. But this follows from 17 C.3. Indeed, if $\mathscr{U}_a$ is the collection of all open subsets of P_a containing $\mathrm{pr}_a\, x$, then $\mathscr{U}_a$ is a local base at $\mathrm{pr}_a\, x$ in P_a because P_a is topological; then, by the second statement of 17 C.3, the collection of all canonical neighborhoods formed of sets from collections $\mathscr{U}_a$ is a local base at x, and obviously each of its elements is open in P.

Obviously the product of any family of accrete spaces is an accrete space. Also finite products of discrete spaces are discrete spaces, but infinite products of discrete spaces need not be discrete spaces. More precisely, if P is the product of an infinite family of at least two-point discrete spaces, then P is not discrete and moreover, P has an infinite local character at each of its points (prove!). Some examples will be given in the concluding part of this subsection, for further examples consult the exercises.

From the corresponding definitions we obtain at once the following simple but often useful result, which asserts that the operation of taking subspaces commutes with the operation of forming products.

17 C.5. *If $\{P_a\}$ is family of closure spaces and Q_a is a subspace of P_a for each a, then $\Pi\{Q_a\}$ is a subspace of $\Pi\{P_a\}$.*

Now we proceed to the two descriptions of product closures promised in the introduction. Recall that the sum closure is the finest closure for the sum of underlying sets rendering all injections continuous.

17 C.6. Theorem. *The projections of a product space into its coordinate spaces are continuous. Moreover, the product closure is the coarsest closure for the product of underlying sets rendering all projections continuous.*

Proof. Let $\langle P, u\rangle$ be the product of a family $\{\langle P_a, u_a\rangle\}$ of closure spaces and let π_a denote the restriction of the relation pr_a to P. – I. The continuity of each $\pi_a : \langle P, u\rangle \to \langle P_a, u_a\rangle$ follows from 16 A.4 because, by the definition of product closure, if $x \in P$ and V is a neighborhood of $\pi_a x$, then $\pi_a^{-1}[V]$ is a canonical neighborhood of x. — II. Now let v be a closure for P such that all mappings $\pi_a : \langle P, v\rangle \to \to \langle P_a, u_a\rangle$ are continuous. Again by 16 A.4, if $x \in P$, $a \in A$ and V is a neighborhood of $\pi_a x$ in $\langle P_a, u_a\rangle$, then $\pi_a^{-1}[V]$ must be a neighborhood of x in $\langle P, v\rangle$. Consequently, every canonical neighborhood of x in $\langle P, u\rangle$ is a neighborhood of x in $\langle P, v\rangle$. Since canonical neighborhoods of x form a local base at x in $\langle P, u\rangle$, u is necessarily coarser than v (by 14 B).

It may be appropriate to notice that the projections have the following significant property:

17 C.7. *If P is the product of a family $\{P_a\}$ of closure spaces and X is a neighborhood of x in P, then $\mathrm{pr}_a[X]$ is a neighborhood of $\mathrm{pr}_a x$ in P_a; in particular, if X is open, then $\mathrm{pr}_a[X]$ is open.*

Remark. A subset $X = \Pi\{X_a\}$ of P is open is and only if all X_a's are open and $X_a = P_a$ except for a finite number of a's. Such open sets are called *canonical open* sets.

Next we shall observe that, if a product is non-void, then every coordinate space admits an embedding into the product.

17 C.8. *Let $P \neq \emptyset$ be the product of a family $\{P_a \mid a \in A\}$ of spaces. Fix an α in A and an x in P. The mapping of P_α into P which assigns to each $z \in P_\alpha$ the point $\{y_a\}$ such that $y_\alpha = z$ and $y_a = x_a$ for $a \neq \alpha$, is an embedding.* – Obvious.

Now we give a description of product closures in terms of convergence.

17 C.9. Theorem. *Let P be the product of a family $\{P_a\}$ of spaces and let $\langle N, \leqq \rangle$ be a net in P. In order that a point x of P be a limit point of $\langle N, \leqq \rangle$ in P it is necessary and sufficient that, for each a, the point $\text{pr}_a x$ be a limit point of the net $\langle \text{pr}_a \circ N, \leqq \rangle$ in $\langle P_a, u_a \rangle$.*

Proof. If x is a limit point of N in $\langle P, u \rangle$, then $\text{pr}_a x$ is a limit point of $\text{pr}_a \circ N$ in $\langle P_a, u_a \rangle$, because of the continuity of the projection of $\langle P, u \rangle$ onto $\langle P_a, u_a \rangle$ (16 A.8, 17 C.6). Conversely, suppose that $\text{pr}_a x$ is a limit point of the net $\text{pr}_a \circ N$ in $\langle P_a, u_a \rangle$ for each a. Since canonical neighborhoods of x form a local base at x, it is sufficient to show that N is eventually in each canonical neighborhood. Recall that a net which is eventually in each member of a finite family of sets is eventually in the intersection of this family. Therefore, it is sufficient to show that N is eventually in each set of the form $\mathbf{E}\{y \mid \text{pr}_a y \in V\}$, where V is a neighborhood of $\text{pr}_a x$ in $\langle P_a, u_a \rangle$. But this is clear because $\text{pr}_a \circ N$ converges to $\text{pr}_a x$ in $\langle P_a, u_a \rangle$.

Remark. Since the convergence of nets uniquely determines the closure of a space, Theorem 17 C.9 can be restated as follows: In order that u be the product closure it is necessary and sufficient that a net N in P converge to a point x in $\langle P, u \rangle$ if and only if the net $\text{pr}_a \circ N$ converges to $\text{pr}_a x$ in $\langle P_a, u_a \rangle$ for each a. For this reason the product closure is often termed the closure of pointwise convergence.

Recall that a mapping f of the sum $P = \Sigma\{P_a\}$ of spaces into a space is continuous if and only if all mappings $f \circ (\text{inj}_a : P_a \to P)$ are continuous. For mappings into products of spaces we shall prove the following result.

17 C.10. Theorem. *A mapping f of a space into the product P of a family $\{P_a\}$ of spaces is continuous if and only if the mapping*

$$(*) \qquad (\text{pr}_a : P \to P_a) \circ f$$

is continuous for each a.

Proof. I. If f is continuous then all mappings of $(*)$ are continuous as compositions of continuous mappings. — II. Conversely, suppose that all mappings $(*)$ are continuous. To prove that f is continuous, by 16 A.8 it is sufficient to show that, if a net N converges to a point x in $\mathbf{D}^*f$, then the net $f \circ N$ converges to the point fx in $\mathbf{E}^*f = P$. Suppose that N converges to x. Since every mapping of $(*)$ is continuous, again by 16 A.8, each net $(\text{pr}_a \circ f) \circ N$ converges to $(\text{pr}_a \circ f)\, x$ in P_a. But $(\text{pr}_a \circ f) \circ N = \text{pr}_a \circ (f \circ N)$, $(\text{pr}_a \circ f)\, x = \text{pr}_a (fx)$, and consequently, by the preceding theorem 17 C.9, the net $f \circ N$ converges to fx; this establishes the continuity of f.

Now let f be a mapping of a product $Q = \Pi\{Q_a \mid a \in A\}$ into a product $P = \Pi\{P_a \mid a \in A\}$. Sometimes there exists a family $\{f_a\}$ such that f_a is a mapping of Q_a into P_a and $\text{gr} f = \{\{x_a\} \to \{f_a x_a\}\}$. Of course such a family, if it exists, is uniquely determined. Now we shall show that, excepting the trivial case $Q = \emptyset$, f is continuous if and only if all the f_a are continuous.

17 C.11. *Let $\{f_a \mid a \in A\}$ be a family, each f_a being a mapping of a space Q_a into a space P_a. If all f_a are continuous then also the mapping f of $Q = \Pi\{Q_a\}$*

into $P = \Pi\{P_a\}$ *which assigns to each* $\{x_a\}$ *the point* $\{f_a x_a\}$ *is continuous. Conversely, if* f *is continuous and* $Q \neq \emptyset$, *then all* f_a *are continuous.*

Proof. I. If all f_a are continuous, then each mapping $(\mathrm{pr}_a : P \to P_a) \circ f$ is continuous since it coincides with the composition $f_a \circ (\mathrm{pr}_a : Q \to Q_a)$ of two continuous mappings. By Theorem 17 C.10 the mapping f is continuous. – II. Now suppose that f is continuous and $Q \neq \emptyset$. Fix an a in A. Choose an x in Q and write the mapping f_a in the form

$$f_a = (\mathrm{pr}_a : P \to P_a) \circ f \circ h$$

where h is the embedding of Q_a into Q, described in 17 C.8, corresponding to x. Since all the mappings on the right side are continuous, f_a is continuous by 16 A.3.

Notice that the graph of f in 17 C.10 is the reduced relational product (5 C.5) of $\{\mathrm{pr}_a \circ \mathrm{gr} f\}$, and the graph of f in 17 C.11 is the relational product (5 C.2) of $\{\mathrm{gr} f_a\}$. According to 17 C.10 and 17 C.11 it may be convenient to introduce the following definition.

17 C.12. Definition. Let $\{f_a\}$ be a family of mappings for closure spaces. The *product of* $\{f_a\}$, denoted by $\Pi\{f_a\}$, is defined to be the mapping

$$\Pi_{\mathrm{rel}}\{\mathrm{gr} f_a\} : \Pi\{\mathbf{D}^* f_a\} \to \Pi\{\mathbf{E}^* f_a\},$$

that is, its graph is the relational product of graphs, i.e. the set of all pairs $\langle\{x_a\}, \{f_a x_a\}\rangle$. If the domain carrier of each f_a is equal to a space R, then we shall define the *reduced product of* $\{f_a\}$ as the mapping

$$\{x \to \{f_a x\} \mid x \in R\} : R \to \Pi\{\mathbf{E}^* f_a\},$$

that is, the graph of the reduced product of mappings is the reduced product of graphs.

Now Theorems 17 C.10 and 17 C.11 can be restated as follows:

17 C.13. Theorem. *The reduced product of a family* $\{f_a\}$ *of mappings for closure spaces is continuous if and only if all* f_a *are continuous. Let* f *be the product of a family* $\{f_a\}$ *of mappings for closure spaces; if all* f_a *are continuous, then* f *is continuous, and conversely if* f *is continuous and* $\mathbf{D} f \neq \emptyset$, *then all* f_a *are continuous.*

17 C.14. Theorem. *Let* f *be the product of a family of mappings* $\{f_a\}$ *for closure spaces. If each* f_a *is a homeomorphism (an embedding) then so is* f. *Conversely, if* $\mathbf{D} f \neq \emptyset$ *and* f *is a homeomorphism (an embedding) then so is each* f_a.

Proof. Applying the second statement of 17 C.13 to both f and f^{-1} we obtain immediately the statements concerning homeomorphisms. The statements concerning embeddings follow from the statements about homeomorphisms and from Proposition 17 C.5 stating that the product of subspaces is a subspace of the product.

Remark. It can be casily proved that the operation of forming products is, in a certain sense, commutative; more precisely, if $\{P_a \mid a \in A\}$ is a family of spaces and φ is a bijective mapping of a set B onto A, then the products $P = \Pi\{P_a \mid a \in A\}$

and $P_1 = \Pi\{P_{\varphi b} \mid b \in B\}$ are homeomorphic. In fact, the mapping $\{\{x_a\} \to \{x_{\varphi b}\}\}$ is a homeomorphism of P onto P_1. Often we will be concerned with products of families all members of which coincide. To this end we introduce a special notation.

17 C.15. Definition. The product of the family of spaces $\{P \mid a \in A\}$ will be denoted by P^A. The product $[\![0, 1]\!]^A$ will be called a *cube*, where $[\![0, 1]\!]$ is taken as a subspace of the reals.

It is to be noted that $|P^A| = |P|^A$ (the symbol X^A, where X is a set, denotes the product of the family of sets $\{X \mid a \in A\}$).

17 C.16. *If P and Q are homeomorphic spaces and A and B are equipollent sets, then the product spaces P^A and Q^B are homeomorphic.* — By 17 C.14.

Convention. Sometimes we shall use the symbol $\mathscr{P}^{\aleph}$, where $\mathscr{P}$ is a closure space and $\aleph$ is a cardinal, to denote any closure space $\mathscr{P}^A$ such that A is a set of cardinal $\aleph$.

In what follows we shall often construct an embedding f of a space Q into a product $P = \Pi\{P_a\}$ of spaces. Clearly f is the reduced product of the family of mappings $\{(\mathrm{pr}_a : P \to P_a) \circ f\}$. To avoid repetition we shall prove the following results which give a sufficient condition for the reduced product of a family of mappings to be an embedding.

17 C.17. Embedding Theorem. *The following three conditions are sufficient for the reduced product f of a family $\{f_a\}$ of mappings of a closure space Q (into spaces) to be an embedding:*

(a) *Each f_a is continuous.*

(b) *The family $\{f_a\}$ distinguishes between points of Q, that is, if $x, y \in Q$, $x \neq y$, then $f_a x \neq f_a y$ for some a.*

(c) *If $x \in Q$, $X \subset Q$, $x \notin \overline{X}$, then, for some a, the point $f_a x$ does not belong to the closure of $f_a[X]$ in $\mathbf{E}^* f_a$.*

Supplement: *If Q is topological, then condition* (c) *can be replaced by the following weaker one:*

(c′) *If X is closed in Q and $x \in Q - X$, then there exists an a such that $f_a x$ is not in the closure of $f_a[X]$ in $\mathbf{E}^* f_a$.*

It is to be noted that conditions (a) and (b) are also necessary. The necessity of (b) is obvious ($fx = fy$ if and only if $f_a x = f_a y$ for each a) and the necessity of (a) follows from 17 C.13 (because every embedding is continuous). On the other hand, condition (c) is not necessary.

Proof. Suppose that f is the reduced product of a family $\{f_a\}$ satisfying conditions (a), (b) and (c), where the space Q is the common domain-carrier of all f_a. To prove that f is an embedding it will suffice to show that f is a continuous injective mapping satisfying the following condition:

$$(*) \qquad x \in Q,\ X \subset Q,\ x \notin \overline{X} \Rightarrow fx \notin \overline{f[X]} \text{ (in } \mathbf{E}^* f).$$

According to 17 C.13 condition (a) implies that f is continuous. Condition (b) implies that f is injective (because $f_a x \neq f_a y$ for some a implies $fx \neq fy$). It remains to prove (*). Since each projection $\pi_a = (\mathrm{pr}_a : \mathbf{E}^* f \to \mathbf{E}^* f_a)$ is continuous, to prove $fx \notin \overline{f[X]}$ in $\mathbf{E}^* f$, it will suffice to show that

$$\mathrm{pr}_a fx \notin \overline{\mathrm{pr}_a [f[X]]} \quad (\text{in } \mathbf{E}^* f_a)$$

for some a. But $\mathrm{pr}_a fx = f_a x$ and $\mathrm{pr}_a [f[X]] = f_a[X]$, and hence it is sufficient to find an a such that $f_a x \notin \overline{f_a[X]}$ in $\mathbf{E}^* f_a$. But such an a exists by condition (c). The proof is complete. The supplement is evident ((c′) implies (c), if Q is topological).

As a rather general example we shall show that every closure space Q is homeomorphic with a subspace of a product $P^{\aleph}$ where P is a certain three-point space independent of Q and $\aleph$ is a suitable cardinal depending on Q.

17 C.18. Theorem. *Suppose that $P = (y_1, y_2, y_3)$ is a three-point set and u is a closure for P such that*

$$u(y_1) = (y_1, y_2), \quad u(y_2) = u(y_3) = P.$$

(Obviously there exists exactly one such closure u.) Then every closure space $\langle Q, v\rangle$ is homeomorphic with a subspace of the product space $\langle P, u\rangle^{\exp Q}$.

Proof. Let f be the reduced product of the family of mappings $\{f_X \mid X \subset Q\}$ where f_X is the mapping of $\langle Q, v\rangle$ into $\langle P, u\rangle$ such that $f_X[X] \subset (y_1)$, $f_X[vX - X] \subset (y_2)$ and $f_X[Q - vX] \subset (y_3)$. Thus f is a mapping of $\langle Q, v\rangle$ into $\langle P, u\rangle^{\exp Q}$. We shall prove that f is an embedding by showing that $\{f_X\}$ fulfils conditions (a), (b), (c) of 17 C.17. The continuity of any f_X will be proved by showing that $x \in vY$ implies $f_X x \in u\, f_X[Y]$. This is evident if $Y \cap (Q - X) \neq \emptyset$ because then $f_X[Y]$ contains y_2 or y_3 and hence $u\, f_X[Y] = P$. In the opposite case we have $Y \subset X$, and hence $x \in vX$ $f_X[Y] = (y_1)$ (because $Y \neq \emptyset$) and finally $f_X x \in u(y_1) = u\, f_X[Y]$, which establishes the continuity of f_X. Condition (b) is fulfilled because $f_{(x)} x \neq f_{(x)} y$ if $x \neq y$. Finally, if $x \notin vX$, then $f_X x = y_3 \notin u(y_1) \supset u\, f_X[X]$ which shows that condition (c) is also fulfilled.

It has already been shown that the operation of forming products is commutative. Now we shall prove that this operation is also associative.

17 C.19. Associativity of products. *If A is the union of a disjoint family of non-void sets $\{A_b \mid b \in B\}$ and if $\{P_a \mid a \in A\}$ is a family of spaces, then the spaces $P = \Pi\{P_a \mid a \in A\}$ and $P' = \Pi\{\Pi\{P_a \mid a \in A_b\} \mid b \in B\}$ are homeomorphic.*

Proof. Consider the mapping f of P into P' which assigns to each $x \in P$ the element $\{\{x_a \mid a \in A_b\} \mid b \in B\}$ (cf. 5 A.13). Obviously f is bijective, and it follows from 17 C.9 that a net N converges to x in P if and only if the net $f \circ N$ converges to fx in P'. By 16 A.8 both f and f^{-1} are continuous which means that f is a homeomorphism.

Corollary. *If $\{P_a \mid a \in A\}$ is a family of spaces and $A_1 \subset A$, then the mapping*

$$\{x \to x \mid A_1\} : \Pi\{P_a \mid a \in A\} \to \Pi\{P_a \mid a \in A_1\}$$

is continuous (this mapping will be called the projection of $\Pi\{P_a \mid a \in A\}$ *into* $\Pi\{P_a \mid a \in A_1\}$).

Proof. Put $A_2 = A - A_1$ and consider the product $P' = \Pi\{\Pi\{P_a \mid a \in A_i\} \mid i = 1, 2\}$. Obviously the mapping in question can be written in the form $g \circ f$, where f is the canonical mapping of $P = \Pi\{P_a \mid a \in A\}$ onto P' and g is the projection of P' into its first coordinate space $P'' = \Pi\{P_a \mid a \in A_1\}$.

D. INDUCTIVE PRODUCTS

Let us consider the product $\langle P, u\rangle \times \langle Q, v\rangle$ of closure spaces $\langle P, u\rangle$ and $\langle Q, v\rangle$. By definition, the collection of all sets of the form $U \times V$, where U is a neighborhood of x in $\langle P, u\rangle$ and V is a neighborhood of y in $\langle Q, v\rangle$, is a local base at $\langle x, y\rangle$ in $\langle P, u\rangle \times \langle Q, v\rangle$ for each $\langle x, y\rangle$. Now, for each $\langle x, y\rangle \in P \times Q$ let $\mathscr{V}_{\langle x,y\rangle}$ be the collection of all sets of the form

$$(*) \qquad ((x) \times V) \cup (U \times (y))$$

where V runs over all neighborhoods of y in $\langle Q, v\rangle$ and U over all neighborhoods U of x in $\langle P, u\rangle$. Obviously, each $\mathscr{V}_{\langle x,y\rangle}$ is a filter base in $P \times Q$ and $\langle x, y\rangle \in \bigcap\mathscr{V}_{\langle x,y\rangle}$. By virtue of **14 B.10** there exists exactly one closure operation for $P \times Q$ such that $\mathscr{V}_z$ is a local base at z for each z in $P \times Q$.

17 D.1. Definition. The *inductive product of two spaces* $\langle P, u\rangle$ and $\langle Q, v\rangle$, denoted by ind $(\langle P, u\rangle \times \langle Q, v\rangle)$, is the set $P \times Q$ endowed with the closure operation defined above. The closure of ind $(\langle P, u\rangle \times \langle Q, v\rangle)$, denoted by ind $(u \times v)$, is called the *inductive product closure*. The neighborhoods of the form $(*)$ are called *canonical neighborhoods for the inductive product*, or simply canonical inductive neighborhoods.

17 D.2. Theorem. *The product closure is coarser than the inductive product closure.*

Proof. Every canonical neighborhood $U \times V$ of a point z for the product contains the canonical neighborhood $(*)$ of z for the inductive product (use **14 B.9**).

If one of the spaces is discrete, then clearly the product closure and the inductive product closure coincide. On the other hand, the reader may show without difficulty that the product closure and the inductive product closure are distinct provided that neither $\langle P, u\rangle$ nor $\langle Q, v\rangle$ is discrete (pick cluster points x of $\langle P, u\rangle$ and y of $\langle Q, v\rangle$ and show that no canonical inductive neighborhood $(*)$ of $\langle x, y\rangle$ contains a neighborhood of the form $U \times V$, that is, a canonical neighborhood for the product).

Recall that the product closure is the coarsest closure for the product of underlying sets making all projections continuous. The inductive product closure can be characterized as follows:

17 D.3. Theorem. *A closure* w *for* $P \times Q$ *is the closure structure of* ind $(\langle P, u\rangle \times \langle Q, v\rangle)$ *if and only if the following two conditions are fulfilled:*

(a) *Each mapping* $\{x \to \langle x, y\rangle\} : \langle P, u\rangle \to \langle P \times Q, w\rangle$, $y \in Q$, *and also each mapping* $\{y \to \langle x, y\rangle\} : \langle Q, v\rangle \to \langle P \times Q, w\rangle$, $x \in P$, *is continuous;*

(b) *If a closure* w_1 *for* $P \times Q$ *fulfils* (a) *with* w *replaced by* w_1, *then* w_1 *is coarser than* w.

Roughly speaking, the inductive product closure is the finest closure for the product of underlying sets rendering all the mappings of (a) *continuous.*

Proof. Since clearly there exists at most one closure for $P \times Q$ satisfying conditions (a) and (b), it is sufficient to show that the inductive product closure w fulfils these conditions. The continuity of mappings of (a) follows from the evident fact that the inverse images of canonical inductive neighborhoods are neighborhoods. We shall prove condition (b). Suppose that w_1 fulfils (a) and let W be a neighborhood of a point $\langle x, y\rangle$ in $\langle P \times Q, w_1\rangle$. We shall prove that W is a neighborhood of $\langle x, y\rangle$ in $\langle P \times Q, w\rangle$, which will imply, by 14 B.8, that w_1 is coarser than w. Since the mapping $\{t \to \langle x, t\rangle\}$ of $\langle Q, v\rangle$ into $\langle P \times Q, w_1\rangle$ is continuous at y, we can choose a neighborhood V of y in $\langle Q, v\rangle$ whose image $(x) \times V$ under this mapping is contained in W. Similarly we can choose a neighborhood U of x in $\langle P, u\rangle$ such that $U \times (y)$ is contained in W. Thus $((x) \times V) \cup (U \times (y)) \subset W$.

17 D.4. *If* $\langle P, u\rangle$ *and* $\langle Q, v\rangle$ *are closure spaces, then all mappings*

$$\{x \to \langle x, y\rangle\} : \langle P, u\rangle \to \operatorname{ind}(\langle P, u\rangle \times \langle Q, v\rangle)$$

and

$$\{y \to \langle x, y\rangle\} : \langle Q, v\rangle \to \operatorname{ind}(\langle P, u\rangle \times \langle Q, v\rangle)$$

are embeddings. They will be called canonical embeddings into the inductive product. — Evident.

Recall that a mapping f into a product space is continuous if and only if the composition of f with each projection is continuous.

17 D.5. *Let* $\mathscr{P}$, $\mathscr{Q}$, *and* $\mathscr{R}$ *be closure spaces. A mapping* f *of* $\operatorname{ind}(\mathscr{P} \times \mathscr{Q})$ *into* $\mathscr{R}$ *is continuous at a point* $\langle x, y\rangle$ *if and only if the mapping* $\{\xi \to f\langle \xi, y\rangle\} : \mathscr{P} \to \mathscr{R}$ *is continuous at* x *and the mapping* $\{\eta \to f\langle x, \eta\rangle\} : \mathscr{Q} \to \mathscr{R}$ *is continuous at* y. *Consequently,* f *is continuous if and only if each mapping*

(*) $\{\xi \to f\langle \xi, y\rangle\} : \mathscr{P} \to \mathscr{R}$, $y \in |\mathscr{Q}|$,

and also each mapping

(**) $\{\eta \to f\langle x, \eta\rangle\} : \mathscr{Q} \to \mathscr{R}$, $x \in |\mathscr{P}|$,

is continuous.

The simple proof is left to the reader.

17 D.6. Example. Let us consider a single-valued relation ϱ on the set $\mathsf{R} \times \mathsf{R}$ ranging in R which assigns zero to the point $\langle 0, 0\rangle$ and the number $xy/(x^2 + y^2)$ to each other point $\langle x, y\rangle$. Now let R denote the space of reals. One can easily show that the function

$$\varrho : \operatorname{ind}(\mathsf{R} \times \mathsf{R}) \to \mathsf{R}$$

is continuous, but the function

$$\varrho : \mathsf{R} \times \mathsf{R} \to \mathsf{R}$$

is not continuous (at the point $\langle 0, 0\rangle$). More interesting examples will be given in Section 19 devoted to continuous algebraic structs.

17 D.7. A point $\langle x, y\rangle$ belongs to the closure of a subset Z of ind $(\langle P, u\rangle \times \langle Q, v\rangle)$ if and only if either $x \in u\, Z^{-1}[(y)]$ or $y \in v\, Z[(x)]$.

Proof. First notice that Z is a relation and hence $Z^{-1}[(y)] = \mathbf{E}\{x \mid \langle x, y\rangle \in Z\}$ and $Z[(x)] = \mathbf{E}\{y \mid \langle x, y\rangle \in Z\}$. It follows from the definition of neighborhoods in the inductive product that $\langle x, y\rangle$ belongs to the closure of Z if and only if it belongs to the closure of the set $(((x) \times Q) \cup (P \times (y))) \cap Z$, and hence to the closure of the set $((x) \times Q) \cap Z$ or the set $(P \times (y)) \cap Z$. Now the result follows from 17 D.4.

Recall that the product of topological spaces is a topological space. On the other hand, it is easily seen that the inductive product of two topological spaces need not be topological.

17 D.8. Definition. The *topological inductive product* of two closure spaces (not necessarily topological) $\mathscr{P}$ and $\mathscr{Q}$ is defined to be the topological modification of the inductive product of $\mathscr{P}$ and $\mathscr{Q}$, i.e. τ ind $(\mathscr{P} \times \mathscr{Q})$.

17 D.9. Theorem. *For any two closure spaces $\mathscr{P}$ and $\mathscr{Q}$ we have* τ ind $(\mathscr{P} \times \mathscr{Q}) = \tau$ ind $(\tau\mathscr{P} \times \tau\mathscr{Q})$.

The proof follows from the following simple result.

17 D.10. *A subset Z of ind $(\mathscr{P} \times \mathscr{Q})$ is open (closed) if and only if $Z[(x)]$ is open (closed) in $\mathscr{Q}$ and $Z^{-1}[(y)]$ is open (closed) in $\mathscr{P}$ for each x in $\mathscr{P}$ and y in $\mathscr{Q}$.*

Proof: 17 D.7.

17 D.11. Theorem. *A closure w for $P \times Q$ is the closure structure of* τ ind $(\langle P, u\rangle \times \langle Q, v\rangle)$ *if and only if w is a topological closure satisfying the following two conditions*:

(a) *Each mapping $\{x \to \langle x, y\rangle\} : \langle P, u\rangle \to \langle P \times Q, w\rangle$, $y \in Q$, and also each mapping $\{y \to \langle x, y\rangle\} : \langle Q, v\rangle \to \langle P \times Q, w\rangle$, $x \in P$, is continuous;*

(b) *if a topological closure w_1 for $P \times Q$ fulfils condition* (a) *with w replaced by w_1, then w_1 is coarser than w.*

Proof: 17 D.3, 16 B.3.

18. SPECIAL SPACES

Most of this section is concerned with defining and developing the properties of pseudometric spaces and of their generalization, semi-pseudometric spaces. In the last two subsections we shall introduce new closure operations for ordered sets which are used to define upper semi-continuous and lower semi-continuous mappings of a closure space into an ordered set, in particular, into the set of reals, and we shall introduce the usual closure operation for the set of all prime ideals of a semi-ring which is a generalization of the closure for the set of all ultrafilters on a set (14 B.12).

In subsection A we shall introduce the concepts of a semi-pseudometric, a pseudometric and a closure operation induced by a semi-pseudometric. In subsection B the concepts of a Lipschitz continuous mapping and a uniformly continuous mapping are introduced and studied. The main results are the metrization lemma 18 B.10 and Theorem 18 B.16 which gives a necessary and sufficient condition for a given semi-pseudometric to be uniformly equivalent with a pseudometric.

In subsection C we shall prove that the class of all pseudometrizable spaces as well as the class of all semi-pseudometrizable spaces is hereditary, countably productive and closed under the operation of forming sums. Various pseudometrizations and semi-pseudometrizations of subspaces, spaces and products are discussed. Subsections A and B are intended to be an introduction to sections 23 and 24 devoted to the examination of uniform spaces.

A. PSEUDOMETRICS

Beginning with this section we shall be much concerned with real numbers. It seems more natural to use a terminology which is current for this situation in preference to that introduced in 10 H.14.

Convention. A real r will be termed positive if $r > 0$, and non-negative if $r \geqq 0$ (the corresponding terminology of 10 H.14 would be strictly positive and positive, respectively).

18 A.1. Definition. A *real-valued relation* is a single-valued relation ranging in the set R of reals. A *semi-pseudometric for a set P* is a real-valued relation d on $P \times P$ which fulfils the following two conditions:

(m 1) for each x in P, $d\langle x, x\rangle = 0$ (i.e. d vanishes on the diagonal of $P \times P$),

(m 2) for all x, y in P, $d\langle x, y\rangle = d\langle y, x\rangle \geqq 0$ (i.e., d is symmetric and non-negative).

A semi-pseudometric d for P is a *pseudometric for* P if

(m 3) for each x, y and z in P, $d\langle x, y\rangle \leqq d\langle x, z\rangle + d\langle z, y\rangle$ (i.e., d fulfils the triangle inequality).

Finally, a semi-pseudometric (a pseudometric) is said to be a *semi-metric* (*metric*) if

(m 4) $d\langle x, y\rangle = 0$ implies $x = y$.

A *semi-pseudometric space* is a pair $\langle P, d\rangle$ where P is a set and d is a semi-pseudometric for P. Similarly we define *pseudometric*, *semi-metric* and *metric spaces*.

Thus a metric fulfils all conditions (m 1)–(m 4), "pseudo" indicates that (m 4) is not assumed and "semi" indicates that the triangle inequality is not assumed. It is to be noted that if d is a semi-pseudometric then there exists exactly one set P such that d is a semi-pseudometric for P (and hence, $\langle P, d\rangle$ is a semi-pseudometric space), namely $P = \mathbf{DD}d$; actually $\mathbf{D}d = P \times P$ for exactly one P and $P = \mathbf{D}(P \times P)$.

18 A.2. Definition. Let $\langle P, d\rangle$ be a semi-pseudometric space. The number $d\langle x, y\rangle$ is called the *distance* from x to y in $\langle P, d\rangle$ or *under* d or the *d-distance* from x to y. If X and Y are subsets of P, then the distance $\operatorname{dist}_d(X, Y)$, or simply $\operatorname{dist}(X, Y)$, from X to Y is defined to be $+\infty$ if $X = \emptyset$ or $Y = \emptyset$ and $\inf\{d\langle x, y\rangle \mid x \in X, y \in Y\} = \inf d[X \times Y]$ otherwise. The distance $\operatorname{dist}(x, X)$ from a point x to a set X is defined to be $\operatorname{dist}((x), X)$. The *diameter* of $X \subset P$ in $\langle P, d\rangle$ is defined to be zero if $X = \emptyset$, $\sup d[X \times X]$ if $d[X \times X]$ is a non-void bounded subset of R and $+\infty$ in the remaining case. If r is a positive real and $x \in P$, then the set $\mathbf{E}\{y \mid d\langle x, y\rangle < r\}$ ($\mathbf{E}\{y \mid d\langle x, y\rangle \leqq r$) is called the *open* (*closed*) *sphere* of radius r about x or briefly, the open (closed) r-sphere about x. A mapping $f : \langle P, d\rangle \to \langle P_1, d_1\rangle$ is said to be *distance-preserving* if $d_1\langle fx, fy\rangle = d\langle x, y\rangle$ for each x and y in P. Two semi-pseudometric spaces are called *isomorphic* if there exists a distance-preserving bijective mapping of one onto the other.

With every semi-pseudometric there is associated a closure operation which will be described now. Let d be a semi-pseudometric for a set P. The relation

(*) $u = \{X \to \mathbf{E}\{x \mid \operatorname{dist}(x, X) = 0\} \mid X \subset P\}$

is a closure operation for the set P. Obviously u is a relation on $\exp P$ ranging in $\exp P$. Conditions (cl i), $i = 1, 2, 3$, of Definition 14 A.1 are verified as follows: since $\operatorname{dist}(x, \emptyset) = +\infty \neq 0$ we obtain $u\emptyset = \emptyset$ which is (cl 1); the self-evident implication $x \in X \Rightarrow \operatorname{dist}(x, X) = 0$ yields $X \subset uX$ which is (cl 2); finally, the additivity of u is a straightforward consequence of the following obvious equality:

$$\operatorname{dist}(x, X \cup Y) = \min(\operatorname{dist}(x, X), \operatorname{dist}(x, Y)).$$

18 A.3. Definition. If d is a semi-pseudometric for a set P, then the closure u of (*) is said to be the *closure induced* by d. Unless otherwise stated every semi-pseudometric space $\langle P, d\rangle$ will be considered as a closure space $\langle P, u\rangle$ where u

is the closure induced by d. For example, if we say that f is a continuous mapping of a semi-pseudometric space $\langle P_1, d_1 \rangle$ into another one $\langle P_2, d_2 \rangle$, it is to be understood that the mapping $f: \langle P_1, u_1 \rangle \to \langle P_2, u_2 \rangle$ is continuous, where u_i is the closure induced by d_i. Similarly we shall speak, e.g., about closed or open subsets of a semi-pseudometric space. A closure operation u (a closure space $\langle P, u \rangle$) is said to be *semi-pseudometrizable* (*semi-metrizable, pseudometrizable, metrizable*) if u is induced by a semi-pseudometric (semi-metric, pseudometric, metric). For convenience, two semi-pseudometrics will be called *topologically equivalent* if they induce the same closure operation (especially, they are for the same set).

For convenience, before going to examples, we shall describe neighborhoods in a semi-pseudometric space.

18 A.4. *Suppose that a closure operation u for a set P is induced by a semi-pseudometric d. Then a subset X of P is a neighborhood of x in $\langle P, u \rangle$ if and only if the d-distance from x to $P - X$ is not zero. Stated in other words, for each x in P, the collection of all open r-spheres about x is a local base at x in $\langle P, u \rangle$. Moreover, if M is any set of positive reals such that* $\inf M$ *is zero, then the open r-spheres about x with r taken in M also form a local base at x in $\langle P, u \rangle$. Since M can be taken countable, every point of $\langle P, u \rangle$ possesses a countable local base, i.e. $\langle P, u \rangle$ is of a countable local character.*

Proof. Obviously, if r and s are reals such that $s < r$, then the open s-sphere about x is contained in the open r-sphere about x. Next, if $M = \mathbf{E}\{n^{-1} \mid n = 1, 2 \ldots\}$, then $\inf M = 0$.

Corollary. *In a semi-pseudometrizable space P a point x belongs to the closure of a subset X of P if and only if there exists a sequence $\{x_n\}$ in X converging to x in P.* – 15 B.9.

Remark. If $0 < r < s$, then the closed r-sphere about x is contained in the open s-sphere about x. In consequence, "open r-spheres" can be replaced by "closed r-spheres" in 18 A.4.

18 A.5. Examples. (a) The relation $d = \{\langle x, y \rangle \to |x - y| \mid \langle x, y \rangle \in \mathrm{R} \times \mathrm{R}\}$ is a metric for R inducing the closure of R. Since $|z| = 0 \Leftrightarrow z = 0$, we obtain $d\langle x, y \rangle = |x - y| = 0$ if and only if $x = y$, and consequently d fulfils (m 1) and (m 4). Next $d\langle x, y \rangle = |x - y| = |y - x| = d\langle y, x \rangle \geqq 0$ which is (m 2). Finally, the triangle inequality $|x - y| \leqq |x - z| + |z - y|$ follows from obvious equality $(x - y) = (x - z) + (z - y)$ because always $|z_1 + z_2| \leqq |z_1| + |z_2|$. Thus d is a metric. Next, the open r-sphere about x is the open interval $]\!] x - r, x + r [\![$ and consequently, by virtue of 18 A.4, the closure induced by d coincides with the order closure for R. In what follows, unless otherwise stated, if R is considered as a metric space, it is to be understood that the metric for R is the metric just defined.

(b) Let P be a set. The relation $\{\langle x, y \rangle \to 0 \mid \langle x, y \rangle \in P \times P\}$ is a pseudometric for P inducing the accrete closure for P. Conversely, if a semi-pseudometric d

induces the accrete closure for P, then necessarily $d\langle x, y\rangle = 0$ for each x and y in P because $d\langle x, y\rangle \neq 0$ implies that x does not belong to the closure of (y).

(c) If d is a semi-pseudometric (pseudometric) for a set P and r is a positive real, then the relation $\{\langle x, y\rangle \to r \,.\, d\langle x, y\rangle\}$, denoted by $r \,.\, d$, is a semi-pseudometric (pseudometric) for P inducing the same closure as d. Since clearly $d = r \,.\, d$ if and only if $d\langle x, y\rangle = 0$ for each $\langle x, y\rangle$, we obtain that the accrete closure for P is the only closure for P induced by exactly one semi-pseudometric (pseudometric).

(d) Given a set P, let us consider the real-valued relation d on $P \times P$ which assigns to a pair $\langle x, y\rangle$ the element 0 if $x = y$ and 1 if $x \neq y$. Obviously d is a metric for P inducing the discrete closure for P.

(e) If d is a semi-pseudometric for a set P, then the relation

$$D = \{\langle X, Y\rangle \to \operatorname{dist}_d(X, Y) \mid \emptyset \neq X \subset P \supset Y \neq \emptyset\}$$

is a semi-pseudometric for the set $\exp P - (\emptyset)$ and $d\langle x, y\rangle = D\langle (x), (y)\rangle$ for each x and y in P. If $d\langle x, y\rangle \neq 0$ for some $\langle x, y\rangle$, that is, if d does not induce the accrete closure for P (by (b)), then D is not a pseudometric and the closure induced by D is not topological. Indeed, put $X = (x, y)$ and notice that $D\langle (x), X\rangle = D\langle (y), X\rangle = 0$ and hence $D\langle (x), (y)\rangle > (D\langle (x), X\rangle + D\langle X, (y)\rangle) = 0$ (which shows that the triangle inequality is not fulfilled), and $(x) \in uu((y))$ but $(x) \notin u((y))$, where u is the closure induced by D.

(f) Let μ be a measure for a set P and let $\mathscr{X}$ be the set of all $X \subset P$ such that μX is finite. Then $d = \{\langle X, Y\rangle \to \mu(X \div Y) \mid X \in \mathscr{X},\ Y \in \mathscr{X}\}$ is a pseudometric for $\mathscr{X}$ (here $X \div Y$ denotes the symmetric difference of X and Y, i.e. the set $X \cup Y - (X \cap Y)$). If $\mu_1(\mu_2)$ is the corresponding outer (inner) measure and $\mathscr{X}_1(\mathscr{X}_2)$ is the set of all $X \subset P$ such that $\mu_1 X(\mu_2 X)$ is finite, then $d_i = \{\langle X, Y\rangle \to \mu_i(X \div Y) \mid X \in \mathscr{X}_i,\ Y \in \mathscr{X}_i\}$ is a semi-pseudometric for $\mathscr{X}_i$, d_1 is a pseudometric but d_2 need not be a pseudometric.

(g) Let μ be a measure for the set R of reals such that every closed interval $[\![x, y]\!]$ has finite measure and $\mu(x) = 0$ for each $x \in \mathrm{R}$. Then $d = \{\langle x, y\rangle \to \mu([\![x, y]\!] \cup [\![y, x]\!]) \mid \langle x, y\rangle \in \mathrm{R} \times \mathrm{R}\}$ is a pseudometric for R. If each interval $[\![x, y]\!]$, $x < y$ has a positive measure, then d is a metric, and moreover d induces the closure of R (i.e. the order closure for R) (in proving this, recall that μX_n converges to $\mu \bigcap\{X_n\}$ in R provided that $\mu X_0 < \infty$ and $\{X_n\}$ is a decreasing sequence of subsets of R).

(h) Let m be the Lebesgue measure for R and A a subset of R such that $m(A \cap [\![x, y]\!]) > 0$ for each interval $[\![x, y]\!]$ with $x < y$, and let

$$d_A = \{\langle x, y\rangle \to m(A \cap ([\![x, y]\!] \cup [\![y, x]\!])) \mid \langle x, y\rangle \in \mathrm{R} \times \mathrm{R}\}\,.$$

Then d_A is a metric inducing the closure of R. Notice that if $\mu = \{X \to m(A \cap X)\}$, then d_A is the pseudometric d from (g).

18 A.6. Theorem. *Let $\langle\{x_a\}, \leqq\rangle$ be a net in a semi-pseudometric space $\langle P, d\rangle$. Then a point x of P is a limit point (an accumulation point) of $\langle\{x_a\}, \leqq\rangle$ in*

$\langle P, d\rangle$ if and only if zero is the limit point (an accumulation point) of the net $\langle\{d\langle x, x_a\rangle\}, \leqq\rangle$ in R.

Proof. Obviously, given a positive real r, the r-sphere U_r about x is the set of all y such that $d\langle x, y\rangle \in]\!] -r, r [\![$. Thus $\langle\{x_a\}, \leqq\rangle$ is eventually (frequently) in U_r if and only if the net $\langle\{d\langle x, x_a\rangle\}, \leqq\rangle$ is eventually (frequently) in $]\!] -r, r [\![$. Now both statements follow from the fact that open spheres about x form a local base at x in $\langle P, d\rangle$ and the intervals $]\!] -r, r [\![$, $r > 0$, form a local base at zero in R.

18 A.7. Theorem. *A mapping f of a semi-pseudometric space $\langle P, d\rangle$ into another one $\langle P_1, d_1\rangle$ is continuous at a point $x \in P$ if and only if the following conditions is fulfilled:*

For each positive real r there exists a positive real s such that $d\langle x, y\rangle < s$ implies $d_1\langle fx, fy\rangle < r$.

Proof. The implication $d\langle x, y\rangle < s \Rightarrow d_1\langle fx, fy\rangle < r$ is equivalent to this assertion: the image under f of the open s-sphere about x in $\langle P, d\rangle$ is contained in the open r-sphere about fx in $\langle P_1, d_1\rangle$. Since open spheres form local bases, the statement follows from 16 ex. 3.

Remark. It is to be noted that we may write $\leqq$ instead of $<$ in 18 A.7.

18 A.8. Definition. A *Lipschitz continuous mapping* or simply a *Lipschitz mapping* of a semi-pseudometric space $\langle P, d\rangle$ into another one $\langle P_1, d_1\rangle$ is a mapping f of $\langle P, d\rangle$ into $\langle P_1, d_1\rangle$ such that there exists a non-negative K, called a *Lipschitz bound* of f, with $K \, . \, d\langle x, y\rangle \geqq d_1\langle fx, fy\rangle$ for each $\langle x, y\rangle \in P \times P$.

As a corollary of 18 A.7 we obtain:

18 A.9. *Every Lipschitz continuous mapping is continuous.*

Proof. Let f be a Lipschitz continuous mapping of $\langle P, d\rangle$ into $\langle P_1, d_1\rangle$ and let always $K \, . \, d\langle x, y\rangle \geqq d_1\langle fx, fy\rangle$ where K is a positive real. Given $r > 0$ put $s = r \, . \, K^{-1}$ and apply 18 A.7.

18 A.10. *The composition of two Lipschitz continuous mappings is a Lipschitz continuous mapping; more precisely, if $f = \langle P, d\rangle \to \langle P_1, d_1\rangle$ and $g : \langle P_1, d_1\rangle \to \to \langle P_2, d_2\rangle$ are Lipschitz continuous, then $g \circ f : \langle P, d\rangle \to \langle P_2, d_2\rangle$ is also Lipschitz continuous.*

Proof. Assuming $K_1 \, . \, d\langle x, y\rangle \geqq d_1\langle fx, fy\rangle$, $K_2 \, . \, d_1\langle z, t\rangle \geqq d_2\langle gz, gt\rangle$, we obtain $K_1 \, . \, K_2 \, . \, d\langle x, y\rangle \geqq d_2\langle g \circ fx, g \circ fy\rangle$.

Before proceeding to an examination of the properties of semi-pseudometric and pseudometric spaces, we derive from the triangle inequality two important inequalities.

18 A.11. *If $\langle P, d\rangle$ is a pseudometric space, x, y, x' and y' are points of P and X is a non-void subset of P, then*

(*) $|\text{dist}(x, X) - \text{dist}(y, X)| \leqq d\langle x, y\rangle$,

(**) $|d\langle x, y\rangle - d\langle x', y'\rangle| \leqq d\langle x, x'\rangle + d\langle y, y'\rangle$.

Proof. I. If $z \in X$, then $\operatorname{dist}(x, X) \leqq d\langle x, z\rangle$ and by the triangle inequality $\operatorname{dist}(x, X) \leqq d\langle x, y\rangle + d\langle y, z\rangle$. Taking the greatest lower bound of $d\langle y, z\rangle$ for z in X, we obtain $\operatorname{dist}(x, X) \leqq d\langle x, y\rangle + \operatorname{dist}(y, X)$ which implies $(\operatorname{dist}(x, X) - \operatorname{dist}(y, X)) \leqq d\langle x, y\rangle$. The same inequality holds with x and y interchanged. Formula (*) follows. — II. Formula (**) follows by a double application of the triangle inequality: $d\langle x, y\rangle \leqq d\langle x, x'\rangle + d\langle x', y\rangle \leqq d\langle x, x'\rangle + d\langle x', y'\rangle + d\langle y', y\rangle$, and consequently $d\langle x, y\rangle - d\langle x', y'\rangle \leqq d\langle x, x'\rangle + d\langle y, y'\rangle$; now the conclusion follows as in the proof of (*).

18 A.12. Theorem. *If $\langle P, d\rangle$ is a pseudometric space, then*

(a) *the function $\{x \to \operatorname{dist}(x, X)\}: \langle P, d\rangle \to \mathsf{R}$ is continuous for each non-void subset X of P;*

(b) *$\langle P, d\rangle$ is a topological space;*

(c) *every open sphere in $\langle P, d\rangle$ is an open subset of $\langle P, d\rangle$ and every closed sphere in $\langle P, d\rangle$ is a closed subset of $\langle P, d\rangle$;*

(d) *if a net $\langle\{x_a \mid a \in A\}, \leqq\rangle$ converges to x in $\langle P, d\rangle$, then the net $\langle\{d\langle x_a, x_b\rangle \mid \langle a, b\rangle \in A \times A\}, \prec\rangle$ converges to zero in R, where $\prec$ is the product order (i.e. $\langle a, b\rangle \succ \langle a_1, b_1\rangle \Leftrightarrow a \geqq a_1, b \geqq b_1$).*

Proof. I. It follows from (*) that each function of (a) is a Lipschitz continuous function with bound 1 (R is a metric space with metric $\{\langle r, s\rangle \to |r - s|\}$, see 18 A.5 (a)). By 18 A.9 each mapping of (a) is continuous. — II. To prove (b) it is sufficient to show that the closure of X is closed for each non-void $X \subset P$. If f is the function of (a) corresponding to X, then clearly $\bar{X} = f^{-1}[(0)]$. Since f is continuous and (0) is a closed subset of R, $\bar{X}$ is a closed subset of $\langle P, d\rangle$ by 16 A.6. — III. The open (closed) r-sphere about an $x \in P$ is clearly the inverse image of the open interval $]\!] -r, r [\![$ (closed interval $[\![-r, r]\!]$) of R under the function $\{y \to d\langle y, x\rangle\} : \langle P, d\rangle \to \mathsf{R}$ which is continuous by (a) because $d\langle y, x\rangle = \operatorname{dist}(y, (x))$. Now statement (c) follows from 16 A.6. — IV. Let r be a positive real. By 18 A.6 we can choose a residual subset B of $\langle A, \leqq\rangle$ such that $b \in B$ implies $d\langle x, x_b\rangle < r$. Clearly $B \times B$ is residual in $\langle A \times A, \prec\rangle$, and by the triangle inequality $d\langle x_a, x_b\rangle \leqq d\langle x_a, x\rangle + d\langle x, x_b\rangle < 2r$ for each $\langle a, b\rangle \in B \times B$.

Corollary. *If P is a pseudometrizable space and U is a neighborhood of a point x of P, then there exists a continuous function f on P such that $fx = 0$ and $fy \geqq 1$ for $y \in (P - U)$.*

In a semi-pseudometric space the statements of 18 A.12 need not hold. We know that a semi-pseudometric space need not be topological (cf. 18 A.5 e). In a semi-pseudometric space which is not topological there exists at least one open r-sphere which is not open. Indeed, at least one point has no local base consisting of open sets and open spheres about each point x form a local base at x. Now it follows from the proof of 18 A.12 that in a semi-pseudometric space which is not topological, the condition (a) is not fulfilled and it is easy to show that (d) may hold. Another example may be in place.

18 A.13. Example. Fix an element x of a set P and consider the real-valued relation d on $P \times P$ such that $d\langle y, z\rangle = 1$ if $x \neq y \neq z \neq x$ and $d\langle y, z\rangle = 0$ otherwise. It is easily seen that d is a semi-pseudometric for P. Let u be the closure induced by d. It is easily seen that

(a) $y \in uY$ if and only if $y \in P$, $Y \subset P$ and $y \in Y$ or $x \in Y$ or both $y = x$ and $Y \neq \emptyset$.

(b) If $y \in P - (x)$, then $u(y) = (x, y)$, $u(x) = P$, and consequently $uu(y) \supset u(x) = P$; thus $u(y) \neq uu(y)$ if P has at least three elements.

(c) Every sphere about x is equal to P, whereas every r-sphere about a $y \in (P - (x))$ with $r < 1$ is equal to (y, x). If $(y, x) \neq P$, then the set (x, y) is neither open nor closed in $\langle P, u\rangle$ (by (a)). It follows that, if P has at least three points, $y \in (P - (x))$ and $0 < r < 1$, then the open r-sphere about y as well as the closed r-sphere about y is neither open nor closed.

(d) Every continuous function on $\langle P, u\rangle$ is constant (compare with the corollary to 18 A.12).

Indeed, if f is a continuous function on $\langle P, u\rangle$, then $fy \in \overline{(fx)} = (fx)$, because $y \in u(x)$, and hence $fy = fx$ for each $y \in P$.

(e) Let $A = P - (x)$ and let $\leqq$ be the identity relation J_A, that is, $\leqq$ is the smallest order for A. If $A \neq \emptyset$, then $\langle J_A, \leqq\rangle$ is a net in $\langle P, d\rangle$ which converges to x (because P is the only neighborhood of x). On the other hand $d\langle y, z\rangle = 1$ if $y \neq z$, $y \in A$, $z \in A$.

Remark. In a pseudometric space the closure of each open r-sphere about a point x is contained in the closed r-sphere about x (because every closed sphere is a closed set in a pseudometric space). This is not true for semi-pseudometric spaces; e.g. in the semi-pseudometric space of 18 A.13 the closed and open r-spheres about a point y of $P - (x)$ with $r < 1$ are equal to (y, x) and the closure of (y, x) is P. Finally, it is to be noted that, in a pseudometric space, the closure of an open r-sphere about a point x need not be identical with the closed r-sphere about x. For example, if P is a set and d is the metric for P which is equal to 1 outside of the diagonal, then the open 1-sphere about each x is (x), but the closed 1-sphere about x is P and the set (x) is closed because the closure induced by d is discrete.

18 A.14. Remark. In a semi-pseudometric space open spheres need not be open even if the induced closure is topological. This will be clear from the proof of the theorem which follows (one can semi-metrize the space of reals such that no open sphere is open). It will be shown in 22 ex. 9 that there exists a topological semi-metrizable space which cannot be semi-metrized in such a manner that each open sphere is open.

18 A.15. Theorem. *A closure space $\langle P, u\rangle$ is semi-pseudometrizable if and only if there exists a sequence $\{U_n\}$ of subsets of $P \times P$ so that $U_n = U_n^{-1}$ for each n and $\{U_n[x] \mid n \in \mathsf{N}\}$ is a local base at x in $\langle P, u\rangle$ for each $x \in P$.*

Proof. I. If a semi-pseudometric d induces u, then we can put $U_n = \mathbf{E}\,\{\langle x, y\rangle \mid d\langle x, y\rangle < (n + 1)^{-1}\}$, because $U_n[x]$ is then the open $(n + 1)^{-1}$-sphere about x. —

II. Conversely, suppose $\{U_n\}$ fulfils the condition of the theorem. Put $V_n = \bigcap\{U_i \mid i \leqq n\}$. Clearly $\{V_i\}$ also fulfils the condition (of course, with U_n replaced by V_n). We shall define a semi-pseudometric for P as follows: if $\langle x, y\rangle$ belongs to no V_n then $d\langle x, y\rangle = 1$, if $\langle x, y\rangle$ belongs to each V_n then $d\langle x, y\rangle = 0$ and if $\langle x, y\rangle \in (V_n - V_{n+1})$ then $d\langle x, y\rangle = (n + 1)^{-1}$. Evidently $V_n[x]$ is the closed $(n + 1)^{-1}$-sphere about x in $\langle P, d\rangle$ for each x and n. It follows that d induces u.

18 A.16. Definition. A *continuous semi-pseudometric* for a closure space $\langle P, u\rangle$ is a semi-pseudometric d for $\langle P, u\rangle$ such that the closure induced by d is coarser than u, i.e., the identity mapping of $\langle P, u\rangle$ onto $\langle P, d\rangle$ is continuous. A *semi-neighborhood of the diagonal* of $\langle P, u\rangle \times \langle P, u\rangle$, where $\langle P, u\rangle$ is a closure space, is defined to be a neighborhood of the diagonal in $\operatorname{ind}(\langle P, u\rangle \times \langle P, u\rangle)$, that is, a subset U of $P \times P$ such that the set

$$(U \cap U^{-1})[x] = (U[x]) \cap (U^{-1}[x]) = \mathbf{E}\{y \mid \langle x, y\rangle \in U, \langle y, x\rangle \in U\}$$

is a neighborhood of x in $\langle P, u\rangle$ for each x in P.

Stated in other words, $U \subset P \times P$ is a semi-neighborhood of the diagonal of $\langle P, u\rangle \times \langle P, u\rangle$ if and only if, for each x in P, there exists a neighborhood V of x in $\langle P, u\rangle$ such that the cross $((x) \times V) \cup (V \times (x))$ is contained in U. On the other hand, $U \subset P \times P$ is a neighborhood of the diagonal in $\langle P, u\rangle \times \langle P, u\rangle$ if and only if, for each x in P, the square $V \times V$ is contained in U for some neighborhood V of x.

18 A.17. Theorem. *Each of the following conditions* (a), (b), (c) *and* (d) *is necessary and sufficient in order that a semi-pseudometric d for a closure space P be continuous*:

(a) *For each x in P and each positive real r there exists a neighborhood V of x in P such that $d\langle x, y\rangle < r$ for each y in V.*

(b) *For each x in P the function $\{y \to d\langle x, y\rangle\}$ on P is continuous at x.*

(c) *The function $d : \operatorname{ind}(P \times P) \to \mathrm{R}$ is continuous at each point of the diagonal.*

(d) *For each positive real r the set*

$$d^{-1}[\,]\!] \leftarrow, r [\![\,] = d^{-1}[<^{-1}[(r)]] = \mathbf{E}\{\langle x, y\rangle \mid d\langle x, y\rangle < r\}$$

is a semi-neighborhood of the diagonal of $P \times P$.

If d is a pseudometric then also each of the following two conditions is necessary and sufficient (*of course, each of these is always sufficient*):

(e) *The function $d : P \times P \to \mathrm{R}$ is continuous;*

(f) *For each positive real r, the set $\mathbf{E}\{\langle x, y\rangle \mid d\langle x, y\rangle < r\}$ is a neighborhood of the diagonal of $P \times P$.*

Proof. I. The conditions (a), (b) and (d) are restatements of the definition. Condition (c) is equivalent to (b) by virtue of 17 D.5 (remember that d is symmetric). – II. Obviously (e) implies (c) and (f) implies (d). Thus both conditions (e) and (f) are sufficient. – III. Condition (e) implies (f) because the inverse image under a continuous mapping of an open set is an open set (by 16 A.6). – IV. It remains to show that some of the conditions (a)–(d) imply (e). The proof of each of these implications requires the triangle inequality. It will be shown that (a) implies (e). Suppose (a), $\langle x, y\rangle \in P$ and $r > 0$. By (a) there exist neighborhoods U of x and V of y in P such that $d\langle x, x'\rangle < 2^{-1}r$ for x' in U and $d\langle y, y'\rangle < 2^{-1}r$ for y' in V. Now if $\langle x', y'\rangle \in U \times V$, then from formula (**) of 18 A.11 we obtain

$$|d\langle x, y\rangle - d\langle x', y'\rangle| \leqq d\langle x, x'\rangle + d\langle y, y'\rangle < 2.2^{-1}r = r$$

which establishes the continuity of $d : P \times P \to \mathsf{R}$ at the point $\langle x, y\rangle$ (by 18 A.7).

B. CLASSES OF SEMI-PSEUDOMETRICS

Pseudometrizable spaces possess many properties which are not possessed by semi-pseudometrizable spaces. In 18 A.12 we proved that every pseudometrizable space is topological and that there are enough (in the sense of the corollary of 18 A.12) continuous functions on it; on the other hand, the semi-pseudometrizable space of 18 A.13 is not topological and every continuous function is constant. Nevertheless, these two properties are far from sufficient for a semi-pseudometrizable space to be pseudometrizable. Now we are able to prove that every open cover of a pseudometrizable space has a locally finite (open) refinement. Nevertheless, an understanding of this property, which added to semi-pseudometrizability "almost" gives pseudometrizability, requires more advanced development and therefore we shall consider this property later. Without this property we cannot satisfactorily solve the so-called metrization problem: under what necessary and sufficient conditions is a given semi-pseudometrizable space pseudometrizable? On the other hand we can find some sufficient conditions which are, perhaps, more important in practice than any solution of the metrization problem.

Let us state the problem to be treated in this subsection as follows: given a semi-pseudometric d, we want to find sufficient conditions, involving no properties of the closure induced by d, for d to be topologically equivalent to a pseudometric.

The exposition will be based upon the fact that, given a semi-pseudometric d for a set P, there exists a pseudometric D for P such that $D\langle x, y\rangle \leqq d\langle x, y\rangle$ for each $\langle x, y\rangle$, and if D' is a pseudometric for P such that $D'\langle x, y\rangle \leqq d\langle x, y\rangle$ for each $\langle x, y\rangle$, then $D'\langle x, y\rangle \leqq D\langle x, y\rangle$ for each $\langle x, y\rangle$ (18 B.3).

For convenience we shall introduce some relations connected with the class of all real-valued relations.

18 B.1. Remark. Let $\leqq$ be the relation consisting of all pairs $\langle f, g\rangle$ such that f and g are real-valued relations, $\mathbf{D}f = \mathbf{D}g$ and $fx \leqq gx$ for each $x \in \mathbf{D}f$. Obviously $\leqq$ is an order for the class of all real-valued relations and, for each set X, the product ordered set R^X is an ordered subset of the class of all real-valued relations ordered by $\leqq$. Next, we shall denote by $+$ (., respectively) the partial composition on the class of all real-valued relations such that $\langle\langle f, g\rangle, h\rangle \in +$ ($\in .$, respectively) if and only if $\mathbf{D}f = \mathbf{D}g$ and $h = \{x \to fx + gx \mid x \in \mathbf{D}f\}$ ($h = \{x \to fx \,.\, gx \mid x \in \mathbf{D}f\}$). Thus $f + g$ is defined if and only if $\mathbf{D}f = \mathbf{D}g$, and if $f + g$ is defined, then $(f + g)\,x = fx + gx$ for each $x \in \mathbf{D}(f + g)$. Obviously both $+$ and $.$ are strongly associative, and hence, by 13 A.4, form categoroid structures. Next, both $+$ and $.$ are commutative. Finally, if r is a real number and f is a real-valued relation, then $r \,.\, f$ denotes the real-valued relation $\{x \to r \,.\, fx \mid x \in \mathbf{D}f\} = \{x \to r \mid x \in \mathbf{D}f\} \,.\, f$. Thus an expression $r \,.\, f$ has two meanings; if both r and f are relations, then $r \,.\, f = \{x \to rx \,.\, fx\}$ and if r is a real number and f is a relation than $r \,.\, f = \{x \to r \,.\, fx\}$.

18 B.2. Theorem. *The ordered class of all real-valued relations is boundedly order-complete. Let $\{f_a \mid a \in A\}$ be a non-void family of real-valued relations with common domain $P \times P$, where P is a set, and let there exist* $\sup \{f_a\}$ *and* $\inf \{f_a\}$. *Then*

(a) *If each f_a is symmetric, then so is* $\sup \{f_a\}$ *and* $\inf \{f_a\}$.

(b) *If $X \subset P \times P$ and $f_a \mid X = g$ for each a, then* $(\sup \{f_a\}) \mid X = (\inf \{f_a\}) \mid X = g$. *In particular, if all the f_a vanish on the diagonal of $P \times P$, then both* $\inf \{f_a\}$ *and* $\sup \{f_a\}$ *vanish on the diagonal.*

(c) *If each f_a fulfils the triangle inequality (i.e. $f_a\langle x, y\rangle \leqq f_a\langle x, z\rangle + f_a\langle z, y\rangle$), then* $\sup \{f_a\}$ *also fulfils the triangle inequality (although* $\inf \{f_a\}$ *need not do so).*

(d) *If each f_a is a semi-pseudometric then so is* $\sup \{f_a\}$ *and* $\inf \{f_a\}$.

(e) *If each f_a is a pseudometric, then* $\sup \{f_a\}$ *is also a pseudometric (although* $\inf \{f_a\}$ *need not be a pseudometric).*

Proof. The statements (a) and (b) are obvious, (d) follows from (a) and (b), (e) follows from (c) and (d). Statement (c) perhaps needs a detailed proof. Let f stand for $\sup \{f_a\}$ and let x, y, z be any elements of P. To prove $f\langle x, y\rangle \leqq f\langle x, z\rangle + f\langle z, y\rangle$ it is sufficient to show that $f\langle x, y\rangle - r \leqq f\langle x, z\rangle + f\langle z, y\rangle$ for each positive real r. Given a positive real r we can choose an a in A so that $f_a\langle x, y\rangle \geqq f\langle x, y\rangle - r$. Clearly, $f\langle x, y\rangle - r \leqq f_a\langle x, y\rangle \leqq f_a\langle x, z\rangle + f_a\langle z, y\rangle \leqq f\langle x, z\rangle + f\langle z, y\rangle$, which concludes the proof.

Remark. On can construct without difficulty two pseudometrics d and d_1 such that $\inf(d, d_1)$ is not a pseudometric. It is easily seen that if d and d_1 induce the same closure operation, say u, then $\inf(d, d_1)$, which is a semi-pseudometric, also induces u. On the other hand, we shall prove in ex. 11 the following surprising result: there exist two metrics d and d_1 inducing the closure structure of the interval $[\![0, 1]\!]$ of reals such that no pseudometric $d_2 \leqq \inf(d, d_1)$ induces the closure structure of $[\![0, 1]\!]$.

18 B.3. Corollary. *For each semi-pseudometric f there exists a greatest pseudometric d smaller than f, i.e. $d \leqq f$ and if a pseudometric d_1 is smaller than f, then $d_1 \leqq d$.*

Proof. Let $\mathscr{M}$ be the set of all pseudometrics smaller than f. Since the constant relation $\{z \to 0 \mid z \in \mathbf{D}f\}$ is a pseudometric smaller than f, the set $\mathscr{M}$ is non-void and by 18 B.3 $d = \sup \mathcal{M}$ is a pseudometric. Obviously d is the greatest pseudometric smaller than f.

We shall need the following description of the greatest pseudometric smaller than a given semi-pseudometric.

18 B.4. Theorem. *Let d be the greatest pseudometric smaller than a given semi-pseudometric f for a set P. Then*

(a) *For each $\langle x, y\rangle \in P \times P$ the number $d\langle x, y\rangle$ is the greatest lower bound of all the numbers*

(*) $\Sigma\{f\langle x_i, x_{i+1}\rangle \mid i \leqq (n-1)\}$,

where $n \in \mathrm{N}$ and $\{x_i \mid i \leqq n\}$ is a finite sequence in P such that $x = x_0$ and $y = x_n$.

(b) *If h is a one-to-one relation such that $\mathbf{D}h = \mathbf{E}h = P$ and $f = f \circ (h \times h)$, then $d = d \circ (h \times h)$ (i.e., if f is invariant under a one-to-one relation h on P ranging on P, then d is invariant under h as well).*

Proof. For brevity, call any finite sequence $\{x_i \mid i \leqq n\}$ in P such that $x_0 = x$ and $x_n = y$ a chain from x to y.

I. Denote by D the real-valued relation on $P \times P$ which assigns to each $\langle x, y\rangle \in \in P \times P$ the infimum of all the numbers (*) where $\{x_i \mid i \leqq n\}$ runs over all finite chains from x to y. If $\{x_i \mid i \leqq n\}$ is any chain from x to y, then the corresponding number (*) is greater than $\Sigma\{d\langle x_i, x_{i+1}\rangle \mid i \leqq n-1\}$ which, by the triangle inequality, is greater than $d\langle x, y\rangle$. As a consequence, $D\langle x, y\rangle \geqq d\langle x, y\rangle$ for each $\langle x, y\rangle \in P \times P$, i.e. $D \geqq d$. Since clearly $D \leqq f$, the proof of (a) will be complete if we show that D is a pseudometric.

II. Obviously D is non-negative and D vanishes on the diagonal. The symmetry of D follows immediately from the following two evident facts: given a pair $\langle x, y\rangle \in \in P \times P$, the single-valued relation ϱ, which assigns to each chain $\{x_i \mid i \leqq n\}$ from x to y the chain $\{x_{n-i} \mid i \leqq n\}$ from y to x, is one-to-one and ranges on the set of all chains from y to x; and the number (*) corresponding to a chain $\{x_i \mid i \leqq n\}$ from x to y is equal to the number (*) corresponding to the chain $\varrho\{x_i\} = \{x_{n-i} \mid i \leqq n\}$. It remains to prove the triangle inequality. Let $x, y, z \in P$. Let $\{x_i \mid i \leqq n\}$ be any chain from x to z, $\{y_i \mid i \leqq m\}$ be any chain from z to y. Consider the chain $\{z_j \mid i \leqq n + m\}$, where $z_j = x_j$ if $j \leqq n$ and $z_j = y_{j-n}$ if $j \geqq n$ (notice that $x_n = y_0$). Clearly $\{z_j\}$ is a chain from x to y and the number (*) corresponding to $\{z_j\}$ is the sum of numbers corresponding to $\{x_i\}$ and $\{y_i\}$. Thus

$$D\langle x, y\rangle \leqq \Sigma\{f\langle x_i, x_{i+1}\rangle\} + \Sigma\{f\langle y_i, y_{i+1}\rangle\}$$

where $\{x_i\}$ is any chain from x to z and $\{y_i\}$ is any chain from z to y. Taking the greatest lower bound over all finite chains $\{x_i\}$ from x to z we obtain

$$D\langle x, y\rangle \leqq D\langle x, z\rangle + \Sigma\{f\langle y_i, y_{i+1}\rangle\}$$

for each chain $\{y_i\}$ from z to y, and finally, taking the greatest lower bound over all $\{y_i\}$ we obtain the required triangle inequality.

III. The proof of (b) follows the proof of the symmetry of D in II. Suppose $f \circ (h \times h) = f$, where h is one-to-one and $\mathbf{D}h = \mathbf{E}h = P$. Fix a pair $\langle x, y \rangle$ in $P \times P$. We must show that $d\langle x, y \rangle = d\langle hx, hy \rangle$. Consider the single-valued relation ϱ which assigns to each chain $\{x_i\}$ from x to y the chain $\{hx_i\}$ from hx to hy. It follows from our assumptions on h that ϱ is one-to-one and ranges on the set of all chains from hx to hy. Now the conclusion follows from the fact that the number $(*)$ corresponding to a chain $\{x_i\}$ from x to y is equal to the number $(*)$ corresponding to the chain $\varrho\{x_i\} = \{hx_i\}$.

18 B.5. Examples. (a) First consider the semi-metric f for N such that $f\langle 0, n \rangle = n^{-1}$, $f\langle n, m \rangle = 1$ for each $n \neq 0 \neq m \neq n$. It is easily seen that 0 is the only accumulation point of $\langle \mathsf{N}, f \rangle$, and that the sequence $\{n + 1 \mid n \in \mathsf{N}\}$ converges to 0. Next, if d is the greatest pseudometric smaller than f, then $d\langle 0, n \rangle = n^{-1}$ and $d\langle n, m \rangle = n^{-1} + m^{-1}$ whenever $n \neq 0 \neq m \neq n$. It follows that f and d are topologically equivalent.

(b) Now we shall show that a semi-metric need not be topologically equivalent to the greatest smaller pseudometric. Let $\langle \mathsf{N}, u \rangle$ be a space such that 0 is the only accumulation point and the sequence $\{n + 1 \mid n \in \mathsf{N}\}$ converges to 0. By (a) the space $\langle \mathsf{N}, u \rangle$ is metrizable. Let Q be the discrete space whose underlying set is the two-point set $(0, 1)$ and let us consider the product space $\langle P, v \rangle = Q \times \langle \mathsf{N}, u \rangle$. The space $\langle P, v \rangle$ is metrizable because it can be metrized, for instance, by the metric d which assigns 1 to each point $\langle\langle 0, n \rangle, \langle 1, m \rangle\rangle$, n^{-1} to each point $\langle\langle i, 0 \rangle, \langle i, n \rangle\rangle$, $n \neq 0$, and $n^{-1} + m^{-1}$ to each point $\langle\langle i, n \rangle, \langle i, m \rangle\rangle$ where $n \neq 0 \neq m \neq n$. Now let f be the semi-metric for P such that $f\langle\langle 0, n \rangle, \langle 1, n \rangle\rangle = n^{-1}$ if $n \neq 0$ and $f\langle x, y \rangle = d\langle x, y \rangle$ otherwise. Clearly f induces the same closure as d but if D is the greatest pseudometric smaller than f, then $D\langle\langle 0, 0 \rangle, \langle 1, 0 \rangle\rangle = 0$ (consider the chains $\langle\langle 0, 0 \rangle, \langle 0, n \rangle, \langle 1, n \rangle, \langle 1, 0 \rangle\rangle$), and consequently, D is not topologically equivalent to f.

18 B.6. Definition. Two semi-pseudometrics d and d_1 are said to be *Lipschitz equivalent* if they are for the same set, say P, and the identity mapping of $\langle P, d \rangle$ onto $\langle P, d_1 \rangle$ as well as its inverse is Lipschitz continuous (see 18 A.8).

18 B.7. Theorem. *Lipschitz equivalent semi-pseudometrics are topologically equivalent. The relation* $\mathbf{E}\{\langle d, d_1 \rangle \mid d_1$ *is Lipschitz equivalent with* $d\}$ *is an equivalence on the class of all semi-pseudometrics. Two semi-pseudometrics d and d_1 are Lipschitz equivalent if and only if there exist positive reals r and s such that* $r \,.\, d \leqq d_1 \leqq s \,.\, d$.

The proof follows at once from the definitions and from properties of Lipschitz continuous mappings (18 A.9, 18 A.10).

Remark. Let $\prec$ be the relation consisting of all pairs $\langle d, d_1 \rangle$ such that d and d_1 are semi-pseudometrics and $d \leqq r \,.\, d_1$ for some positive real r. It is easy to see that $\prec$ is a quasi-order on the class of all semi-pseudometrics and $d \prec d_1 \prec d$ if and only if d and d_1 are Lipschitz equivalent.

18 B.8. *A semi-pseudometric d is Lipschitz equivalent to a pseudometric if and only if d is Lipschitz equivalent to the greatest pseudometric D smaller than d.*

Proof. "If" is self-evident. Conversely, suppose that $r . d_1 \leqq d \leqq s . d_1$ for some pseudometric d_1 and positive reals r and s. Since $r . d_1$ is a pseudometric smaller than d, we have $r.d_1 \leqq D$; since obviously $D \leqq d$, we find that D is Lipschitz equivalent to d_1, and hence to d.

Suppose that a semi-pseudometric f for a set P is Lipschitz equivalent to the greatest pseudometric d smaller than f, that is, $L.f \leqq d \leqq f$ for some positive real L. If $\{x_i \mid i \leqq n\}$ is a finite sequence in P, then $f\langle x_0, x_n\rangle \leqq L^{-1} . d\langle x_0, x_n\rangle \leqq \leqq L^{-1} \Sigma\{d\langle x_i, x_{i+1}\rangle \mid i \leqq (n-1)\} \leqq L^{-1} . \Sigma\{f\langle x_i, x_{i+1}\rangle \mid i \leqq (n-1)\}$. Conversely, if

$$f\langle x_0, x_n\rangle \leqq L^{-1} . \Sigma\{f\langle x_i, x_{i+1}\rangle \mid i \leqq n-1\},$$

for each finite sequence $\{x_i \mid i \leqq n\}$ in P, then $Lf \leqq d$ by 18 B.4. Thus we have proved:

18 B.9. *A semi-pseudometric for a set P is Lipschitz equivalent to a pseudometric if and only if*

$$f\langle x_0, x_n\rangle \leqq K . \Sigma\{f\langle x_i, x_{1+i}\rangle \mid i \leqq (n-1)\}$$

for some real K and each finite sequence $\{x_i \mid i \leqq n\}$ in P.

Now we shall prove that the inequality of 18 B.9 is fulfilled with $K = 2$ whenever f fulfils a very curious inequality involving sequences of four points. Another sufficient condition for the inequality of 18 B.9 will be given in ex. 7.

18 B.10. Lemma. *Let f be a real-valued relation on a class $P \times P$ such that $f\langle x, x\rangle = 0$ and $f\langle x, y\rangle \geqq 0$ for each $x \in P$, $y \in P$ (i.e. f is non-negative and vanishes on the diagonal) and*

$(*)$ $f\langle x_0, x_3\rangle \leqq 2 \max (f\langle x_0, x_1\rangle, f\langle x_1, x_2\rangle, f\langle x_2, x_3\rangle)$
for each sequence $\{x_i \mid i \leqq 3\}$ in P. Then

$(**)$ $f\langle x_0, x_{n+1}\rangle \leqq 2 \Sigma\{f\langle x_i, x_{i+1}\rangle \mid i \leqq n\}$
for each finite sequence $\{x_i \mid i \leqq n + 1\}$ in P.

Proof. Assuming that $(**)$ is not true, let us consider the smallest n in $\mathbf{N}$ such that $(*)$ is not true for an appropriate sequence $\{x_i \mid i \leqq n + 1\}$ and put $r = \Sigma\{f\langle x_i, x_{i+1}\rangle \mid i \leqq n\}$. Thus $f\langle x_0, x_{n+1}\rangle > 2r \geqq 0$. It is easily seen that $n > 3$ (if $n = -1$, then it is to be understood that the sum of a void family is zero). Let us consider the greatest integer m such that $-1 \leqq m \leqq n$ and $\Sigma\{f\langle x_i, x_{i+1}\rangle \mid i \leqq m\} \leqq \frac{1}{2}r$.

If $m = n$, then clearly $r = 0$ and hence $f\langle x_i, x_{i+1}\rangle = 0$ for each $i \leqq n$ (f is non-negative) which implies that $f\langle x_0, x_{n+1}\rangle = 0$ (because $(**)$ is true for $n \leqq 3$) and this contradicts our assumption $f\langle x_0, x_{n+1}\rangle > 0$. Thus $m < n$. According to the choice of n we have $f\langle x_0, x_{m+1}\rangle \leqq 2 \Sigma\{f\langle x_i, x_{i+1}\rangle \mid i \leqq m\} \leqq 2 . \frac{1}{2}r = r$. Clearly $f\langle x_{m+1}, x_{m+2}\rangle \leqq r$ and $\Sigma\{f\langle x_i, x_{i+1}\rangle \mid i = m + 2, \ldots, n\} \leqq \frac{1}{2}r$, and therefore, by our choice of n, $f\langle x_{m+2}, x_{n+1}\rangle \leqq 2 . \frac{1}{2} r = r$. Thus $f\langle x_0, x_{n+1}\rangle \leqq \leqq 2 \max (f\langle x_0, x_{m+1}\rangle, f\langle x_{m+1}, x_{m+2}\rangle, f\langle x_{m+2}, x_{n+1}\rangle) \leqq 2r$ which contradicts our assumption. The proof is complete.

Remark. If f is a symmetric real-valued relation on $P \times P$ vanishing on the diagonal and satisfying condition $(*)$, then f is non-negative (put $x_0 = x_2 = x$ and $x_1 = x_3 = y$ in $(*)$). Thus $(**)$ is true whenever f is symmetric, vanishes on the diagonal and satisfies $(*)$.

18 B.11. Theorem. *Let f be a semi-pseudometric for a set P such that $f\langle x_0, x_3\rangle \leqq \leqq 2 \max \{f\langle x_i, x_{i+1}\rangle \mid i \leqq 2\}$ for each sequence $\{x_i \mid i \leqq 3\}$ in P. Then there exists a pseudometric d such that $2^{-1} . f \leqq d \leqq f$, and if h is a one-to-one relation such that $\mathbf{D}h = \mathbf{E}h = P$ and $f \circ (h \times h) = f$, then $d \circ (h \times h) = d$. For d one can take the greatest pseudometric smaller than f.*

Proof. Consider the greatest pseudometric d smaller than f and apply 18 B.10 and 18 B.4.

18 B.12. Corollary. *Let P be a set and let $\{U_n\}$ be a sequence of subsets of $P \times P$ such that $U_0 = P \times P$, $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subset U_n = U_n^{-1} \supset \mathsf{J}_P$ for each n. Then there exists a pseudometric d for P such that*

$(*)$ $U_{n+1} \subset \mathbf{E}\{\langle x, y\rangle \mid d\langle x, y\rangle < 2^{-(n+1)}\} \subset U_n$

for each n, and if h is a one-to-one relation such that $\mathbf{D}h = \mathbf{E}h = P$ and $(h \times h)[U_n] = = U_n$ for each n, then $d \circ (h \times h) = d$.

Proof. Let us consider the real-valued relation f on $P \times P$ which is 2^{-n} on each set $U_n - U_{n+1}$ and zero on $\bigcap\{U_n\}$. Since $U_n = U_n^{-1} \supset \mathsf{J}_P$ for each n, f is a semi-pseudometric for P. If $\langle x_i, x_{i+1}\rangle \in U_{n+1}$ for each $i \leqq 2$, then $\langle x_0, x_3\rangle \in U_{n+1} \circ \circ U_{n+1} \circ U_{n+1} \subset U_n$ and hence $f\langle x_0, x_3\rangle \leqq 2^{-n} = 2 . 2^{-(n+1)}$, which proves that $f\langle x_0, x_3\rangle \leqq 2 \max \{f\langle x_i, x_{i+1}\rangle \mid i \leqq 2\}$ for each sequence $\{x_i \mid i \leqq 3\}$ in P. Let us consider the pseudometric d of 18 B.11. Clearly $(*)$ holds.

The foregoing result enables us to give a necessary and sufficient condition for a closure space to be pseudometrizable. Recall that by 18 A.15 a closure space P is semi-pseudometrizable if and only if there exists a sequence $\{U_n\}$ of symmetric subsets of $P \times P$ such that $\{U_n[x] \mid n \in \mathsf{N}\}$ is a local base at x for each x in P.

18 B.13. Theorem. *A closure space $\mathscr{P}$ is pseudometrizable if and only if there exists a sequence $\{V_n\}$ of symmetric subsets of $\mathscr{P} \times \mathscr{P}$ such that $V_{n+1} \circ V_{n+1} \subset V_n$ for each n and $\{V_n[x] \mid n \in \mathsf{N}\}$ is a local base at x for each x in $\mathscr{P}$.*

Proof. If the closure structure of a space $\mathscr{P}$ is induced by a pseudometric d then we can take $V_n = \mathbf{E}\{\langle x, y\rangle \mid d\langle x, y\rangle < 2^{-n}\}$. Conversely, given $\{V_n\}$, let us consider the sequence $\{U_n\}$ where $U_0 = |\mathscr{P}| \times |\mathscr{P}|$ and $U_n = V_{2n}$ for $n > 1$, and the pseudometric d of 18 B.12. It is easily seen that d induces the closure structure of $\mathscr{P}$.

18 B.14. Definition. A mapping f of one semi-pseudometric space $\langle P, d\rangle$ into another $\langle P_1, d_1\rangle$ is said to be *uniformly continuous* if for each positive real r there is a positive real s such that $d\langle x, y\rangle < s$ implies $d_1\langle fx, fy\rangle < r$, that is $(f \times f)[\mathbf{E}\{\langle x, y\rangle \mid d\langle x, y\rangle < s\}] \subset \mathbf{E}\{\langle x_1, y_1\rangle \mid d_1\langle x_1, y_1\rangle < r\}$. Semi-pseudometrics d and d_1 are said to be *uniformly equivalent* if they are for the same set, say P, and the mappings $\mathsf{J} : \langle P, d \rangle \to \langle P, d_1 \rangle$ and $\mathsf{J} : \langle P, d_1 \rangle \to \langle P, d \rangle$ are uniformly continuous.

18 B.15. Every Lipschitz continuous mapping is uniformly continuous and every uniformly continuous mapping is continuous. Any two Lipschitz equivalent semi-pseudometrics are uniformly equivalent and any two uniformly equivalent semi-pseudometrics are topologically equivalent. The composition of two uniformly continuous mappings is a uniformly continuous mapping. Thus the composition of mappings is a strongly associative partial composition on the class of all uniformly continuous mappings. The relation $\mathsf{E}\{\langle d, d_1 \rangle \mid d$ is uniformly equivalent with $d_1\}$ is an equivalence on the class of all semi-pseudometrics.

The proof is simple and therefore left to the reader.

Now we shall prove a very useful consequence of 18 B.11 and 18 B.12. The meaning of this result will be seen in Section 24.

18 B.16. Theorem. *A semi-pseudometric g for a set P is uniformly equivalent to a pseudometric if and only if the following condition is fulfilled:*

for each real $r > 0$ there exists a real $s > 0$ such that $g\langle x, y \rangle < s$, $g\langle y, z \rangle < s$ imply $g\langle x, z \rangle < r$.

Proof. I. Suppose that g is uniformly equivalent with a pseudometric d and let r be a positive real. Choose a positive real r_1 such that $d\langle x, y \rangle < r_1$ implies $g\langle x, y \rangle < r$, and then a positive real s such that $g\langle x, y \rangle < s$ implies $d\langle x, y \rangle < \frac{1}{2}r_1$. If $g\langle x, y \rangle < s$, $g\langle x, z \rangle < s$, then $d\langle x, z \rangle \leqq d\langle x, y \rangle + d\langle y, z \rangle \leqq 2 \cdot \frac{1}{2} \cdot r_1 = r_1$ and hence $g\langle x, z \rangle < r$. – II. Conversely, assuming that the condition is fulfilled, let us consider a sequence $\{r_n\}$ of positive reals such that $g\langle x_0, x_1 \rangle < r_{n+1}$, $g\langle x_0, x_2 \rangle < r_{n+1}$, $g\langle x_2, x_3 \rangle < r_{n+1}$ imply $g\langle x_0, x_3 \rangle < r_n$, and also the sequence $\{U_n\}$ where $U_0 = P \times P$ and $U_n = \mathsf{E}\{\langle x, y \rangle \mid g\langle x, y \rangle < r_n\}$ for $n \geqq 1$. Let d be the pseudometric of 18 B.12. It follows immediately from 18 B.12 that d is uniformly equivalent to g.

Corollary. *If a semi-pseudometric g is uniformly equivalent to some pseudometric and if h is a bijective relation on P with $g \circ (h \times h) = g$, then g is uniformly equivalent to a pseudometric d with $d \circ (h \times h) = d$.*

In conclusion we shall state a few simple results which will be used without any reference.

18 B.17. *If d is a semi-pseudometric (pseudometric) and r is a positive real, then $r \cdot d$ is a semi-pseudometric (pseudometric) Lipschitz equivalent to d, and $D = \{x \to \min(r, dx) \mid x \in \mathsf{D}d\}$ is a semi-pseudometric (pseudometric) uniformly equivalent to d but not Lipschitz equivalent to d provided that d is not bounded (i.e. such that $\mathsf{E}d$ is not a bounded subset of R).*

18 B.18. *Let φ be a non-negative real-valued relation whose domain is the set of all non-negative reals and let $\varphi 0 = 0$. Then*

(a) *If d is a semi-pseudometric then $\varphi \circ d$ is a semi-pseudometric, but $\varphi \circ d$ need not be a pseudometric if d is a pseudometric.*

(b) *If $\varphi(x + y) \leqq \varphi x + \varphi y$ for all non-negative reals x and y (a real-valued relation satisfying this condition is often said to be subadditive) and d is a pseudometric, then $\varphi \circ d$ is also a pseudometric.*

(c) *If $\varphi x \neq 0$ whenever $x \neq 0$ and if d is a semi-metric, then $\varphi \circ d$ is also a semi-metric.*

(d) *If $K . x \leqq \varphi x \leqq L . x$ for each $x \geqq 0$ and some reals K and $L, K > 0$, then $\varphi \circ d$ is Lipschitz equivalent to d for each semi-pseudometric d.*

(e) *If $\varphi : [\![0, \rightarrow]\!] \rightarrow [\![0, \rightarrow]\!]$ is continuous at 0 and d is a semi-pseudometric for a set P, then the identity mapping of $\langle P, d \rangle$ onto $\langle P, \varphi \circ d \rangle$ is uniformly continuous.*

C. SEMI-PSEUDOMETRIZATION OF SUBSPACES, SUMS AND PRODUCTS

Various semi-pseudometrizations of subspaces, products and sums of semi-pseudometrizable spaces will be discussed. The topological parts of the results of this section are summarized in the following theorem.

18 C.1. Theorem. *Each of the classes of all pseudometrizable, metrizable, semi-pseudo-metrizable and semi-metrizable spaces is hereditary and closed under countable products and arbitrary sums. The inductive product of two semi-pseudometrizable (semi-metrizable) spaces is a semi-pseudometrizable (semi-metrizable) space.*

18 C.2. *If Q is a subset of a set P and if d is a pseudometric (metric, semi-pseudometric, semi-metric) for P, then the* domain-restriction *d_Q of d to $Q \times Q$ is a pseudometric (metric, semi-pseudometric, semi-metric) for Q, called the* relativization *of d to Q or the* restriction *of d to a semi-pseudometric for Q, and the closure induced by d_Q is a relativization of the closure induced by d. The space $\langle Q, d_Q \rangle$ is sometimes called a* subspace *of $\langle P, d \rangle$.*

The proof is simple and therefore left to the reader.

18 C.3. Remark. If $\langle Q, v \rangle$ is a subspace of a semi-pseudometrizable space $\langle P, u \rangle$ and if v is induced by a semi-pseudometric d, then d is a relativization of some semi-pseudometric inducing the closure u. Actually, suppose that a semi-pseudometric D induces u and consider the real-valued relation d_1 on $P \times P$ such that $d_1\langle x, y \rangle = d\langle x, y \rangle$ if $\langle x, y \rangle \in Q \times Q$ and $d_1\langle x, y \rangle = D\langle x, y \rangle$ otherwise; obviously d is a restriction of d_1 to d and one can prove without difficulty that d_1 is a semi-pseudometric inducing u. On the other hand, if d is a pseudometric and $\langle P, u \rangle$ is pseudometrizable, then, in general, d is not a relativization of any pseudometric inducing u. For example, let us consider a non-void discrete subspace $\langle Q, v \rangle$ of a pseudometrizable space $\langle P, u \rangle$ such that Q has an accumulation point, say x, in $\langle P, u\rangle$ (thus $x \in P - Q$). Clearly v is induced by the metric d the only values of which are the numbers 0 and 1 (thus $d\langle x, y\rangle$ equals 0 or 1 for $x = y$ or $x \neq y$ respectively). Suppose that d is the relativization of a continuous pseudometric D for $\langle P, u\rangle$. Choose a sequence $\{x_n\}$ in Q which converges to x in $\langle P, u\rangle$. It follows that the sequence converges to x in $\langle P, D\rangle$. By 18 A.12 we find that the net $\{D\langle x_n, x_m\rangle \mid \langle n, m\rangle \in \mathsf{N} \times \mathsf{N}\}$ converges to zero in R. But this is impossible because $D\langle x_n, x_m\rangle = d\langle x_n, x_m\rangle = 1$ for each n and m such that $x_m \neq x_n$.

18 C.4. *If Q is a subset of a set P and a pseudometric D for Q is Lipschitz equivalent to the relativization to Q of a pseudometric d for P, then D is the relativization of a pseudometric Lipschitz equivalent to d.*

Proof. Suppose that there exist positive reals K and L such that $L \,.\, d\langle x, y\rangle \leqq D\langle x, y\rangle \leqq K \,.\, d\langle x, y\rangle$ for each x and y in Q. Consider the real-valued relation f on $P \times P$ which assigns to each $\langle x, y\rangle$ the number $D\langle x, y\rangle$ if $\langle x, y\rangle \in Q \times Q$ and $K \,.\, d\langle x, y\rangle$ otherwise. Clearly f is a semi-pseudometric for P and $L \,.\, d \leqq f \leqq K \,.\, d$. Let d_1 be the greatest pseudometric smaller than f. Thus $d_1 \leqq f$ and, $L \,.\, d$ being a pseudometric, $L \,.\, d \leqq d_1$, so that $L \,.\, d \leqq d_1 \leqq K \,.\, d$. On the other hand, using the description of d given in 18 B.4, we see immediately that D is a relativization of d_1.

18 C.5. Suppose that P is the sum of a family $\{P_a \mid a \in A\}$ of sets and $\{d_a\}$ is a family such that d_a is a semi-pseudometric for P_a. Choose a positive real number r and put, for each $\langle a, x\rangle$ and $\langle b, y\rangle$ in P,

$$\begin{aligned} d\langle\langle a, x\rangle, \langle b, y\rangle\rangle &= r \qquad &&\text{if} \quad a \neq b\,, \\ &= d_a\langle x, y\rangle &&\text{if} \quad a = b\,. \end{aligned}$$

It is easily seen that d is a semi-pseudometric, and the closure induced by d is the sum of the induced closures. Next, if each d_a is a semi-metric, then d is also a semi-metric. On the other hand, if all the d_a are pseudometrics then d need not be a pseudometric. For example, if there exist $a, b \in A$, $a \neq b$, $x \in P_a$ and $y, z \in P_b$ such that $d_b\langle y, z\rangle > 2r$, then $d\langle \mathrm{inj}_b\, y, \mathrm{inj}_b\, z\rangle = d_b\langle y, z\rangle > 2r = d\langle \mathrm{inj}_b\, y, \mathrm{inj}_a\, x\rangle + d\langle \mathrm{inj}_a\, x, \mathrm{inj}_b\, z\rangle$. Nevertheless, if $2r \geqq d_a\langle x, y\rangle$ for each a and $\langle x, y\rangle \in P_a \times P_a$, then clearly d is indeed a pseudometric. Now if $\{\langle P_a, u_a\rangle\}$ is a family of pseudometrizable spaces then we can find a family $\{d_a\}$ such that d_a is a pseudometric inducing u_a and $d_a\langle x, y\rangle \leqq 1$ for each x and y in P_a, and the semi-pseudometric d constructed above with $r = 1$ will be a pseudometric inducing the closure of $\Sigma\{\langle P_a, u_a\rangle\}$. It is to be noted that the mappings $\mathrm{inj}_a : \langle P_a, d_a\rangle \to \langle P, d\rangle$ are distance-preserving (i.e. isometric).

18 C.6. *Let $\{\langle P_a, d_a\rangle \mid a \in A\}$ be a finite family of semi-pseudometric spaces and let u_a be the closure induced by d_a. Each of the following three real-valued relations on $\Pi\{P_a\}$ is a semi-pseudometric for P inducing the closure of $\Pi\{\langle P_a, u_a\rangle\}$:*

(*) $d_1\langle\{x_a\}, \{y_a\}\rangle = \max\{d_a\langle x_a, y_a\rangle \mid a \in A\}$,

(**) $d_2\langle\{x_a\}, \{y_a\}\}\rangle = (\Sigma\{(d_a\langle x_a, y_a\rangle)^2 \mid a \in A\})^{\frac{1}{2}}$,

(***) $d_3\langle\{x_a\}, \{y_a\}\rangle = \Sigma\{d_a\langle x_a, y_a\rangle \mid a \in A\}$.

If all the d_a are pseudometrics, metrics or semi-metrics, then d_i have the same property. We always have

(†) $d_1 \leqq d_2 \leqq d_3 \leqq \operatorname{card} A \,.\, d_1$,

and consequently all three semi-pseudometrics are Lipschitz equivalent.

Proof. Obviously all the d_i are semi-pseudometrics, and if all the d_a are semi-metrics, then all the d_i are again semi-metrics. Also it is clear that d_1 and d_3 are pseudometrics if all the d_a are pseudometrics. The triangle inequality for d_2 follows at once from the following well-known Cauchy inequality

$$(\Sigma\{(r_a + s_a)^2\})^{\frac{1}{2}} \leqq (\Sigma\{r_a^2\})^{\frac{1}{2}} + (\Sigma\{s_a^2\})^{\frac{1}{2}},$$

which holds for each finite family $\{r_a\}$ and $\{s_a\}$ in R. The inequality (†) follows from the inequality

$$\max \{r_a\} \leqq (\Sigma\{r_a^2\})^{\frac{1}{2}} \leqq \Sigma\{r_a\} \leqq \operatorname{card} A \,.\, \max \{r_a\}$$

which holds for each finite family $\{r_a \mid a \in A\}$ of non-negative reals. Since all the d_i are Lipschitz equivalent, they are topologically equivalent, and consequently, to prove that each d_i induces the product closure $u = \Pi\{u_a\}$, it is sufficient to show that one of them induces u. It is clear that d_1 induces u because for each $\{x_a\} \in P$ and each positive real r we have $U = \Pi\{U_a \mid a \in A\}$, where U is the open r-sphere about $\{x_a\}$ in $\langle P, d_1\rangle$ and U_a is the open r-sphere about x_a in $\langle P_a, d_a\rangle$.

18 C.7. Remarks. (a) Under the assumptions of 18 C.6, if all the d_a are pseudometrics, then d_3 is the greatest pseudometric for P such that, for each α in A and each $\{x_a\} \in P$, the mapping of $\langle P_\alpha, d_\alpha\rangle$ into $\langle P, d_3\rangle$ which assigns to each $y \in P_\alpha$ the element $\{y_a\} \in P$ where $y_\alpha = y$ and $y_a = x_a$ for $a \in A - (\alpha)$, is a distance-preserving mapping.

(b) Under the assumptions of 18 C.6 let us consider the semi-pseudometrics $d_a^* = \{\langle x, y\rangle \to d_a\langle \mathrm{pr}_a\, x, \mathrm{pr}_a\, y\rangle \mid \langle x, y\rangle \in P \times P\}$, $a \in A$, for P. Clearly $d_1 = \sup \{d_a^*\}$, $d_2 = (\Sigma\{d_a^* \,.\, d_a^*\})^{\frac{1}{2}}$, $d_3 = \Sigma\{d_a^*\}$.

18 C.8. Theorem. *The mappings $\{\langle x, y\rangle \to x + y\} : \mathsf{R} \times \mathsf{R} \to \mathsf{R}$, and $\{\langle x, y\rangle \to x \,.\, y\} : \mathsf{R} \times \mathsf{R} \to \mathsf{R}$ are continuous (roughly speaking, the addition and the multiplication of reals are continuous).*

Proof. We know that $\{\langle x, y\rangle \to |x - y| \mid \langle x, y\rangle \in \mathsf{R} \times \mathsf{R}\}$ is a metric inducing the closure of R. By 18 C.6, $d = \{\langle\langle x_1, y_1\rangle, \langle x_2, y_2\rangle\rangle \to |x_1 - x_2| + |y_1 - y_2|\}$ is a metric inducing the closure of R × R. Now to prove that addition is continuous it is sufficient to show that $+ : \langle \mathsf{R} \times \mathsf{R}, d\rangle \to \mathsf{R}$ is a Lipschitz continuous mapping, and this is true because evidently

$$|(x_1 + y_1) - (x_2 + y_2)| \leqq |x_1 - x_2| + |y_1 - y_2| = d\langle\langle x_1, y_1\rangle, \langle x_2, y_2\rangle\rangle \,.$$

Multiplication is not Lipschitz continuous on the whole $\langle \mathsf{R} \times \mathsf{R}, d\rangle$, but its restriction to each subspace $Q = \langle [\![-M, M]\!] \times [\![-M, M]\!], d_Q\rangle$, $M > 0$, is

Lipschitz continuous. Indeed, from the obvious inequality $|x_1 . y_1 - x_2 . y_2| \leqq \leqq |y_2| . |x_1 - x_2| + |x_1| . |y_1 - y_2|$ we obtain

$$|x_1 . y_1 - x_2 . y_2| \leqq Md\langle\langle x_1, y_1\rangle, \langle x_2, y_2\rangle\rangle$$

for each $\langle\langle x_1, y_1\rangle, \langle x_2, y_2\rangle\rangle \in Q$. Thus multiplication is continuous on each indicated subspace Q of $\mathsf{R} \times \mathsf{R}$. But such Q interiorly cover $\mathsf{R} \times \mathsf{R}$, and thus multiplication is continuous.

From the last theorem we shall derive two important results; the first of these has already been proved directly and will be proved in a more general situation in the next section. It is to be noted that the proof is so arranged as to be applicable to the case where the range is a topological ring.

18 C.9. Corollary. *If the functions $f : P \to \mathsf{R}$ and $g : P \to \mathsf{R}$ are continuous, then so are the functions $f + g : P \to \mathsf{R}$ and $f . g : P \to \mathsf{R}$.*

Proof. Since the composition of two continuous mappings is continuous, the proof follows at once from the following obvious equalities:

$$f + g : P \to \mathsf{R} = (\{\langle x, y\rangle \to x + y\} : \mathsf{R} \times \mathsf{R} \to \mathsf{R}) \circ (\{x \to \langle fx, gx\rangle\} : P \to \mathsf{R} \times \mathsf{R})$$
$$f . g : P \to \mathsf{R} = (\{\langle x, y\rangle \to x . y\} : \mathsf{R} \times \mathsf{R} \to \mathsf{R}) \circ (\{x \to \langle fx, gx\rangle\} : P \to \mathsf{R} \times \mathsf{R})$$

18 C.10. *Let f be a real-valued relation on a closure space P. The function $f : P \to \to \mathsf{R}$ is continuous if and only if*

$$d = \{\langle x, y\rangle \to |fx - fy| \mid \langle x, y\rangle \in P \times P\}$$

is a continuous pseudometric for P.

Proof. I. Suppose that d is a continuous pseudometric for P. Clearly $(f : P \to \mathsf{R}) = (f : \langle|P|, d\rangle \to \mathsf{R}) \circ (\mathsf{J}_{|P|} : P \to \langle|P|, d\rangle)$. The identity mapping of P onto $\langle|P|, d\rangle$ is continuous by our assumption, and the mapping f of $\langle|P|, d\rangle$ into R is clearly Lipschitz continuous and hence continuous. Thus $f : P \to \mathsf{R}$ is continuous by 16 A.3. — II. Now suppose that $f : P \to \mathsf{R}$ is continuous. By 18 A.17 it is enough to show that $d : P \times P \to \mathsf{R}$ is continuous, and this follows from the following factorization of d:

$$d = g_3 \circ g_2 \circ g_1 \circ (f \times f),$$

where $g_1 = \{\langle x, y\rangle \to \langle x, -y\rangle\}$ (product of $\mathsf{J}_{|\mathsf{R}|}$ and $\{y \to -y\}$), $g_2 = \{\langle x, y\rangle \to \to x + y\}$ and $g_3 = \{z \to |z| \mid z \in \mathsf{R}\}$ — the latter is continuous because $||z_1| - |z_2|| \leqq \leqq |z_1 - z_2|$.

18 C.11. Remark. The product P of an uncountable family $\{P_a \mid a \in A\}$ of semi-pseudometrizable spaces need not be semi-pseudometrizable. Indeed, if $P \neq \emptyset$ and if P_a is not an accrete space for an uncountable number of a's in A, then P is not of a countable local character by 17 ex. 3. For example, if m is an uncountable cardinal, then the spaces R^m and $(0, 1)^m$ are not semi-pseudometrizable.

Now if $\{d_a \mid a \in A\}$ is a countable infinite family of semi-pseudometrics, then the numbers (*), (**) and (***) in 18 C.6 may be infinite and therefore d_i need not be meaningful. Nevertheless, if the d_a are sufficiently small, then d_1, d_2 and d_3 can be defined by the same formulae (in (*) one must write sup). To prove this we need some elementary current results on the convergence of series of real numbers which will be derived in the next section. According to the commutativity of the operation of taking products we may restrict ourselves to the case where $A = \mathsf{N}$. For the sake of completeness we shall give a direct proof of facts following from the elementary properties of series; the reader familiar with these can omit the first part of the proof.

18 C.12. *Let $\{\langle P_n, d_n\rangle\}$ be a sequence of semi-pseudometric spaces such that $d_n\langle x, y\rangle \leqq 2^{-n}$ for each $\langle x, y\rangle$ in $P_n \times P_n$ and n in N. Let u_n be the closure induced by d_n and let $\langle P, u\rangle$ be the product of the sequence $\{\langle P_n, u_n\rangle\}$. Let D_i, $i = 1, 2, 3$ be the real-valued relation on P defined by the following formulae:*

(*) $D_1\langle x, y\rangle = \sup\{d_n\langle \mathrm{pr}_n\, x, \mathrm{pr}_n\, y\rangle \mid n \in \mathsf{N}\}$,
(**) $D_2\langle x, y\rangle = (\Sigma\{(d_n\langle \mathrm{pr}_n\, x, \mathrm{pr}_n\, y\rangle)^2 \mid n \in \mathsf{N}\})^{\frac{1}{2}}$,
(***) $D_3\langle x, y\rangle = \Sigma\{d_n\langle \mathrm{pr}_n\, x, \mathrm{pr}_n\, y\rangle \mid n \in \mathsf{N}\}$.

Then D_i, $i = 1, 2, 3$, is a semi-pseudometric inducing the closure u, and if all d_n are semi-metrics, pseudometrics or metrics, then D_i has the same property.

Proof. I. Let D_i^k, $i = 1, 2, 3$, $k \in \mathsf{N}$, be defined by the corresponding formula for D_i, where the indices are restricted to the set of all $n \leqq k$. As in the proof of 18 C.6 one can show that $D_1^k \leqq D_2^k \leqq D_3^k$ and clearly $D_3^k\langle x, y\rangle \leqq \Sigma\{2^{-n} \mid n \leqq k\} \leqq 2$. Next, obviously, for each $\langle x, y\rangle$ in $P \times P$, the sequence $\{D_i^k\langle x, y\rangle \mid k \in \mathsf{N}\}$ is non-decreasing; being bounded, there exists its supremum which is equal to the only limit point of this sequence (which is defined to be the sum of corresponding series) i.e. $D_i\langle x, y\rangle$. Hence $D_1 \leqq D_2 \leqq D_3$, and if each D_i^k is a semi-pseudometric, or pseudometric then so is D_i. But clearly each D_i^k is a semi-pseudometric, and if each d_n is a pseudometric, then each D_i^k is also a pseudometric (the verification follows the proof of 18 C.6). Finally, if each d_a is a semi-metric and $x, y \in P$, $x \neq y$, then $\mathrm{pr}_n\, x \neq \mathrm{pr}_n\, y$ for some n, and consequently $D_i^n\langle x, y\rangle \geqq d_n\langle \mathrm{pr}_n\, x, \mathrm{pr}_n\, y\rangle > 0$.

II. It remains to show that each D_i induces the closure u. Because of the inequality $D_1 \leqq D_2 \leqq D_3$ it is sufficient to prove that D_3 is a continuous semi-pseudometric for $\langle P, u\rangle$, i.e. the closure induced by D_3 is coarser than u, and the closure induced by D_1 is finer than u (notice that $\mathsf{J}_P : \langle P, D_3\rangle \to \langle P, D_1\rangle$ is Lipschitz continuous). Since the open r-spheres about x in a semi-pseudometric space form a local base, it is sufficient to show that, given an element x of P, each open r-sphere about x in $\langle P, D_3\rangle$ is a neighborhood of x in $\langle P, u\rangle$, and that every neighborhood of x in $\langle P, u\rangle$ contains an open r-sphere about x in $\langle P, D_1\rangle$. Fix an $r > 0$ and choose an n_0 so that $\Sigma\{2^{-n} \mid n > n_0\} \leqq 2^{-n_0} < 2^{-1} . r$; choose a positive real s such that $(n_0 + 1) . s \leqq \leqq 2^{-1} . r$. It is clear that the set $U = \mathsf{E}\{y \mid n \leqq n_0 \Rightarrow d_n\langle \mathrm{pr}_n\, x, \mathrm{pr}_n\, y\rangle < s\}$ is contained in the open r-sphere about x in $\langle P, D_3\rangle$. But U is clearly a canonical neighborhood of x in $\langle P, u\rangle$ (each d_n induces u_n). Now let V be any neighborhood of x in $\langle P, u\rangle$. By definition of the product closure, the set V contains a neighborhood W of x in $\langle P, u\rangle$ of the form

$$\mathsf{E}\{y \mid n \leqq n_0 \Rightarrow \mathrm{pr}_n\, y \in W_n\},$$

where $n_0 \in \mathrm{N}$ and W_n is a neighborhood of $\mathrm{pr}_n\, x$ in $\langle P_n, u_n\rangle$ for each $n \leqq n_0$. Since d_n induces u_n, W_n contains an open r_n-sphere about $\mathrm{pr}_n\, x$ in $\langle P, d_n\rangle$. Now, if r is a positive real, $r < \min\{r_n \mid n \leqq n_0\}$, then clearly the open r-sphere about x in $\langle P, D_1\rangle$ is contained in W and hence in V, which completes the proof.

18 C.13. *Suppose that P_1 and P_2 are spaces and d_i is a semi-pseudometric inducing the closure structure of P_i, $i = 1, 2$. Then the closure structure of* ind $(P_1 \times P_2)$ *is induced by the semi-pseudometric d which assigns to each pair $\langle\langle x_1, x_2\rangle, \langle y_1, y_2\rangle\rangle$ the number 1 if $x_1 \neq y_1$ and $x_2 \neq y_2$, the number $d_1\langle x_1, y_1\rangle$ if $x_2 = y_2$ and $d_2\langle x_2, y_2\rangle$ if $x_1 = y_1$.*

Proof. Notice that, given a positive real $r < 1$, the open r-sphere about $\langle x_1, x_2\rangle$ in $\langle P_1 \times P_2, d\rangle$ is equal to the set $((x_1) \times U_2) \cup (U_1 \times (x_2))$ where U_i is the open r-sphere about x_i in $\langle P_i, d_i\rangle$.

D. SEMI-CONTINUOUS FUNCTIONS

Occasionally we shall need the concepts of upper and lower semi-continuous functions. It turns out that semi-continuity can be considered as continuity in usual sense relative to an appropriate closure operation for the range carrier, and therefore it can serve as a useful exercise on continuity. The proofs are omitted.

18 D.1. Definition. Let $\langle P, \leqq\rangle$ be a monotone ordered set. The collection consisting of the set P and of all intervals $]\!] \leftarrow, x [\![$, $x \in P$, is a base for open sets of a topological closure operation for P which will be denoted by u_+. Similarly, let u_- be the topological closure operation for P such that the collection consisting of the set P and all intervals $]\!] x, \rightarrow [\![$, $x \in P$ is an open base. We shall say that a mapping f of a closure space $\mathcal{Q}$ into $\langle P, \leqq\rangle$ is *upper semi-continuous* (*lower semi-continuous*) *at a point* $y \in \mathcal{Q}$ if the mapping $f: \mathcal{Q} \to \langle P, u_+\rangle$ ($f: \mathcal{Q} \to \langle P, u_-\rangle$) is continuous at y. We shall say that f is *upper semi-continuous* (*lower semi-continuous*) if f is upper semi-continuous (lower semi-continuous) at each point $y \in \mathcal{Q}$, i.e. if the mapping $f: \mathcal{Q} \to \langle P, u_+\rangle$ ($f: \mathcal{Q} \to \langle P, u_-\rangle$) is continuous.

It what follows the symbols P, $\leqq$, u_+ and u_- have the meaning introduced in 18 D.1.

18 D.2. The order closure u for $\langle P, \leqq\rangle$ is the coarsest closure for P finer than both u_+ and u_-. – Observe that $]\!] x, y [\![\,=\,]\!] \leftarrow, y [\![\cap]\!] x, \rightarrow [\![$.

18 D.3. A mapping f of a space $\mathcal{Q}$ into $\langle P, u\rangle$ is continuous (at y) if and only if the mapping $f: \mathcal{Q} \to \langle P, \leqq\rangle$ is simultaneously upper and lower semi-continuous (at y). – 18 D.2.

18 D.4. Each of the following two conditions is equivalent to the upper semi-continuity of a mapping f of a space $\mathcal{Q}$ into $\langle P, \leqq\rangle$:

(a) the set $f^{-1}[⟦\leftarrow, x ⟦] = \mathbf{E}\{y \mid fy < x\}$ is open in $\mathcal{Q}$ for each x in P;
(b) the set $f^{-1}[⟦ x, \rightarrow ⟧] = \mathbf{E}\{y \mid fy \geqq x\}$ is closed in $\mathcal{Q}$ for each x in P.

Similar result holds for lower semi-continuity. It may be worth-while to point out that:

18 D.5. A mapping $f : \mathcal{Q} \to \langle P, \leqq \rangle$ is upper semi-continuous (lower semi-continuous) at y if and only if, for each $x > fy$ $(x < fy)$ there exists a neighborhood U of y in $\mathcal{Q}$ such that $z \in U \Rightarrow fz < x$ $(z \in U \Rightarrow fz > x)$.

18 D.6. *Let X be a subset of a closure space $\mathcal{Q}$ and let f be the function on $\mathcal{Q}$ which is* 1 *on X and* 0 *on $|Q| - X$ (this function is often called the* characteristic function of the set *X in $\mathcal{Q}$). Then f is lower (upper) semi-continuous if and only if X is open (closed). It follows that f is continuous if and only if X is simultaneously closed and open.* — Use 18 D.4.

18 D.7. Let $\{f_a \mid a \in A\}$ be a family of mappings of a closure space $\mathcal{Q}$ into $\langle P, \leqq \rangle$. If f is a mapping of $\mathcal{Q}$ into $\langle P, \leqq \rangle$ such that $fy = \sup \{f_a y\}$ $(fy = \inf \{f_a y\})$ for each y in $\mathcal{Q}$, then f is lower (upper) semi-continuous whenever all the mappings f_a are lower (upper) semi-continuous.

Apply 18 D.5 or observe that

$$f^{-1}[⟧ \leftarrow, x ⟧] = \bigcap\{f_a^{-1}[⟧ \leftarrow, x ⟧] \mid a \in A\}$$
$$f^{-1}[⟦ x, \rightarrow ⟦] = \bigcap\{f_a^{-1}[⟦ x, \rightarrow ⟦] \mid a \in A\}$$

and apply 18 D.4.

18 D.8. Let f and g be functions on a space $\mathcal{Q}$ and $y \in \mathcal{Q}$. If both f and g are upper (lower) semi-continuous then so is $f + g$, and if in addition $f \geqq 0$, $g \geqq 0$, then fg is also upper (lower) semi-continuous. Next, f is upper semi-continuous if and only if $-f$ is lower semi-continuous. Finally, if $fy > 0$ for each y in $\mathcal{Q}$, then f is upper semi-continuous if and only if $1/f = \{y \to (fy)^{-1}\} : \mathcal{Q} \to \mathsf{R}$ is lower semi-continuous.

Corollary. *If f and g are continuous functions on $\mathcal{Q}$, then each of the mappings $f + g$, $f \cdot g$, $-f$ and if $fy > 0$ for each y, then also $1/f$ is continuous.*

Hint. The upper-semi-continuity of $f + g$: If U is a neighborhood of y in $\mathcal{Q}$ such that $fz < fy + r$ and $gz < gy + r$ for each $z \in U$, then $(f + g) z = fz + gz < < (f + g) y + 2r$ for each z in U.

E. STRUCTURE SPACES

In this subsection the structure space $\mathfrak{P}(\mathcal{R})$ of all prime ideals of a semi-ring $\mathcal{R}$ and the structure space $\mathfrak{M}(\mathcal{R})$ of all maximal ideals of a unital semi-ring $\mathcal{R}$ will be introduced. The structure space $\mathfrak{P}(\mathcal{R})$ of $\mathcal{R}$ is the set of all prime ideals of $\mathcal{R}$ endowed with the so-called *hull-kernel closure operation.* It turns out that the hull-kernel closure operation is uniquely determined by the set of all ideals ordered by the inclusion $\subset$. Therefore we shall define the structure space of an ordered set (subject to some conditions) and the structure space $\mathfrak{P}(\mathscr{R})$ will be defined as a subspace of the structure space of the ordered set of all ideals in $\mathscr{R}$.

We shall show that the ultrafilter space βX of a set X is a homeomorph of the structure space of the semi-ring $\langle \exp X, \cap, \cup \rangle$. In 41 E we shall investigate the structure space of the ring of all bounded continuous functions on a given closure space.

18 E.1. Let $\langle P, \leqq \rangle$ be a non-void ordered set such that each non-void subset X of P has a unique greatest lower bound which will be denoted by $\bigwedge X$. We shall write $x_1 \wedge x_2$ instead of $\bigwedge(x_1, x_2)$. Clearly $\bigwedge(X \cup Y) = (\bigwedge X) \wedge (\bigwedge Y)$ for all non-void X and Y.

Let us consider the single-valued relation v on $\exp P$ ranging in $\exp P$ which is defined as follows: $v\emptyset = \emptyset$ and $vX = \leqq [\bigwedge X]$ if $X \neq \emptyset$. By definition $v\emptyset = \emptyset$. and it is easily seen that $X \subset vX = vvX$ for each $X \subset P$, and $X \subset Y$ implies that $vX \subset vY$. In particular, $vX \cup vY \subset v(X \cup Y)$ for each X and Y. On the other hand, the inclusion $vX \cup vY \subset v(X \cup Y)$ need not be true, and therefore v need not be a closure operation (this inclusion obtains, e.g. if $\leqq$ is a monotone order).

Suppose that $x \in (v(X \cup Y) - (vX \cup vY))$. Clearly $X \neq \emptyset \neq Y$, $x \geqq \bigwedge(X \cup Y)$, but neither $x \geqq \bigwedge X$ nor $x \geqq \bigwedge Y$. If follows that

$$x \in v(X \cup Y) \Rightarrow x \in vX \text{ or } x \in vY$$

provided that x has the following property:

(*) if $x \geqq y \wedge z$, then $x \geqq y$ or $x \geqq z$.

In fact, $\bigwedge(X \cup Y) = \bigwedge X \wedge \bigwedge Y$ for all non-void X and Y. Let S be the set of all x with property (*) and let u be the single-valued relation on $\exp S$ ranging in $\exp S$ which assigns $S \cap vX$ to each $X \subset S$. It follows from the properties of v that u is a closure operation for S.

18 E.2. Definition. If $\langle P, \leqq \rangle$ is a non-void ordered set such that each non-void set has a unique greatest lower bound, then the topological space $\langle S, u \rangle$ defined in 18 E.1 is called the *structure space of* $\langle P, \leqq \rangle$.

18 E.3. Example. Let $\langle P, \leqq \rangle$ be a monotone ordered set satisfying the assumptions of 18 E.2, that is to say, $\mathscr{P}$ is boundedly order-complete and $\mathscr{P}$ has a least element. Let us consider the structure space $\langle S, u \rangle$ of $\langle P, \leqq \rangle$. Clearly each element x of P has property (*) of 18 E.1, and therefore $S = P$. The set uX, $X \neq \emptyset$, is the interval $[\![\bigwedge X, \rightarrow [\![$. Thus u is the closure of upper semi-continuity for $\langle P, \leqq \rangle$ (as defined in 18 D.1).

18 E.4. Let $\mathscr{R} = \langle R, +, . \rangle$ be a commutative semi-ring and let $\langle P, \leqq \rangle$ be the set of all ideals of $\mathscr{R}$ ordered by $\subset$. Suppose that $P \neq \emptyset$. If X is a non-void set of ideals of $\mathscr{R}$, i.e. if X is a non-void subset of P, then $\bigcap X = \bigcap\{x \mid x \in X\}$ is an ideal, and clearly $\bigcap X$ is the unique greatest lower bound of X in $\langle P, \subset \rangle$. Thus $\langle P, \subset \rangle$ satisfies the assumptions of 18 E.2 and therefore the structure space $\langle S, u\rangle$ of $\langle P, \subset\rangle$ is well-defined.

Recall that an ideal x is said to be prime if $a \,.\, b \in x$ implies that either $a \in x$ or $b \in x$. We shall prove that each prime ideal x belongs to S, i.e. has property (*) of 18 E.1. If $x \supset x_1 \cap x_2$ and $a_i \in x_i - x$ ($-$ denotes set-difference), then $a_1 \,.\, a_2 \in x_1 \cap \cap x_2$ (because $b_i a_i = a_i b_i \in x_i$ for each $b_i \in R$) and so certainly $a_1 \,.\, a_2 \in x$; since x is a prime ideal we have $a_1 \in x$ or $a_2 \in x$ which contradicts our assumption $a_i \notin x$. Thus the set of all prime ideals is contained in S.

18 E.5. Definition. If $\mathscr{R}$ is a commutative semi-ring and $\langle S, u\rangle$ is the space of 18 E.4, then the subspace of $\langle S, u\rangle$ consisting of all prime ideals will be called the *structure space of prime ideals* of $\mathscr{R}$ and will be denoted by $\mathfrak{P}(\mathscr{R})$.

18 E.6. *The collection of all sets of the following form is a closed base for* $\mathfrak{P}(\mathscr{R}) : h(a) = \mathsf{E}\{x \mid x \in \mathfrak{P}(\mathscr{R}), a \in x\},\ a \in |\mathscr{R}|$.

Proof. Evidently these sets are closed. Assuming that y does not belong to the closure of X in $\mathfrak{P}(\mathscr{R})$, we shall show that $y \notin h(a)$, $h(a) \supset X$ for some a. By definition the prime ideal y does not contain $\bigcap\{x \mid x \in X\}$ and hence we can choose an a in $\bigcap X - y$. Clearly the element a has the required properties.

18 E.7. Let $\mathscr{R} = \langle R, +, .\rangle$ be a commutative semi-ring with unity which will be denoted by 1. We know that each ideal in $\mathscr{R}$ is contained in a maximal ideal in $\mathscr{R}$. We shall prove that each maximal ideal is a prime ideal. Assuming x maximal, $a_1, a_2 \notin x$, $a_1 a_2 \in x$, we shall derive a contradiction. Since x is a maximal ideal we have $1 = b_1 a_1 + c_1 = b_2 a_2 + c_2$ for some $b_i \in R$ and $c_i \in x$. Since $a_1 a_2 \in x$, we find that $1 = 1 \,.\, 1 = (b_1 a_1 + c_1)(b_2 a_2 + c_2) \in x$ and this contradicts the fact that $1 \notin x$.

18 E.8. Definition. If $\mathscr{R}$ is a commutative semi-ring with unity, then the subspace of the structure space $\mathfrak{P}(\mathscr{R})$ of $\mathscr{R}$ consisting of all maximal ideals of $\mathscr{R}$ will be called the *structure space of maximal ideals* of $\mathscr{R}$ and will be denoted by $\mathfrak{M}(\mathscr{R})$.

18 E.9. Ultrafilter space. The ultrafilter space βX of a given set X was defined in 14 B.12. We shall prove that βX is a homeomorph of $\mathfrak{M}(\mathscr{R})$ where $\mathscr{R} = \langle \exp X, \cap, \cup\rangle$. Recall that $\emptyset$ is the unit element of $\mathscr{R}$, X is the zero of $\mathscr{R}$, ideals of $\mathscr{R}$ coincide with filters on X, and maximal ideals of $\mathscr{R}$ coincide with ultrafilters on X. Next, recall that a proper filter $\mathscr{X}$ on X is an ultrafilter if and only if, for each subset Y of X, either Y or $X - Y$ belongs to $\mathscr{X}$. It follows that each prime ideal is a maximal ideal (and, of course, each maximal ideal is a prime ideal). Thus $\mathfrak{P}(\mathscr{R}) = \mathfrak{M}(\mathscr{R})$, and the underlying set of $\mathfrak{M}(\mathscr{R})$ is the set ult X of all ultrafilters on X. Now we shall construct a homeomorphism f of βX onto $\mathfrak{M}(\mathscr{R})$. The elements of $\beta X - X$ are free ultrafilters on X, i.e. ultrafilters $\mathscr{X}$ such that $\bigcap \mathscr{X} = \emptyset$. Put $f\mathscr{X} = \mathscr{X}$ for $\mathscr{X}$ in $\beta X - X$. If $x \in X$, then the collection of all $Y \subset X$, $x \in Y$, is an ultrafilter on X; let fx be this ultrafilter.

We know that any non-free ultrafilter is necessarily of the form fx, $x \in X$, and clearly $x \neq y$ implies that $fx \neq fy$. Thus f is a bijective mapping. It remains to show that the mappings f and f^{-1} are continuous. It follows from the definition of βX (and it was stated in 15 ex. 12) that the sets of the form

$$(*)\ \mathrm{o}Y = Y \cup \mathbf{E}\{\mathscr{X} \mid Y \in \mathscr{X} \in (\beta X - X)\}, \quad Y \subset X,$$

form an open base for βX. Since $\mathrm{o}Y = \beta X - \mathrm{o}(P - Y)$ for each $Y \subset X$, the sets $(*)$ form a closed base for βX. Clearly

$$f[\mathrm{o}Y] = \mathbf{E}\{\mathscr{X} \mid Y \in \mathscr{X} \in \mathrm{ult}\, X\}.$$

By 18 E.6 the sets $f[\mathrm{o}Y]$ form a closed base for $\mathfrak{M}(\mathscr{R})$. Thus f carries a closed base for βX onto the closed base for $\mathfrak{M}(\mathscr{R})$; it follows that f is a homeomorphism.

18 E.10. Let $\mathscr{R} = \langle R, +, . \rangle$ be a ring with unity, consisting of bounded real-valued relations on a set X, i.e. the elements of R are real-valued relations on X and the addition and the multiplication are defined pointwise; that is to say, $\mathscr{R}$ is a subring of the product ring R^X consisting of bounded relations (not necessarily all). In 41 E we shall examine the structure space $\mathfrak{M}(\mathscr{R})$ for the case when R consists of all bounded f such that $f: \langle X, u \rangle \to \mathsf{R}$ is continuous for a fixed closure operation u for X. In this case the ring $\mathscr{R}$ has the following property: for each $x \in X$ the set

$$(*) \qquad \mathbf{E}\{f \mid fx = 0\}$$

is a maximal ideal of $\mathscr{R}$. Thus we have a mapping φ of $\langle X, u \rangle$ into $\mathfrak{M}(\mathscr{R})$ which assigns to each x the maximal ideal $(*)$. We shall show in 41 E that this mapping is continuous, and if φ is one-to-one, then φ is a homeomorphism.

19. TOPOLOGIZED ALGEBRAIC STRUCTS

This section contains definitions and theorems relating to general functional analysis which are needed at various places throughout the book. Although the reader is not presumed to have made a study of functional analysis a prerequisite for studying this book, some acquaintance with basic facts is needed; our development rather formally points out the basic ideas and applies the purely topological results of earlier sections. Sometimes general results will be applied to the particular case of real numbers without any detailed treatment of elementary facts. In the later development we shall always try to apply topological results to general functional analysis.

The reader is possibly familiar with the fact that a topological group is a struct $\langle G, \sigma, u \rangle$ such that $\langle G, \sigma \rangle$ is a group, u is a closure for G and the mappings $\sigma : \langle G, u \rangle \times \langle G, u \rangle \to \langle G, u \rangle$ and $\{x \to x^{-1}\} : \langle G, u \rangle \to \langle G, u \rangle$ (x^{-1} denotes the inverse of x in $\langle G, \sigma \rangle$) are continuous. Similarly, a topological ring is a struct $\langle R, \sigma, \mu, u \rangle$ such that $\langle R, \sigma, \mu \rangle$ is a ring, $\langle R, \sigma, u \rangle$ is a topological group and the mapping $\mu : \langle R, u \rangle \times \langle R, u \rangle \to \langle R, u \rangle$ is continuous. The last condition is sometimes replaced by the weaker condition that μ: ind $(\langle R, u \rangle \times \langle R, u \rangle) \to \langle R, u \rangle$ be continuous. Thus the topological requirements consist of conditions which demand that certain mappings be continuous, disregarding the algebraic relations among the various compositions involved, and sometimes algebraic properties of single compositions (e.g., of being a group structure). Therefore it is natural to begin with a pair $\langle \sigma, u \rangle$ where σ is an internal composition on a set G and u is a closure operation for G (such a pair will be termed a topologized internal composition), and to study the topological requirements for such a pair. Then some results concerning more complicated topologized internal structures will follow by combining the results concerning individual topologized compositions. A similar development will be carried out for structs containing an external composition.

An auxiliary study of topologized internal compositions in A is followed by an examination of topological groups, rings and fields in B. For the later development, the normed groups and rings are of particular interest. An auxiliary study of topologized external compositions in C is followed by a study of topological modules and algebras. Particular attention is given to normed modules and algebras because the normed algebras of bounded continuous, uniformly continuous or proximally continuous functions will be studied in detail in later sections (25 E, F, 41 C, E).

Combinations of topological and algebraic requirements imply some topological properties of topologized algebraic structs (e.g., every topological group is a topological space) and their mappings. Therefore continuous homomorphisms will be examined in E.

Subsection F contains some results concerning series in a commutative topologized semi-group which will be needed later.

A. TOPOLOGIZED INTERNAL COMPOSITIONS

At first sight it might seem natural to define a continuous internal compotision as a triple $\langle \sigma, u, v \rangle$ such that σ is an internal composition, u is a closure for $\mathbf{D}\sigma$, v is a closure for $\mathbf{E}\sigma$ and the mapping $\sigma : \langle \mathbf{D}\sigma, u \rangle \to \langle \mathbf{E}\sigma, v \rangle$ is continuous. Nevertheless this definition is too general. Usually, a composition σ on a set P and a closure operation u for P are given, and we need to know whether the mapping $\sigma : \langle P \times P, v \rangle \to \langle P, u \rangle$ is continuous, where v is the product closure $u \times u$ or the inductive product closure $\operatorname{ind}(u \times u)$. More precisely, the continuity of σ relative to thc product closure or relative to the inductive product closure usually must be assumed in various theorems. It is to be noted that sometimes there are situations in which some other "product" closures are of certain interest, for example the sequential product which will be considered in Section 35. Here we restrict ourselves to the product closure and the inductive product closure.

Thus our basic concept will be a pair $\langle \sigma, u \rangle$ where σ is an internal composition and u is a closure for $\mathbf{DD}\sigma$, and we will define the continuity of such a pair in various ways.

19 A.1. Definition. A *topologized internal composition* is a pair $\langle \sigma, u \rangle$ such that σ is an internal composition and u is a closure for $\mathbf{DD}\sigma$. A *topologized semi-group* is a struct $\langle P, \sigma, u \rangle$ such that $\langle \sigma, u \rangle$ is a topologized internal composition and $\langle P, \sigma \rangle$ is a semi-group. A *topologized internal composition* $\langle \sigma, u \rangle$ is said to be *continuous* (*inductively continuous*) if the mapping

$$\sigma : \langle \mathbf{DD}\sigma, u \rangle \times \langle \mathbf{DD}\sigma, u \rangle \to \langle \mathbf{DD}\sigma, u \rangle$$
$$(\sigma : \operatorname{ind}(\langle \mathbf{DD}\sigma, u \rangle \times \langle \mathbf{DD}\sigma, u \rangle) \to \langle \mathbf{DD}\sigma, u \rangle)$$

is continuous. A *topologized semi-group* $\langle P, \sigma, u \rangle$ is said to be *continuous* or *inductively continuous* if $\langle \sigma, u \rangle$ is continuous or inductively continuous, respectively. A *composition* σ on a space $\langle P, u \rangle$ is said to be *continuous* or *inductively continuous* if $\langle \sigma, u \rangle$ has the corresponding property.

We shall use the term continuous (inductively continuous) internal composition instead of continuous (inductively continuous) topologized internal composition, and similarly continuous (inductively continuous) semi-group instead of continuous (inductively continuous) topologized semi-group. Instead of internal composition we shall say composition.

For example, a topologized composition $\langle\sigma, u\rangle$ is continuous provided that u is a discrete or an accrete closure (recall that the product of two discrete closures is a discrete closure, and a mapping of a discrete space into any space as well as a mapping of any space into an accrete space is continuous). Addition is a continuous group structure on the space R of reals and multiplication is a continuous semi-group structure on the space R of reals (by 18 C.8).

It is to be noted that there is another term which frequently appears in the literature and which is used with two different meanings, namely a topological semi-group is usually either a continuous or inductively continuous semi-group such that the closure structure is topological.

19 A.2. Every continuous composition is an inductively continuous composition. – The inductive product closure is finer than the product closure.

Because of the importance of the concepts introduced, we shall describe them more directly in the two propositions which follow.

19 A.3. A composition σ on a space $\langle P, u\rangle$ is inductively continuous if and only if the following condition is fulfilled:

The mappings $\{y \to y\sigma x\} : \langle P, u\rangle \to \langle P, u\rangle$ and $\{y \to x\sigma y\} : \langle P, u\rangle \to \langle P, u\rangle$ are continuous for each x in P (that is, the left translations and the right translations are continuous).

Corollary. *If σ is a continuous composition on a space $\mathscr{P}$ then the translations are continuous.*

19 A.4. *Let σ be a composition on a space $\langle P, u\rangle$. In order that σ be continuous it is necessary and sufficient that for each x and y in P and each neighborhood U of $x\sigma y$ there exist a neighborhood V of x and a neighborhood W of y such that $[V]\,\sigma\,[W] \subset U$. In order that σ be inductively continuous it is necessary and sufficient that for each x and y in P and each neighborhood U of $x\sigma y$ there exist a neighborhood V of x and a neighborhood W of y so that $[V]\,\sigma y \subset U$ and $x\sigma[W] \subset U$.*

Corollary. *If σ is continuous and e is the neutral element of σ, then for each neighborhood U of e there exists a neighborhood V of e such that $[V]\,\sigma\,[V] \subset U$.*

19 A.5. Examples. (a) Let $\langle P, u\rangle$ be a non-void space, and denote the product space $\langle P, u\rangle^P$ by $\langle Q, v\rangle$. Let us consider the composition $\sigma = \{\langle f, g\rangle \to \to g \circ f\}$ on Q; thus σ is the restriction of the composition $\circ$ to Q. It is easily seen that σ is a semi-group structure on Q. Consider the triple $\langle Q, \circ, v\rangle$.

(α) The right translation by g is continuous if and only if $g : \langle P, u\rangle \to \langle P, u\rangle$ is a continuous mapping.

First suppose that g is continuous; we shall prove that the mapping $\{f \to g \circ f\} : : \langle Q, v\rangle \to \langle Q, v\rangle$ is continuous. Let $\{f_a \mid a \in A\}$ be a net converging to f in $\langle Q, v\rangle$, that is, for each $x \in P$ the net $\{f_a x \mid a \in A\}$ converges to fx in $\langle P, u\rangle$. We must show that the net $\{g \circ f_a\}$ converges to $g \circ f$ in $\langle Q, v\rangle$, i.e., that $\{(g \circ f_a)\, x \mid a \in A\}$ converges to $(g \circ f)\, x$ in $\langle P, u \rangle$ for each x in P. But $(g \circ f_a)\, x = g(f_a x)$ and $(g \circ f)\, x = g(fx)$. Since g is continuous and $\{f_a x\}$ converges to fx, then necessarily $\{g(f_a x)\}$ converges to $g(fx)$.

Conversely, suppose that $g : \langle P, u \rangle \to \langle P, u \rangle$ is not continuous. There exists a point x of P and a net $\{x_a \mid a \in A\}$ which converges to x in $\langle P, u \rangle$ such that the net $\{gx_a\}$ does not converge to gx (by 16 A.8). For each a let f_a be the constant relation $\{y \to x_a\}$ and f be the constant relation $\{y \to x\}$. Clearly the net $\{f_a\}$ converges to $\{f\}$ in $\langle Q, v \rangle$. On the other hand, the net $\{g \circ f_a\}$ does not converge to $g \circ f$. Indeed, $(g \circ f_a)\, y = gx_a$ and $(g \circ f)\, y = gx$ for each y in P, i.e. $g \circ f_a$ and $g \circ f$ are constant relations; but $\{gx_a\}$ does not converge to gx by assumption.

(β) The left translations are continuous, i.e. for each $g \in Q$ the mapping $\{f \to f \circ g\}$ of $\langle Q, v \rangle$ into $\langle Q, v \rangle$ is continuous.

This is almost self-evident. Suppose that a net $\{f_a\}$ converges to f in $\langle Q, v \rangle$. To prove that the net $\{f_a \circ g\}$ converges to $f \circ g$ in $\langle Q, v \rangle$ it is sufficient to show that the net $\{(f_a \circ g)\, x\}$ converges to $(f \circ g)\, x$ in $\langle P, u \rangle$ for each $x \in P$. But $(f_a \circ g)\, x = f_a(gx)$, $(f \circ g)\, x = f(gx)$ and $\{f_a y\}$ converges to fy in $\langle P, u \rangle$ for each y in P by our assumption, which gives the result.

(b) Let $\mathscr{P}$ be a closure space. Since the composite of two continuous mappings is a continuous mapping (16 A.3), the composition of mappings $\circ$ is an internal composition on the set $\mathbf{C}(\mathscr{P}, \mathscr{P})$ of continuous mappings of $\mathscr{P}$ into itself, and hence $\langle \mathbf{C}(\mathscr{P}, \mathscr{P}), \circ \rangle$ is a semi-group. It follows from (a) that this semi-group endowed with the closure of pointwise convergence is an inductively continuous semi-group. On the other hand, in general this semi-group is not continuous. This will be proved in the exercises.

19 A.6. *If σ is an inductively continuous semi-group structure on a space $\langle P, u \rangle$ and an element a of P has an inverse element, then the translations $(\{x \to a\sigma x\} : \langle P, u \rangle \to \langle P, u \rangle$ and $\{x \to x\sigma a\} : \langle P, u \rangle \to \langle P, u \rangle)$ are homeomorphisms.*

Proof. Let a^{-1} be the inverse element of a. The mappings $\{y \to a^{-1}\sigma y\} : \langle P, u \rangle \to \langle P, u \rangle$ and $\{y \to y\sigma a^{-1}\} : \langle P, u \rangle \to \langle P, u \rangle$ are inverses of the mappings in question because $a\sigma(a^{-1}\sigma x) = (a\sigma a^{-1})\, \sigma x = x$ and $(x\sigma a^{-1})\, \sigma a = x\sigma(a^{-1}\sigma a) = x$ for each x. Since all four of the mappings considered are continuous (by 19 A.3), they are homeomorphisms.

Corollary. *If σ is a continuous semi-group structure on a space $\mathscr{P}$ and an element a of $\mathscr{P}$ possesses an inverse, then the left translation by a as well as the right translation by a are homeomorphisms.*

Let σ be a composition on a set P and let us consider the relation ϱ consisting of all pairs $\langle a, b \rangle$ such that b is the inverse of a. If σ is a semi-group structure, then ϱ is a single-valued relation (cf. 6 B.10). Now if σ is enriched by a closure u, it is sometimes important to know whether the mapping ϱ of the subspace $\mathbf{D}\varrho$ of $\langle P, u \rangle$ into itself is continuous. For convenience we shall introduce the following terminology.

19 A.7. Definition. The *inversion of a topologized semi-group structure* $\langle \sigma, u \rangle$ is the mapping $\{x \to x^{-1}\}$ of the subspace of $\langle \mathbf{DD}\sigma, u \rangle$ consisting of all elements x of P possessing an inverse into itself. The inversion of a topologized semi-group $\langle P, \sigma, u \rangle$ is the inversion of $\langle \sigma, u \rangle$.

19 A.8. *If the inversion f of a topologized semi-group structure is continuous, then f is a homeomorphism.*

Proof. If b is the inverse of a, then a is the inverse of b, and consequently $f = f^{-1}$. It follows that if f is continuous, then f is a homeomorphism.

For example, the inversion of $\langle \mathsf{R}, + \rangle$ is continuous; in fact the inversion of $\langle \mathsf{R}, + \rangle$ is an isometry of the metric space R.

The concluding part is devoted to the definitions of the restriction of a topologized composition and the product of topologized compositions.

19 A.9. Definition. A *restriction of a topologized composition* $\langle \sigma, u \rangle$ is a topologized internal composition $\langle \varrho, v \rangle$ such that ϱ is a restriction of σ and v is a relativization of u. The definitions of a *restriction of a topologized semi-group structure* or a *group structure* are evident. A *sub-semi-group or a subgroup of a topologized semi-group or group* $\langle P, \sigma, u \rangle$ is a sub-semi-group or subgroup $\langle Q, \varrho \rangle$ of $\langle P, \sigma \rangle$ enriched by the relativization v of u to Q.

19 A.10. Theorem. *Restrictions of a continuous or inductively continuous composition are continuous or inductively continuous respectively. If the inversion of a topologized semi-group structure $\langle \sigma, u \rangle$ is continuous, then the inversion of each restriction of $\langle \sigma, u \rangle$ is continuous.* — Evident.

19 A.11. Definition. Let $\{\langle \sigma_a, u_a \rangle \mid a \in A\}$ be a family of topologized internal compositions. The *compositional product*, or simply *product*, of $\{\langle \sigma_a, u_a \rangle\}$ is the compositional product σ of $\{\sigma_a\}$ (6 E.6) enriched by the product closure u, that is, $\langle \mathbf{DD}\sigma, u \rangle = \Pi\{\langle \mathbf{DD}\sigma_a, u_a \rangle \mid a \in A\}$ and $\{x_a\} \sigma \{y_a\} = \{x_a \sigma_a y_a\}$. The product of a family $\{\langle P_a, \sigma_a, u_a \rangle \mid a \in A\}$ where each $\langle \sigma_a, u_a \rangle$ is a composition on P_a enriched by a closure is defined to be the triple $\langle P, \sigma, u \rangle$ where $\langle \sigma, u \rangle$ is the product of $\{\langle \sigma_a, u_a \rangle\}$ and $P = \mathbf{DD}\sigma$.

19 A.12. Theorem. *Let $\langle \sigma, u \rangle$ be the compositional product of a family $\{\langle \sigma_a, u_a \rangle\}$ of topologized compositions. If all $\langle \sigma_a, u_a \rangle$ are continuous or inductively continuous then so is $\langle \sigma, u \rangle$. If each σ_a is a semi-group structure and the inversion of each $\langle \sigma_a, u_a \rangle$ is continuous, then so is the inversion of $\langle \sigma, u \rangle$.*

Corollary. *The product of a family of continuous or inductively continuous semi-groups (groups) is a continuous or inductively continuous semi-group (group, respectively).*

Proof. All statements follow from the fact that the product of any family of continuous mappings is a continuous mapping (17 C.13). The inversion of $\langle \sigma, u \rangle$ is the product of the family $\{f_a\}$ where each f_a is the inversion of $\langle \sigma_a, u_a \rangle$, and any translation g of $\langle \sigma, u \rangle$ is the product of a family $\{g_a\}$ where each g_a is an appropriate translation of $\langle \sigma_a, u_a \rangle$ (e.g. if g is the left translation by $\{x_a\}$ then g_a is the left translation by x_a). Thus we have proved the statements concerning the inversion and the inductive continuity of $\langle \sigma, u \rangle$. For the proof of the statement concerning continuity of $\langle \sigma, u \rangle$ we shall need some auxiliary mappings. For brevity write $P_a = \mathbf{D}\mathbf{D}\sigma_a$, $P = \mathbf{D}\mathbf{D}\sigma$; thus $P = \Pi\{P_a\}$. Let f be the product of the family $\{\sigma_a : \langle P_a, u_a \rangle \times \langle P_a, u_a \rangle \to \langle P, a \rangle\}$; thus f is continuous. Consider the mapping h of $\langle P, u \rangle \times \langle P, u \rangle$ into $\Pi\{\langle P_a, u_a \rangle \times \langle P_a, u_a \rangle\}$ which assigns to each $\langle \{x_a\}, \{y_a\} \rangle$ the point $\{\langle x_a, y_a \rangle\}$. By 17 C.19 the mapping h is a homeomorphism. Clearly $\sigma : \langle P, u \rangle \times \langle P, u \rangle \to \langle P, u \rangle = f \circ h$. The proof is complete.

19 A.13. Example. Let $\mathscr{P}$ be a closure space and let $\mathbf{C}(\mathscr{P}, \mathscr{P})$ be the set of all continuous mappings of $\mathscr{P}$ into itself endowed with the restriction of the composition of mappings as a semi-group structure and with the closure of pointwise convergence. By 19 A.5 (b), $\mathbf{C}(\mathscr{P}, \mathscr{P})$ is an inductively continuous semi-group. Consider the subset H of $\mathbf{C}(\mathscr{P}, \mathscr{P})$ consisting of all homeomorphisms of $\mathscr{P}$ onto itself. Clearly H endowed with the restriction of compositions of mappings as a semi-group structure and with the closure of pointwise convergence is a sub-semi-group of $\mathbf{C}(\mathscr{P}, \mathscr{P})$. Since $\mathbf{C}(\mathscr{P}, \mathscr{P})$ is inductively continuous, H is inductively continuous as well. On the other hand H is a group (because the inverse h^{-1} of a homeomorphism h is the inverse of the element h of H). Thus H is an inductively continuous group. It will be shown in the exercises that neither H nor the inversion of H need be continuous.

B. TOPOLOGICAL GROUPS, RINGS AND FIELDS

In the foregoing subsection we introduced the concepts of an inductively continuous group, a continuous group, an inductively continuous group with continuous inversion and a continuous group with continuous inversion. The latter is particularly important and will be studied in this section under the current name topological group. It will be shown here that the underlying space of a topological group is a topological space; moreover, in Section 24, we will see that the underlying space of a topological group is uniformizable.

Now let $\langle P, \sigma, \mu, u \rangle$ be a ring $\langle P, \sigma, \mu \rangle$ enriched by a closure operation u (simply a topologized ring). It may be shown on examples that all of the following cases are possible: each of the compositions $\langle \sigma, u \rangle$ and $\langle \mu, u \rangle$, independently of each other, may be continuous or inductively continuous, need not be inductively continuous, and the inversion may but need not be continuous. One of all these possibilities, namely that $\langle P, \sigma, u \rangle$ is a topological group and $\langle \mu, u \rangle$ is continuous is currently studied under the term topological ring.

19 B.1. Definition. A *group structure* σ on a set G and a *closure* u for the set G are said to be *compatible* if $\langle \sigma, u \rangle$ is a continuous composition with a continuous inversion, that is, if the following two conditions are fulfilled:

(g 1) The mapping $\sigma : (\langle G, u \rangle \times \langle G, u \rangle) \to \langle G, u \rangle$ is continuous.
(g 2) The mapping $\{x \to x^{-1}\} : \langle G, u \rangle \to \langle G, u \rangle$ is continuous.

If a group structure σ on G and a closure u for G are compatible, then σ is said to be an *admissible group structure for the space* $\langle G, u \rangle$ and u is said to be an *admissible closure for the group* $\langle G, \sigma \rangle$. A *topological group* is a triple $\langle G, \sigma, u \rangle$ such that $\langle G, \sigma \rangle$ is a group and u is an admissible closure for the group $\langle G, \sigma \rangle$.

As a trivial example notice that the discrete and the accrete closures for a set G are admissible for any group $\langle G, \sigma \rangle$.

In what follows the notation will be as abbreviated as possible. The group composition will usually be written multiplicatively, that is, it will be denoted by . . If we say that $\mathcal{G}$ is a topological group, then the group structure of $\mathcal{G}$ will be denoted by . and the closure structure will be indicated, as usual, by u.

Sometimes it is convenient to replace the conditions (g 1) and (g 2) by a single condition.

19 B.2. *A closure u for G is admissible for a group $\langle G, . \rangle$ if and only if the following condition is fulfilled:*

(g) *the mapping $\{\langle x, y \rangle \to x \,.\, y^{-1}\}$ of the product $\langle G, u \rangle \times \langle G, u \rangle$ into $\langle G, u \rangle$ is continuous.*

Proof. I. First let us suppose (g 1) and (g 2). The mapping $f = \{\langle x, y \rangle \to \langle x \,.\, y^{-1} \rangle\}$ of $\langle G, u \rangle \times \langle G, u \rangle$ onto itself is continuous as the product of two continuous mappings $\{x \to x\}$ and $\{y \to y^{-1}\}$ the latter being continuous by (g 2). The mapping under question is continuous as the composition of f and $\{\langle x, z \rangle \to x \,.\, z\}$: $\langle G, u \rangle \times \langle G, u \rangle \to \langle G, u \rangle$ the latter being continuous by (g 1). – II. Conversely, suppose (g). Let 1 be the neutral element of the group G. Since clearly $\{x \to \langle 1, x \rangle \mid x \in G\} : \langle G, u \rangle \to \langle G, u \rangle \times \langle G, u \rangle$ is continuous and $\{\langle 1, x \rangle \to 1 \,.\, x^{-1} = x^{-1}\}$ is continuous by (g), their composition $\{x \to x^{-1}\}$ is continuous, yielding (g 2). Finally, the mapping in (g 1) is continuous as the composite of two continuous mappings, namely $\{\langle x, y \rangle \to \langle x, y^{-1} \rangle\}$ and $\{\langle x, z \rangle \to x \,.\, z^{-1}\}$, the former being continuous as the product of two continuous mappings and the latter by (g).

19 B.3. *If $\mathcal{G}$ is a topological group, then the mappings $\{x \to x^{-1}\} : \mathcal{G} \to \mathcal{G}$ and $\{x \to a \,.\, x \,.\, b\} : \mathcal{G} \to \mathcal{G}$, where a and b belong to $\mathcal{G}$, are homeomorphisms.*

Proof. Since in a group every element has an inverse, the statement follows from 19 A.8 and 19 A.6.

Corollary. *Let $\mathcal{G}$ be a topological group and a and b be elements of $\mathcal{G}$. A subset U of G is a neighborhood of x if and only if $a \,.\, U \,.\, b$ is a neighborhood of $a \,.\, x \,.\, b$ (notice the cases of $a = 1$ or $b = 1$). A subset U of $\mathcal{G}$ is a neighborhood of a point x if and only if U^{-1} is a neighborhood of x^{-1} (notice the special case of $x = 1$).*

19 B.4. *Every topological group is a topological space, more precisely, the underlying closure space of a topological group is topological.*

Proof. Let $\mathscr{G}$ be a topological group. According to 15 A.2 it is sufficient to show that for each neighborhood U of a point x of $\mathscr{G}$ there exists a neighborhood V of x such that U is a neighborhood of each point of V. By virtue of the foregoing proposition it is sufficient to prove this for $x = 1$. Since the group structure is continuous at $\langle 1, 1 \rangle$, given a neighborhood U of 1, we can choose a neighborhood V of 1 so that $V . V \subset U$. It follows from the corollary of the preceding proposition that U is a neighborhood of each point of V. Indeed, $y . V$ is a neighborhood of y for each $y \in |\mathscr{G}|$ by the corollary, and clearly $y . V \subset V . V$ provided that $y \in V$.

Remark. In Section 24 we will see that the closure structures of topological groups possess further important properties (e.g. they are uniformizable).

19 B.5. Examples. (a) According to 18 C.8 the natural closure for the reals is admissible for the additive group $\langle \mathsf{R}, + \rangle$ of real numbers. In what follows the letter R will also be used to denote the additive group of reals, endowed with the order closure. (b) According to 18 C.8 and 18 ex. 10 the relativization of the natural closure for R to $\mathsf{R} - (0)$ is admissible for the multiplicative group $\mathsf{R} - (0)$ of the reals.

Sometimes it is convenient to define an admissible closure for a group by neighborhood systems at points. According to the corollary of 19 B.3 it is sufficient to define the neighborhood system $\mathscr{U}$ at the neutral element, and the neighborhood system at x is then $x . [\mathscr{U}]$. If $\mathscr{U}$ is a filter in $\exp |\mathscr{G}|$ such that $1 \in U$ for each U in $\mathscr{U}$, then $x . [\mathscr{U}]$ is a filter such that $x \in V$ for each V in $x . [\mathscr{U}]$. By Theorem 14 B.10 there exists exactly one closure for the set $|\mathscr{G}|$ such that $x . [\mathscr{U}]$ is the neighborhood system at x. But this closure need not be compatible with the group structure of $\mathscr{G}$. Now we give necessary and sufficient conditions on $\mathscr{U}$ for this closure to be admissible.

19 B.6. Theorem. *Let $\mathscr{G}$ be a group. A filter $\mathscr{U}$ on $\mathscr{G}$ is a neighborhood system at the neutral element* 1 *with respect to an admissible closure for the group $\mathscr{G}$ if and only if the following conditions are fulfilled*:

(gn 1) *for each U in $\mathscr{U}$ there exists a V in $\mathscr{U}$ with $V . V \subset U$;*

(gn 2) *if $U \in \mathscr{U}$ then $U^{-1} \in \mathscr{U}$;*

(gn 3) 1 *belongs to each U in $\mathscr{U}$;*

(gn 4) *for each x in $\mathscr{G}$ and U in $\mathscr{U}$, $(x . U . x^{-1}) \in \mathscr{U}$.*

Proof. First let U be the neighborhood system at 1 with respect to an admissible closure u for the group $\mathscr{G}$. Conditions (gn 1) and (gn 2) follow readily from (g 1) and (g 2), respectively. The third condition is obviously fulfilled, for a neighborhood of a point x contains x. Finally, the last condition is an immediate consequence of 19 B.3, because $\{y \to x . y . x^{-1}\}$ is a homeomorphism of $\langle |\mathscr{G}|, u \rangle$ onto itself which carries 1 into 1 and V into $x . V . x^{-1}$.

Conversely, let $\mathscr{U}$ be a filter on $\mathscr{G}$ satisfying all the conditions (gn i). By 14 B.10 there exists a closure u for $\mathscr{G}$ such that $x . [\mathscr{U}]$ is the neighborhood system at x for each x in $\mathscr{G}$. It will be shown that u is admissible for the group $\mathscr{G}$. We shall prove (g), i.e. that $\{\langle x, y \rangle \to x . y^{-1}\}$ is continuous. Let $x_0, y_0 \in \mathscr{G}$ and let $U = x_0 . y_0^{-1} . V$ be a neighborhood of $x_0 . y_0^{-1}$; thus $V \in \mathcal{U}$. We want to find a neighborhood W of 1 such that $x \in x_0 . W$, $y \in y_0 . W$ implies $x . y^{-1} \in U$. If $x = x_0 . u$, $y = y_0 . v$, then

$$(*) \qquad (x_0 . y_0^{-1})^{-1} . (x . y^{-1}) = y_0 . u . v^{-1} . y_0^{-1}.$$

By (gn 1), (gn 2) and (gn 4) we can find a neighborhood W of 1 such that $y_0 . W . W^{-1} . y_0^{-1} \subset V$ (prove!). From (*) it follows at once that W possesses the required properties.

Sometimes we shall define an admissible closure for a group by a local base at the neutral element. The corresponding theorem, which is an immediate consequence of the preceding result, is as follows.

19 B.7. *Let $\mathcal{G}$ be a group. A base $\mathcal{V}$ of a filter on $\mathcal{G}$ is a local base at the neutral element with respect to an admissible closure for the group $\mathcal{G}$ if and only if the following four conditions are fulfilled*:

(gnb 1) *for each V in $\mathcal{V}$ there exists a U in $\mathcal{V}$ with $U . U \subset V$;*

(gnb 2) *for each V in $\mathcal{V}$ there exists a U in $\mathcal{V}$ with $U^{-1} \subset V$;*

(gnb 3) *each $V \in \mathcal{V}$ contains the neutral element of $\mathcal{G}$;*

(gnb 4) *for each V in $\mathcal{V}$ and x in $\mathcal{G}$ there exists a U in $\mathcal{V}$ with $U \subset x . V . x^{-1}$.*

Remark. A neighborhood V of the neutral element 1 of a topological group $\mathcal{G}$ is said to be symmetric if $V = V^{-1}$. If V is any neighborhood of 1, then $V . V^{-1}$, $V^{-1} \cap V$, $V^{-1} \cup V$ are symmetric neighborhoods of 1. It is easy to see that the collection of all symmetric neighborhoods of the neutral element 1 is a local base at 1. If n is a positive integer, then the collection of all sets of the form V^n, where V is a neighborhood of 1, is also a local base at 1 (V^n is defined by induction: $V^{p+1} = V^p . V$).

Remark. If $\mathcal{G}$ is a commutative group, then the postulates (gn 4) of 19 B.6 and (gnb 4) of 19 B.7 are automatically fulfilled. On the other hand, if $\mathcal{G}$ is not commutative, then the first three conditions may be fulfilled but not the fourth. For example, if $\mathcal{H}$ is a subgroup of a group $\mathcal{G}$ and if $\mathcal{U}$ is the collection of all subsets of $\mathcal{G}$ containing $\mathcal{H}$, then $\mathcal{U}$ satisfies the conditions (gn i), $i = 1, 2, 3$ and (gn 4) is satisfied if and only if $\mathcal{H}$ is an invariant subgroup of $\mathcal{G}$. Recall that by Definition 8 D.7 (and Remark 8 D.8) a subgroup $\mathcal{H}$ of a group $\mathcal{G}$ is invariant if (and only if) $x . y . x^{-1} \in \mathcal{H}$ for each x in $\mathcal{G}$ and y in $\mathcal{H}$. An *inner automorphism* of $\mathcal{G}$ is defined to be either a relation of the form $\varrho_x = \{y \to x . y . x^{-1} \mid y \in \mathcal{G}\}$ with x in $\mathcal{G}$, or a mapping $\varrho_x : \mathcal{G} \to \mathcal{G}$. It is easily seen that a subgroup $\mathcal{H}$ of $\mathcal{G}$ is invariant if and only if $\varrho[|\mathcal{H}|] \subset |\mathcal{H}|$ for each inner automorphism ϱ of $\mathcal{G}$.

19 B.8. Definition. If $\langle G, ., u\rangle$ is a topological group, $\langle H, *\rangle$ is a subgroup of $\langle G, .\rangle$ and v is the relativization of u to H, then by 19 A.10 $\langle H, *, v\rangle$ is a topological group which will be called a *subgroup of the topological group* $\langle G, ., u\rangle$.

19 B.9. Theorem. *If H is a subgroup of a topological group $\mathcal{G}$, then $\overline{H}$ is also a subgroup of G. If H is an invariant subgroup, then $\overline{H}$ is also an invariant subgroup.*

Proof. To prove the first statement we must show that $x, y \in \bar{H}$ implies $x \,.\, y^{-1} \in \bar{H}$. If $x, y \in \bar{H}$, then clearly $\langle x, y \rangle \in \bar{H} \times \bar{H}$ which implies, according to the continuity of the mapping $\{\langle x, y \rangle \to x \,.\, y^{-1}\}$, that $x \,.\, y^{-1} \in \bar{H}$. Now let H be an invariant subgroup of $\mathscr{G}$. Thus $f[H] \subset H$ for each inner automorphism f of $\mathscr{G}$. Since every inner automorphism f is continuous by 19 B.3, we have $f[\bar{H}] \subset \bar{H}$ for each inner automorphism f of $\mathscr{G}$. But this means that $\bar{H}$ is an invariant subgroup.

19 B.10. *A subgroup H of a topological group $\mathscr{G}$ is closed in $\mathscr{G}$ if and only if $U \cap H = U \cap \bar{H} \neq \emptyset$ for some open subset U of $\mathscr{G}$.*

Proof. The "only if" part is obvious. Conversely, suppose that $U \cap H = U \cap \bar{H} \neq \emptyset$ for some open subset U of $\mathscr{G}$. Let x be any point of $\bar{H}$. We have to prove $x \in H$. First let us choose a point y in $U \cap H$ and a neighborhood V of the neutral element 1 so that $y \,.\, V \subset U$. Since $x \in \bar{H}$, it follows that the set $(V \,.\, x) \cap H$ is non-void, and consequently we can choose a point z in it. Now clearly we have $z \,.\, x^{-1} \in V$ and hence $y \,.\, z \,.\, x^{-1} \in y \,.\, V \subset U$. Since $\bar{H}$ is a group and $y, z, x^{-1} \in \bar{H}$, the element $y \,.\, z \,.\, x^{-1}$ must belong to $\bar{H}$. As a consequence, $y \,.\, z \,.\, x^{-1} \in U \cap \bar{H}$ and by our hypothesis, $y \,.\, z \,.\, x^{-1} \in U \cap H$, in particular $y \,.\, z \,.\, x^{-1} \in H$. Since y and z also belong to H, x must belong to H, which concludes the proof.

19 B.11. *A subgroup of a topological group is open if and only if its interior is non-void. Every open subgroup is closed.*

Proof. If a subgroup H of a topological group $\mathscr{G}$ is a neighborhood of a point x, then H is a neighborhood of each of its points, i.e. H is open, because $f = \{z \to y \,.\, z \,.\, x^{-1}\}$ is a homeomorphism of $\mathscr{G}$ onto $\mathscr{G}$ (by 19 B.3) which carries x in y, and if moreover $y \in H$, then $f[H] = H$. The converse implication is obvious. The second statement is a straightforward consequence of 19 B.10 (put $U = H$).

19 B.12. Definition (and proposition). If $\{\langle G_a, \sigma_a, u_a \rangle\}$ is a family of topological groups, then by 19 A.12 $\langle \Pi\{G_a\}, \Pi\{\sigma_a\}, \Pi\{u_a\} \rangle$ is a topological group; this topological group is termed the product of the family $\{\langle G_a, \sigma_a, u_a \rangle\}$.

19 B.13. Examples. (a) If $m \neq 0$ is any cardinal then R^m, considered as a product of the additive topological group R, is a topological group. (b) If $m \neq 0$, then $(\mathsf{R} - (0))^m$, considered as a product of the multiplicative group $\mathsf{R} - (0)$, is a topological group. (c) Let 2 be the discrete group consisting of two points 0 and 1 such that 0 is the neutral element (in consequence, $1 + 1 = 0$ if $+$ denotes the group composition). For any $m \neq 0$, 2^m is a topological group.

If $\{\mathscr{G}_a \mid a \in A\}$ is a non-void family of commutative topological groups such that $\mathscr{G}_a = \mathscr{G}$ for each a in A, then in addition to the product group $\mathscr{G}^A$ we can define another closure which is admissible for the underlying abstract group G^A, the so called closure of uniform convergence.

19 B.14. Definition (and proposition). *Let us consider a commutative topological group $\mathscr{G} = \langle G, \cdot, u \rangle$ and a non-void set P. Let $\mathscr{U}$ be the neighborhood system at the neutral element of $\mathscr{G}$ and let $\mathscr{V}$ be the collection of all subsets V of G^P such that*

$V \supset U^P$ *for some* U *in* $\mathcal{U}$. *It is easily seen that* $\mathcal{V}$ *fulfils conditions* (gn i), $i = 1, 2, 3, 4$ *of* 19 B.6 (*relative to* $\langle G, \cdot \rangle^P$), *and consequently, there exists exactly one closure* v *admissible for the group* $\langle G, \cdot \rangle^P$ *such that* $\mathcal{V}$ *is the neighborhood system at the neutral element of* $\langle G, \cdot \rangle^P$. *This closure will be called* the closure of uniform convergence for $\mathcal{G}^P$, *and the topological group* $\langle\langle G, \cdot \rangle^P, v\rangle$ *will be denoted by* unif $\mathcal{G}^P$.

19 B.15. For any commutative topological group $\mathcal{G}$ and any non-void set P, the closure structure of $\mathcal{G}^P$ is coarser than that of unif $\mathcal{G}^P$; stated in other words, the closure of pointwise convergence is coarser than the closure of uniform convergence. If P is finite, then $\mathcal{G}^P$ = unif $\mathcal{G}^P$ (show, for instance, that each neighborhood of an element x in $\mathcal{G}^P$ is a neighborhood of x in unif $\mathcal{G}^P$ and, if P is finite, then each neighborhood of an x in unif $\mathcal{G}^P$ is a neighborhood of x in $\mathcal{G}^P$).

Corollary. *For any positive integer* n *the groups* R^n *and* unif R^n *coincide.*

19 B.16. Theorem. *Suppose that* $\mathcal{G} = \langle G, \cdot, u\rangle$ *is a commutative group and* P *is a non-void set. If* w *is any closure for* P, *then the set* H *of all elements* f *of* G^P *such that* $f : \langle P, w\rangle \to \langle G, u\rangle$ *is a continuous mapping, is a closed subset of* unif $\mathcal{G}^P$. *In addition,* H *is a stable subset of* $\langle G, \cdot \rangle^P$ *and hence* H *is a closed subgroup of* unif $\mathcal{G}^P$.

Before giving the proof it may be in place to restate the result for groups of mappings.

19 B.17. Suppose that $\mathcal{G} = \langle G, \cdot, u\rangle$ is a group and $\mathcal{P}$ is a non-void closure space. Let us consider the relation ν on $G^{|\mathcal{P}|}$ which assigns to each f the mapping $f : \mathcal{P} \to \mathcal{G}$. Clearly ν is one-to-one and ranges on $\mathbf{F}(\mathcal{P}, \mathcal{G})$. Thus we can define a group structure on $\mathbf{F}(\mathcal{P}, \mathcal{G})$ in such a manner that $\nu : \langle G, \cdot \rangle^{|\mathcal{P}|} \to \mathbf{F}(\mathcal{P}, \mathcal{G})$ is an isomorphism. The set $\mathbf{F}(\mathcal{P}, \mathcal{G})$ with this group structure will be called the *group of mappings of* $\mathcal{P}$ *into* $\mathcal{G}$. Next, we can endow the group $\mathbf{F}(\mathcal{P}, \mathcal{G})$ with a closure operation so that $\nu : \mathcal{G}^{|\mathcal{P}|} \to \mathbf{F}(\mathcal{P}, \mathcal{G})$ becomes a homeomorphism. Clearly this closure is admissible for the group $\mathbf{F}(\mathcal{P}, \mathcal{G})$. This topological group will be called the *group of mappings of* $\mathcal{P}$ *into* $\mathcal{G}$ *endowed with the closure of pointwise convergence.* Finally, we can endow the group $\mathbf{F}(\mathcal{P}, \mathcal{G})$ with a closure operation so that the mapping ν : unif $\mathcal{G}^{|\mathcal{P}|} \to \mathbf{F}(\mathcal{P}, \mathcal{G})$ becomes a homeomorphism. This closure is admissible for $\mathbf{F}(\mathcal{P}, \mathcal{G})$ and the resulting topological group, denoted by unif $\mathbf{F}(\mathcal{P}, \mathcal{G})$ will be called the *group of mappings of* $\mathcal{P}$ *into* $\mathcal{G}$ *endowed with the closure of uniform convergence.* Now we are prepared to give the required restatement of Theorem 19 B.16.

19 B.16a. Theorem. *Let* $\mathcal{G}$ *be a topological group and let* $\mathcal{P}$ *be a non-void closure space. The set* $\mathbf{C}(\mathcal{P}, \mathcal{G})$ *of all continuous mappings of* $\mathcal{P}$ *into* $\mathcal{G}$ *is a closed subset of the topological group* unif $\mathbf{F}(\mathcal{P}, \mathcal{G})$ (= *the group of mappings of* $\mathcal{P}$ *into* $\mathcal{G}$ *endowed with the closure of uniform convergence*); *in addition,* $\mathbf{C}(\mathcal{P}, \mathcal{G})$ *is stable, and hence* $\mathbf{C}(\mathcal{P}, \mathcal{G})$ *is a closed subgroup of* unif $\mathbf{F}(\mathcal{P}, \mathcal{G})$.

According to 15 B.4 the fact that $\mathbf{C}(\mathcal{P}, \mathcal{G})$ is closed in unif $\mathbf{F}(\mathcal{P}, \mathcal{G})$ is equivalent to the following: if a net N converges to f in unif $\mathbf{F}(\mathcal{P}, \mathcal{G})$ and if N ranges in $\mathbf{C}(\mathcal{P}, \mathcal{G})$, then $f \in \mathbf{C}(\mathcal{P}, \mathcal{G})$; stated in other words, if a net N of continuous mappings converges to a mapping f in unif $\mathbf{F}(\mathcal{P}, \mathcal{G})$ then f is continuous. Sometimes it is convenient to have the following currently employed convention:

19 B.18. Convention. Let $\mathcal{P}$ be a struct and $\mathcal{G}$ be a topological group. It is said that a *net N in* $\mathbf{F}(\mathcal{P}, \mathcal{G})$ *converges to f pointwise* or *uniformly* accordingly as N converges to f in the group $\mathbf{F}(\mathcal{P}, \mathcal{G})$ endowed with the closure of pointwise convergence or uniform convergence.

Now the main result of **19 B.16a** can be restated as follows:

19 B.16b. *If a net of continuous mappings of a space $\mathcal{P}$ into a topological group $\mathcal{G}$ converges to a mapping f uniformly, then f is a continuous mapping. Roughly speaking, a uniform limit of continuous mappings is a continuous mapping.*

Proof of Theorem **19 B.16.** I. To prove that H is closed it is sufficient to show that the complement of H is open. The composition in unif $\mathcal{G}^{|P|}$ will also be denoted by $\cdot$. Let f be any element of the complement of H. There exists a point x of P such that $f: P \to G$ is not continuous at x. As a consequence, there exists a neighborhood U of fx such that $(f[W] - U) \neq \emptyset$ for each neighborhood W of x. Let us choose a symmetric neighborhood V of the neutral element of $\mathcal{G}$ so that $fx\,.\,[V]\,.\,[V]\,.\,[V] \subset U$. It will be shown that the neighborhood $f\,.\,[V^P]$ (of f in unif $\mathcal{G}^{|P|}$) is contained in the complement of H which will establish that H is closed. Moreover, we shall prove that no $g : P \to \mathcal{G}$ with g in $f\,.\,[V^P]$ is continuous at x. Suppose that $g \in f\,.\,[V^P]$ and $g : P \to \mathcal{G}$ is continuous at x. There exists a neighborhood W of x such that $g[W] \subset gx\,.\,[V]$. Since $g \in f\,.\,[V^P]$ and V is symmetric we have $f[W] \subset \bigcup\{gy\,.\,[V] \mid y \in W\} = [g[W]]\,.\,[V]$. But $g[W] \subset gx\,.\,[V]$ and hence $f[W] \subset gx\,.\,[V]\,.\,[V]$. Finally, since $g \in f[V^P]$ and hence $gx \in fx\,.\,[V]$, we obtain

$$f[W] \subset fx\,.\,[V]\,.\,[V]\,.\,[V] \subset U$$

which contradicts our hypothesis $(f[W] - U) \neq \emptyset$ for each neighborhood W of x. — II. H is stable. Suppose $f, g \in H$, k is the inverse for g in $\mathcal{G}^{|P|}$ and $h = f\,.\,k$. We must show that $h \in H$, i.e. that h is continuous. Let x be any point of $\mathcal{P}$ and U be any neighborhood of hx in $\mathcal{G}$. We must find a neighborhood W of x so that $h[W] \subset U$. By the remark following **19 B.7** we can find a symmetric neighborhood V of the neutral element in $\mathcal{G}$ so that $fx\,.\,[V]\,.\,kx\,.\,[V] \subset U$. Since $f: \mathcal{P} \to \mathcal{G}$ and $g : \mathcal{P} \to \mathcal{G}$ are continuous and the sets $fx\,.\,[V]$ and $[V]\,.\,gx$ are neighborhoods of fx and gx respectively, there exists a neighborhood W of x such that $f[W] \subset fx\,.\,V$ and $g[W] \subset [V]\,.\,gx$. Since $([V]\,.\,gx)^{-1} = kx\,.\,[V^{-1}] = kx\,.\,[V]$ we obtain $h[W] \subset [f[W]]\,.\,[k[W]] \subset fx\,.\,[V]\,.\,kx\,.\,[V] \subset U$, which completes the proof.

Now we proceed to topological rings and fields.

19 B.19. Definition. A *topological ring* is a quadruple $\mathcal{R} = \langle R, \sigma, \mu, u\rangle$ such that $\langle R, \sigma, \mu\rangle$ is a ring (called the *underlying ring*), $\langle R, \sigma, u\rangle$ is a topological group (called the *underlying additive topological group*) and $\langle \mu, u \rangle$ is a continuous internal composition. A *topological field* is a topological ring $\mathscr{F} = \langle F, \sigma, \mu, u \rangle$ such that $\langle F, \sigma, \mu \rangle$ is a field (called the *underlying field*) and the multiplicative group of $\langle F, \sigma, \mu \rangle$ endowed with the relativization of u is a topological group (called the *multiplicative topological group of* $\mathscr{F}$). If $\langle R, \sigma, \mu, u \rangle$ is a topological ring (field), then the ring (field) structure $\langle \sigma, \mu \rangle$ and the closure u are said to be *compatible*, u is said to be *admissible for the ring (field)* $\langle R, \sigma, \mu \rangle$ and the *ring (field) structure* $\langle \sigma, \mu \rangle$ is said to be *admissible for the space* $\langle R, u \rangle$.

Stated more directly, $\langle R, \sigma, \mu, u \rangle$ is a topological ring if and only if $\langle R, \sigma, \mu \rangle$ is a ring, both topologized compositions $\langle \sigma, u \rangle$ and $\langle \mu, u \rangle$ are continuous and the inversion of $\langle \sigma, u \rangle$ is continuous; $\langle F, \sigma, \mu, u \rangle$ is a topological field if and only if $\langle F, \sigma, \mu \rangle$ is a field, both topologized compositions $\langle \sigma, u \rangle$ and $\langle \mu, u \rangle$ are continuous and the inversion of $\langle \sigma, u \rangle$ as well as the inversion of $\langle \mu, u \rangle$ are continuous.

Convention. If a is an m-tuple $\langle a_1, \ldots, a_m \rangle$ but not an n-tuple, $n > m > 1$, then $\langle a; b \rangle$ stands for $\langle a_1, \ldots, a_m, b \rangle$; if a is not a pair then $\langle a; b \rangle = \langle a, b \rangle$. Finally, $\langle x_1, \ldots, x_p; y \rangle$ stands for $\langle \langle x_1, \ldots, x_p \rangle; y \rangle$ as defined above. – Thus e.g. if $\mathscr{G} = \langle G, \sigma \rangle$ and σ is not a pair, then $\langle \mathscr{G}; u \rangle = \langle G, \sigma, u \rangle$; if $\mathscr{R} = \langle R, r \rangle$, $r = \langle \sigma, \mu \rangle$ and μ is not a pair, then $\langle \mathscr{R}; u \rangle = \langle R, \sigma, \mu, u \rangle$.

19 B.20. Examples. (a) If $\mathscr{R} = \langle R, \sigma, \mu \rangle$ is a ring (field) and u is either the discrete closure for R or the accrete closure for R, then $\langle \mathscr{R}; u \rangle$ is a topological ring (field). – (b) The field R of reals endowed with the usual closure (= order closure) is a topological field (by 19 B.5). In what follows the letter R will also stand for this topological field.

19 B.21. (a) *If* $\mathscr{R} = \langle R, \sigma, \mu, u \rangle$ *is a topological ring (field) and* $\mathscr{R}_1 = \langle R_1, \sigma_1, \mu_1 \rangle$ *is a subring (subfield) of* $\mathscr{R}$, *then* $\langle \mathscr{R}_1; v \rangle$ *is a topological ring (field) where* v *is the relativization of* u *to* R_1. – 19 A.10.

(b) *If* $\{\mathscr{R}_a\}$ *is a non-void family of topological rings, where* $\mathscr{R}_a = \langle R_a, \sigma_a, \mu_a, u_a \rangle$, *then*

$$\mathscr{R} = \langle \Pi\{R_a\}, \Pi_{\mathrm{comp}}\{\sigma_a\}, \Pi_{\mathrm{comp}}\{\mu_a\}, \Pi\{u_a\} \rangle$$

is a topological ring called the product ring *and denoted by* $\Pi\{\mathscr{R}_a\}$. – 19 A.12.

19 B.22. Remark. If $\{\langle \mathscr{R}_a; u_a \rangle \mid a \in A\}$ is a family of topological rings such that $\langle \mathscr{R}_a; u_a \rangle = \langle \mathscr{R}; u \rangle$ for each a in A, then the product $\Pi\{\langle \mathscr{R}_a; u_a \rangle\}$ is usually denoted by $\langle \mathscr{R}; u \rangle^A$. For example, R^A is a topological ring (if $A \neq \emptyset$).

19 B.23. *The underlying closure space of a topological ring (field) is topological.* – 19 B.4.

The proof of the fact that R is a topological field was based upon the existence of the absolute value $\{x \to |x|\}$ on R (see 18 C.8). Now we shall prove that it is possible to introduce an admissible closure for a group, ring or field by means of a function satisfying certain conditions corresponding to properties of the absolute value used in proving that R is a topological field. The main result is contained in the following lemma.

19 B.24. Lemma. *Suppose that* $\langle R, +, \cdot \rangle$ *is a ring and* φ *is a function on* R *satisfying the following three conditions*:

(1) $\varphi 0 = 0$; (2) $\varphi(-x) = \varphi x$ *for each* $x \in R$; *and*

(3) $\varphi(x + y) \leqq \varphi x + \varphi y$ *for each* x *and* y *in* R.

Then $\varrho = \{\langle x, y \rangle \to \varphi(x - y) \mid \langle x, y \rangle \in R \times R\}$ *is a pseudometric for* R *and the closure* u *induced by* ϱ *is admissible for the group* $\langle R, + \rangle$. *If, moreover,*

(4) $\varphi(x \,.\, y) \leqq \varphi(x) \,.\, \varphi(y)$ *for each* x *and* y *in* R,

then u *is admissible for the ring* $\langle R, +, \cdot \rangle$. *If* (4) *is replaced by the following stronger condition*

(5) $\varphi(x \,.\, y) = (\varphi x) \,.\, (\varphi y)$ *for each* x *and* y *in* R,

then the inversion of $\langle \cdot, u \rangle$ *is continuous and consequently, if* $\langle R, +, \cdot \rangle$ *is a field, then* $\langle R, +, \cdot, u \rangle$ *is a topological field.*

Proof. I. The function ϱ is a pseudometric. From (1), (2) and (3) we obtain at once that $2\varphi x = (\varphi x) + (\varphi x) \geqq \varphi(x - x) = \varphi 0 = 0$ and hence $\varphi x \geqq 0$ for each $x \in R$ which shows that φ is a non-negative function. It follows that ϱ is also non-negative. The symmetry of ϱ follows from (2), and the triangle inequality is obtained from (3) as follows:

$$\varrho\langle x, y \rangle = \varphi(x - y) \leqq \varphi(x - z) + \varphi(z - y) = \varrho(x, z) + \varrho(z, y) .$$

II. The proof of the remaining statements preceeds similarly to that of the corresponding statements for R in 18 C.8, 18 ex. 10, and is therefore left to the reader.

Corollary. *The field* C *of complex numbers endowed with the usual closure is a topological field.*

Proof. Let $\varphi(x + iy) = \sqrt{(x^2 + y^2)}$. Then $\varphi = \{x \to \varphi x\}$ fulfils conditions (1), (2), (3) and (5), and ϱ induces the usual closure for C.

19 B.25. Definition. A *norm* for a group $\langle R, + \rangle$ is a function φ on $\langle R, + \rangle$ satisfying conditions (1), (2) and (3) of 19 B.24. Next, a *norm* for a ring $\langle R, +, . \rangle$ is a function φ on $\langle R, +, . \rangle$ satisfying conditions (1) to (4) of 19 B.24. Finally, a *norm* for a field $\langle R, +, \cdot \rangle$ is a function φ on $\langle R, +, \cdot \rangle$ satisfying conditions (1), (2), (3) and (5) of 19 B.24. A *normed group, ring* or *field* is a struct $\langle \mathscr{R}; \varphi \rangle$ where $\mathscr{R}$ is a group, ring or field and φ is a norm for $\mathscr{R}$. If $\langle \mathscr{R}; \varphi \rangle$ is a normed group, ring or field, then the pseudometric $\varrho = \{\langle x, y \rangle \to \varphi(x - y)\}$ is said to be *induced* by φ, and the closure induced by ϱ will be said to be *induced* by φ or to be the *normed closure* for $\langle \mathscr{R}; \varphi \rangle$.

Lemma 19 B.24 can be now restated as follows:

19 B.26. Theorem. *If* $\langle \mathscr{R}; \varphi \rangle$ *is a normed group, ring or field, then the normed closure is admissible for* $\mathscr{R}$.

19 B.27. The normed group (ring) of mappings of a struct $\mathscr{P} = \langle P, \xi \rangle$ into a normed group (ring) $\langle \mathscr{R}; \varphi \rangle$. *Let* $\mathscr{P} = \langle P, \xi \rangle$ *be a non-void struct,* P *being a set, and let* $\langle \mathscr{R}; \varphi \rangle$ *be a normed group or ring. The underlying set of* $\mathscr{R}$ *is* *denoted by* R *and the addition by* $+$. *Let us consider the set* $\mathscr{F}$ *of all mappings* f *of* $\mathscr{P}$ *into* $\langle\mathscr{R}; \varphi\rangle$ (*that is, all* $f \in \mathbf{F}(\mathscr{P}, \langle\mathscr{R}; \varphi\rangle)$) *such that*

$$\psi f = \sup\{\varphi f x \mid x \in P\}$$

exists. Let ψ *be the relation* $\{f \rightarrow \psi f \mid f \in \mathscr{F}\}$. *Then*

(α) $\mathscr{F}$ *is a stable subset of the group (ring)* $\mathbf{F}(\mathscr{P}, \mathscr{R})$ *of mappings of* $\mathscr{P}$ *into* $\mathscr{R}$, *i.e.* $\mathscr{F}$ *is a group (ring) under the algebraic structure of* $\mathbf{F}(\mathscr{P}, \mathscr{R})$.

(β) *The function* $\langle\psi, \mathscr{F}, \mathsf{R}\rangle$ *is a norm for the group (ring)* $\mathscr{F}$.

(γ) *The underlying topological group of* $\mathscr{F}$ *is a subgroup of the group* unif $\mathbf{F}(\mathscr{P}, \mathscr{G})$ *where* $\mathscr{G}$ *is the underlying topological group of* $\langle\mathscr{R}; u\rangle$ *and where* u *is the normed closure for* $\langle\mathscr{R}; \varphi\rangle$.

Proof. Addition (and multiplication) of the group (or ring) $\mathbf{F}(\mathscr{P}, \mathscr{R})$ as well as of the group (ring) $\mathscr{R}$ will be denoted by $+$ (and $\cdot$, respectively).

I. It is self-evident that if $f \in \mathscr{F}$ then $(-f) \in \mathscr{F}$ and $\psi(-f) = \psi f$, $0 \in \mathscr{F}$ and $\psi 0 = 0$.

II. Now prove that if $f, g \in \mathscr{F}$, then $(f + g) \in \mathscr{F}$ and $\psi(f + g) \leqq \psi f + \psi g$. If $x \in P$ then $\varphi((f + g)x) = \varphi(fx + gx) \leqq \varphi f x + \varphi g x \leqq \psi f + \psi g$. It follows that $\psi(f + g)$ exists and is less than or equal to $\psi f + \psi g$.

III. Now let $\mathscr{R}$ be a ring and $f, g \in \mathscr{F}$. The same argument as in II shows that $\psi(f . g)$ exists and is less than or equal to $\psi f . \psi g$.

IV. Statements (α) and (β) follow from I, II and III.

V. Statement (γ) is an immediate consequence of the corresponding definitions.

19 B.28. Theorem. *Let* $\mathbf{C}^*(\mathscr{P}, \mathsf{R})$ *be the ring of all bounded continuous functions on a closure space* $\mathscr{P}$. *Then*

$$\|\cdot\| = \{f \rightarrow \sup\{|fx| \mid x \in |\mathscr{P}|\} \mid f \in \mathbf{C}^*(\mathscr{P}, \mathsf{R})\}$$

is a norm for $\mathbf{C}^*(\mathscr{P}, \mathsf{R})$. *In what follows, the symbol* $\mathbf{C}^*(\mathscr{P}, \mathsf{R})$ *will be used to denote the normed ring* $\langle\mathbf{C}^*(\mathscr{P}, \mathsf{R}); \|\cdot\|\rangle$.

Proof. 19 B.27.

C. TOPOLOGIZED EXTERNAL COMPOSITIONS

Let ϱ be an external composition on a struct $\mathscr{P} = \langle P, \pi\rangle$ over a struct $\mathscr{A} = \langle A, \alpha\rangle$. By definition, this is to mean here that ϱ is a single-valued relation with domain $|\mathscr{A}| = A$ such that $\varrho[(a)]$ is a single-valued relation on $|\mathscr{P}|$ $(=P)$ ranging in $|\mathscr{P}|$ for each a in A. Thus $|\mathscr{A}| = \mathbf{D}\varrho$ and $|\mathscr{P}| = \mathbf{DE}\varrho$. Instead of $\varrho[a]\,x$ we shall write $a\varrho x$. Often we consider the struct $\langle P, \pi, \varrho, \alpha\rangle$ whose underlying class is P and whose structure is the triple $\langle\pi, \varrho, \alpha\rangle$. It is to be noted that the structs $\mathscr{P}$ and $\mathscr{A}$ are uniquely determined by $\langle\pi, \varrho, \alpha\rangle$, namely $\mathscr{P} = \langle\mathbf{DE}\varrho, \pi\rangle$ and $\mathscr{A} = \langle\mathbf{D}\varrho, \alpha\rangle$. For example, a module over a ring $\mathscr{A}$ is a struct $\langle P, \pi, \varrho, \alpha\rangle$ satisfying certain conditions connecting π, ϱ and α. Here we will be concerned with the case where π and α are closure structures.

Now we shall define the continuity of an external composition ϱ on a space $\langle P, u\rangle$ over a closure space $\langle A, v\rangle$. The definition will require certain mappings to be continuous. We can construct the following mappings:

$$\{\langle a, x\rangle \to a\varrho x\} : A \times P \to P,$$
$$\{x \to a\varrho x\} : P \to P\,, \quad a \in A$$

and

$$\{a \to a\varrho x\} : A \to P\,, \quad x \in P\,.$$

If we want to speak about continuity we must endow the domain carriers and the range carriers with a closure operation. All the mappings in question have a common range carrier, namely P, and undoubtedly the only candidate for the closure structure for P is the given closure u. The domains of the last two mappings are P and A, and the only candidates for the corresponding closure structures are the given closure operations, u for P and v for A. It remains to topologize the domain $A \times P$ of the first mapping. Naturally, the closure structure for $A \times P$ must be an important closure operation constructed from the given closures v for A and u for P, e.g. the product closure or the inductive product closure. It turns out that the product closure as well as the inductive product closure yield continuities which are often required.

19 C.1. Definition. We shall say that an external composition ϱ on a closure space $\langle \mathscr{P}, u\rangle$ over a closure space $\langle \mathscr{A}, v\rangle$ is *continuous* (*inductively continuous*) *on* $\langle \mathscr{P}, u\rangle$ *over* $\langle \mathscr{A}, v\rangle$ if the mapping

$$\{\langle a, x\rangle \to a\varrho x\} : \langle \mathscr{A}, v\rangle \times \langle \mathscr{P}, u\rangle \to \langle \mathscr{P}, u\rangle$$
$$(\{\langle a, x\rangle \to a\varrho x\} : \operatorname{ind}(\langle \mathscr{A}, v\rangle \times \langle \mathscr{P}, u\rangle) \to \langle \mathscr{P}, u\rangle)$$

is continuous.

Notice that we require continuity of mappings whose graphs are not ϱ but a closely related relation $\{\langle a, x\rangle \to a\varrho x\}$. This relation, called the *external multiplication associated with* ϱ, occurs very frequently in topological investigations of external compositions and therefore it will be convenient to adopt the following definition.

19 C.2. If ϱ is any external composition then any mapping f, whose graph is the external multiplication σ associated with ϱ, is also termed an *external multiplication.* If the structures of $\mathbf{D}^*f$ and $\mathbf{E}^*f$ are clearly given by the context then we shall simply speak about the properties of σ instead of f.

It is worth noticing that the external multiplication associated with ϱ entirely determines ϱ, namely $\varrho = \{\langle a, \langle x, a\varrho x\rangle\rangle \mid a \in \mathscr{A}, x \in \mathscr{P}\}$. Now Definition 19 C.1 can be stated as follows: An external composition ϱ on a space $\langle \mathscr{P}, u\rangle$ over a space $\langle \mathscr{A}, v\rangle$ is continuous (inductively continuous) if the external multiplication associated with ϱ is continuous under the product closure $v \times u$ (inductive product closure $\operatorname{ind}(v \times u)$) and u. If the closures u and v are clear from the context then we shall say simply that the external multiplication is continuous (inductively continuous).

The following two statements follow immediately from Definition 19 C.1 and the description of continuous mappings on inductive products.

19 C.3. Theorem. *An external composition ϱ on a space $\langle \mathscr{P}, u\rangle$ over a space $\langle \mathscr{A}, v\rangle$ is inductively continuous if and only if all the following mappings are continuous*:

$$\{x \to a\varrho x\} : \langle \mathscr{P}, u\rangle \to \langle \mathscr{P}, u\rangle\,, \quad a \in \mathscr{A}\,,$$
$$\{a \to a\varrho x\} : \langle \mathscr{A}, v\rangle \to \langle \mathscr{P}, u\rangle\,, \quad x \in \mathscr{P}\,.$$

19 C.4. Every continuous external composition is inductively continuous; more precisely, if ϱ is a continuous external composition on $\mathscr{P}$ over $\mathscr{G}$, then ϱ is an inductively continuous composition on $\mathscr{P}$ over $\mathscr{G}$.

It is to be remarked that the continuity and the inductive continuity of an internal composition can be reduced to the corresponding continuity of a certain external composition. Namely, let σ be an internal composition on a closure space $\langle \mathscr{P}, u\rangle$ and let us consider the relation

$$\varrho = \{x \to \langle y, x\sigma y\rangle \mid \langle x, y\rangle \in \mathscr{P} \times \mathscr{P}\}$$

Clearly ϱ is an external composition on $\langle \mathscr{P}, u\rangle$ over $\langle \mathscr{P}, u\rangle$ and the external multiplication associated with ϱ coincides with σ. It follows that σ is a continuous (inductively continuous) internal composition on $\langle \mathscr{P}, u\rangle$ if and only if ϱ is a continuous (inductively continuous) external composition on $\langle \mathscr{P}, u\rangle$ over $\langle \mathscr{P}, u\rangle$.

19 C.5. Remarks. Let ϱ be an external composition on a set P over a set A. (a) If u is a closure for P and v is the discrete closure for A, then ϱ is a continuous external composition on $\langle P, u\rangle$ over $\langle A, v\rangle$ if and only if all the mappings $\{x \to a\varrho x\} : \langle P, u\rangle \to \langle P, u\rangle$, $a \in A$, are continuous. Indeed, in this case the product $\langle A, v\rangle \times \langle P, u\rangle$ coincides with the sum $\Sigma\{\langle P, u\rangle \mid a \in A\}$ and the statement follows from 17 B.4. Thus the case where there is given a closure u for P such that all the mappings $\{x \to a\varrho x\} : \langle P, u\rangle \to \langle P, u\rangle$, $a \in A$, are continuous is reduced to continuous external compositions by endowing A with the discrete closure. (b) If u is the accrete closure for P and v is any closure for A, then ϱ is a continuous external composition on $\langle P, u\rangle$ over $\langle A, v\rangle$. This is evident because every mapping into an accrete space is continuous. Thus given a closure v for A we can choose a closure u for P so that ϱ becomes a continuous external composition on $\langle P, u\rangle$ over $\langle A, v\rangle$. (c) Let u be a closure for P and v be a closure for A such that all the mappings $\{a \to a\varrho x\} : \langle A, v\rangle \to \langle P, u\rangle$, $x \in P$ are continuous. The mappings $\{x \to a\varrho x\} : \langle P, u\rangle \to \langle P, u\rangle$ need not be continuous as will be seen in 19 C.6. Therefore this case does not reduce to the notion of an inductively continuous or a continuous external composition.

19 C.6. Example. Let $\mathscr{M}$ be a set of mappings of a closure space $\mathscr{P} = \langle P, u\rangle$ into itself and let us consider the relation

$$\varrho = \Sigma\{\operatorname{gr} f \mid f \in \mathscr{M}\}$$

Clearly ϱ is an external composition on $\langle P, u\rangle$ over $\mathscr{M}$ and $f\varrho x = fx$ for each f in $\mathscr{M}$ and x in P. Thus $f = \{x \to f\varrho x\} : \langle P, u\rangle \to \langle P, u\rangle$ for each f in $\mathscr{M}$. Let us consider a closure v for $\mathscr{M}$.

(a) All the mappings $\{f \to fx\} : \langle \mathcal{M}, v \rangle \to \langle P, u \rangle$, $x \in P$ are continuous if and only if the closure v is finer than the closure of pointwise convergence.

Recall that the closure w of pointwise convergence for $\mathcal{M}$ is the closure such that the mapping $\{f \to \operatorname{gr} f\} : \langle \mathcal{M}, w \rangle \to \langle P, u \rangle^P$ is an embedding. Thus a net $\{f_a\}$ in $\langle \mathcal{M}, w \rangle$ converges to f in $\langle \mathcal{M}, w \rangle$ if and only if the net $\{f_a x\}$ converges to fx in $\langle P, u \rangle$ for each x in P. Now the statement is obvious.

(b) The external composition ϱ on $\langle P, u \rangle$ over $\langle \mathcal{M}, v \rangle$ is inductively continuous if and only if $\mathcal{M} \subset \mathbf{C}(\langle P, u \rangle, \langle P, u \rangle)$ (i.e. each $f \in \mathcal{M}$ is continuous) and v is finer than the closure of pointwise convergence. – This follows from **19 C.3**.

(c) The external composition ϱ is continuous on $\langle P, u \rangle$ over $\langle \mathcal{M}, v \rangle$ if and only if the following condition is fulfilled: if a net $\{f_a \mid a \in A\}$ converges to f in $\langle \mathcal{M}, v \rangle$ and a net $\{x_a \mid a \in A\}$ converges to x in $\langle P, u \rangle$ then $f_a x_a$ converges to fx in $\langle P, u \rangle$. This is precisely the description of continuity of the mapping $\{\langle f, x \rangle \to fx\}$: $(\langle \mathcal{M}, v \rangle \times \langle P, u \rangle) \to \langle P, u \rangle$ by means of convergent nets (see **16 A.8** and **17 C.9**).

(d) If the closure u is compatible with a commutative group structure $+$ for P and $\langle \mathcal{M}, v \rangle$ is a subspace of the underlying space of the group of continuous mappings of $\langle P, u \rangle$ into the topological group $\langle P, +, u \rangle$ endowed with the closure of uniform convergence, then ϱ is a continuous external composition on $\langle P, u \rangle$ over $\langle \mathcal{M}, v \rangle$.

For the proof we may assume that $\langle \mathcal{M}, v \rangle$ coincides with the underlying space of the group unif $\mathbf{C}(\langle P, u \rangle, \langle P, +, u \rangle)$ which, for brevity, will be denoted by $\mathfrak{G} = \langle \mathcal{M}, +, v \rangle$. Suppose that $\{f_a\}$ converges to f in $\mathfrak{G}$, $\{x_a\}$ converges to x in $\langle P, u \rangle$ and W is a neighborhood of fx. Let $\mathcal{V}$ be the neighborhood system at the neutral element 0 of P and for each V in $\mathcal{V}$ let

$$V^* = \mathbf{E}\{f \mid f \in \mathfrak{G}, \mathbf{E} f \subset V\}.$$

By definition the set $\mathcal{V}^*$ of all V^*, $V \in \mathcal{V}$ is a local base at the neutral element in $\mathfrak{G}$. Let us choose a symmetric V in $\mathcal{V}$ so that $fx + [V] + [V] \subset W$. Since f is continuous we can choose a neighborhood U of x so that $f[U] \subset fx + [V]$. Let us consider the neighborhood $f + [V^*]$ of f. Clearly if $g \in f + [V^*]$, then

$$g[U] \subset [f[U]] + [V] \subset fx + [V] + [V] \subset W.$$

Thus

$$g \in (f + [V^*]) \Rightarrow g[U] \subset W.$$

Now since $\{f_a\}$ converges to f and $\{x_a\}$ converges to x we can choose a residual set A_0 of indexes so that $f_a \in (f + [V^*])$ and $x_a \in U$ for each index a in A_0, and consequently, by the above implication, $f_a x_a \in W$ for each a in A_0 which proves that the net $\{f_a x_a\}$ converges to fx. (It should be noted that one may eliminate the use of nets in the proof and obtain a more direct proof.)

(e) If v is the closure of pointwise convergence for $\mathcal{M} \subset \mathbf{C}(\langle P, u \rangle, \langle P u \rangle)$, then ϱ is inductively continuous on $\langle P, u \rangle$ over $\langle \mathcal{M}, v \rangle$ but ϱ need not be continuous. For example, let $\langle P, u \rangle$ be the unit interval $[\![0, 1]\!]$ of real numbers endowed with the order closure (i.e. $[\![0, 1]\!]$ is a subspace of R), and let us choose a sequence $\{f_n\}$ of continuous mappings of $[\![0, 1]\!]$ into itself so that $f_n x_n = 1$, where $x_n = 1/(n + 1)$

and $f_n x = 0$ for $|x - x_n| > (4n + 4)^{-2}$. Such a sequence obviously exists. Next, evidently the sequence $\{x_n\}$ converges to 0, $\{f_n\}$ pointwise converges to the constant mapping $\{x \to 0\}$ (for each x, at most one $f_n x$ is different from 0), but $f_n x_n = 1$ for each n, and hence the sequence $\{f_n x_n\}$ converges to 1. By (c) ϱ is not continuous.

Let us recall that we have defined a continuous internal composition for a closure space $\langle P, u\rangle$ as an internal composition for $\langle P, u\rangle$ such that the mapping $\sigma : \langle P, u\rangle \times \langle P, u\rangle \to \langle P, u\rangle$ is continuous, and a continuous internal composition for a set P as a pair $\langle \sigma, u\rangle$ such that σ is a continuous composition for $\langle P, u\rangle$. In the case of an external composition ϱ the situation is more complicated because the continuity depends on two closures, one for $\mathbf{D}\varrho$ and one for $\mathbf{DE}\varrho$.

19 C.7. Definition. A *topologized external composition* is a triple $\langle u, \varrho, v\rangle$ where ϱ is an external composition, say on P over A, and u is a closure for P and v is a closure for A; we shall say that $\langle u, \varrho, v\rangle$ is *continuous* (*inductively continuous*) if ϱ is a continuous (inductively continuous) external composition on $\langle P, u\rangle$ over $\langle A, v\rangle$. If ϱ is a continuous (inductively continuous) external composition on $\langle P, u\rangle$ over $\langle A, v\rangle$, then we shall say that $\langle \varrho, v\rangle$ is a *continuous* (*inductively continuous*) *external composition on* $\langle P, u\rangle$, and that $\langle u, \varrho\rangle$ is a *continuous* (*inductively continuous*) *external composition over* $\langle A, v\rangle$.

Remark. One can define the *external multiplication associated with a given topologized external composition* $\langle u, \varrho, v\rangle$ to be the mapping of $\langle \mathbf{D}\varrho, v\rangle \times \langle \mathbf{DE}\varrho, u\rangle$ into $\langle \mathbf{DE}\varrho, u\rangle$ whose graph is the external multiplication associated with ϱ. Then $\langle u, \varrho, v\rangle$ is continuous or inductively continuous if and only if the corresponding multiplication is continuous or inductively continuous.

The concept of a topologized external composition requires no comment. The pair $\langle \varrho, v\rangle$ is very appropriate if v is fixed and we examine various u. For example, there is given a module L over the topological field of reals and we examine various closures making L a topological linear space (19 D.2); here the pair $\langle \varrho, v\rangle$ is fixed. The pair $\langle u, \varrho\rangle$ occurs frequently in another situation. There is given a closure space $\langle P, u\rangle$ and a subset $\mathcal{M}$ of $\mathbf{F}(\langle P, u\rangle, \langle P, u\rangle)$ and we examine various closures v for $\mathcal{M}$ such that, e.g., the external composition $\varrho = \Sigma\{\operatorname{gr} f \mid f \in \mathcal{M}\}$ (considered in 19 C.6) is continuous; here the pair $\langle u, \varrho\rangle$ is fixed.

In conclusion we shall introduce the definitions of the restriction of a topologized external composition and of the product of topologized external compositions.

19 C.8. Definition. The *restriction of a topologized external composition* $\langle u, \varrho, v\rangle$ is a topologized external composition $\langle u_1, \varrho_1, v_1\rangle$ such that ϱ_1 is a restriction of ϱ, u_1 is a relativization of u and v_1 is a relativization of v.

19 C.9. Theorem. *The restriction of a continuous (inductively continuous) external composition is continuous (inductively continuous).* — Immediate consequence of earlier results (Sections 16, 17).

The product of a family $\{\langle u_b, \varrho_b, v_b\rangle \mid b \in B\}$ of topologized external compositions will be defined only in the case that all the v_b coincide and hence all the domains $\mathbf{D}\varrho_b$ coincide.

19 C.10. Definition. The *product* of a non-void family $\{\langle u_b, \varrho_b, v\rangle \mid b \in B\}$ of topologized external compositions, denoted by $\Pi\{\langle u_b, \varrho_b, v\rangle\}$, is defined as the topological external composition $\langle u, \varrho, v\rangle$ where ϱ is the product $\Pi\{\varrho_b\}$ (see 8 A.15) of the family $\{\varrho_b\}$ and u is the product closure operation $\Pi\{u_b\}$. Thus

$$\Pi\{\langle u_b, \varrho_b, v\rangle\} = \langle \Pi\{u_b\}, \Pi\{\varrho_b\}, v\rangle .$$

It is to be noted that the symbols Π occurring in the equality all have different meanings.

19 C.11. Theorem. *The product of continuous (inductively continuous) external compositions is a continuous (inductively continuous) external composition.*

Proof. Let $\{u, \varrho, v\}$ be the product of a non-void family $\{\langle u_b, \varrho_b, v\rangle \mid b \in B\}$ of topologized external compositions; let each ϱ_b be an external composition on P_b over A, and let ϱ be on P over A, i.e. $P = \Pi\{P_b\}$. Finally put $\mathscr{P} = \langle P, u\rangle$, $\mathscr{P}_b = \langle P_b, u_b\rangle$ and $\mathscr{A} = \langle A, v\rangle$. Thus $\mathscr{P}$ is the product of the family $\{\mathscr{P}_b\}$ of closure spaces. — I. First suppose that each $\langle u_b, \varrho_b, v\rangle$ is a continuous external composition, i.e. that the mapping

$$f_b = \{\langle a, y\rangle \to a\varrho_b y\} : \mathscr{A} \times \mathscr{P}_b \to \mathscr{P}_b$$

is continuous for each b in B. We must show that the mapping

$$f = \{\langle a, x\rangle \to a\varrho x\} : \mathscr{A} \times \mathscr{P} \to \mathscr{P}$$

is continuous. By 17 C.10 it suffices to show that the mapping $\mathrm{pr}_b \circ f : \mathscr{A} \times \mathscr{P} \to \mathscr{P}_b$ is continuous for each b in B. But clearly $(\mathrm{pr}_b \circ f)\langle a, x\rangle = f_b\langle a, \mathrm{pr}_b\, x\rangle$ for each $\langle a, x\rangle \in A \times P$ and hence

$$((\mathrm{pr}_b \circ f) : \mathscr{A} \times \mathscr{P} \to \mathscr{P}_b) = f_b \circ ((\mathrm{J} : \mathscr{A} \to \mathscr{A}) \times (\mathrm{pr}_b : \mathscr{P} \to \mathscr{P}_b)) .$$

But the right side of the equality is a composite of two continuous mappings, f_b being continuous by our assumption and the product mapping continuous (17 C.13) because the identity mapping $\mathrm{J} : \mathscr{A} \to \mathscr{A}$ as well as the projection $\mathrm{pr}_b : \mathscr{P} \to \mathscr{P}_b$ (17 C.6) is continuous. – II. Now suppose that each $\langle u_b, \varrho_b, v\rangle$ is an inductively continuous external composition, that is, by 19 C.3, the mapping $\{a \to a\varrho_b y\} : \mathscr{A} \to \mathscr{P}_b$ is continuous for each b in B and y in P_b, and the mapping $\{y \to a\varrho_b y\} : \mathscr{P}_b \to \mathscr{P}_b$ is continuous for each b in B and a in A. To prove that the composition $\langle u, \varrho, v\rangle$ is inductively continuous, by 19 C.3 it suffices to show that the mapping $\{a \to a\varrho x\} : \mathscr{A} \to \mathscr{P}$ is continuous for each a in A, and the mapping $\{x \to a\varrho x\} : \mathscr{P} \to \mathscr{P}$ is continuous for each x in P. But, given an $x \in P$, the mapping $\{a \to a\varrho x\} : \mathscr{A} \to \mathscr{P}$ is the reduced product of the family of continuous mappings $\{\{a \to a\varrho_b\, \mathrm{pr}_b\, x\} : \mathscr{A} \to \mathscr{P}_b\}$ and therefore it is continuous by 17 C.13, and given an $a \in A$, the mapping $\{x \to a\varrho x\} : \mathscr{P} \to \mathscr{P}$ is the product of the family $\{\{y \to a\varrho_b y\} : \mathscr{P}_b \to \mathscr{P}_b\}$ of continuous mappings and therefore it is continuous by 17 C.13.

19 C.12. Convention. The product of a family $\{\langle u, \varrho, v\rangle \mid b \in B\}$ is denoted by $\langle u, \varrho, v\rangle^B$.

D. TOPOLOGICAL MODULES AND ALGEBRAS

The purpose of this subsection is to introduce the concepts of a topological module and a topological algebra over a topological ring or field. Particular attention turns to normed algebras of bounded functions which will be discussed in Sections 25 and 41. We point out that we do not intend to build a general theory of topological algebraic structs. Nevertheless a general introductory remark may be in place.

First, let a topologized internal algebraic struct be a struct $\langle P, \alpha; u\rangle$ such that P is a set, u is a closure for P and α is a multiplet $\langle\alpha_1, \ldots, \alpha_n\rangle$ such that each α_i is an internal composition on P; $\langle\alpha_1, \ldots, \alpha_n, u\rangle$ is called a topologized internal structure on P. Now let $\langle P, \alpha; u\rangle$ be a topologized internal struct, ϱ be an external composition on P over a set A and finally, let $\langle\beta; v\rangle$ be a topologized internal structure for A. It is natural to form the struct $\langle P, \alpha; u; \langle\varrho, \beta; v\rangle\rangle$ and to term it a topologized algebraic struct (with only one external composition) over the topologized algebraic struct $\langle \mathbf{D}\varrho, \beta, v\rangle$. Now we can speak about the continuity or the inductive continuity of the internal compositions $\langle\alpha_i, u\rangle$ (recall that $\alpha = \langle\alpha_1, \ldots, \alpha_n\rangle$) and $\langle\beta_i, v\rangle$ (where $\beta = \langle\beta_1, \ldots, \beta_z\rangle$), about the continuity of the inversion of $\langle\alpha_i, u\rangle$ or $\langle\beta_i, v\rangle$) and finally, about the continuity or inductive continuity of the topologized external composition $\langle u, \varrho, v\rangle$. Besides these topological requirements some algebraic conditions are usually involved. We restrict our attention to two important cases, namely to topological modules and topological algebras over a topological ring or a topological field.

19 D.1. Definition. A *topological module over a topological ring* $\mathscr{R} = \langle R, \sigma', \mu', v\rangle$ ($\langle R, \sigma', u'\rangle$ is the underlying ring of the topological ring $\mathscr{R}$) is a multiplet $\mathscr{L} = \langle L, \sigma, u, \varrho, \sigma', \mu'\rangle$ such that $\langle L, \sigma, \varrho, \sigma', \mu'\rangle$ is a module over $\langle R, \sigma', \mu'\rangle$ (called the *underlying module over the underlying ring of* $\mathscr{R}$), $\langle L, \sigma, u\rangle$ is a topological group (called the *underlying topological group*) and $\langle u, \varrho, v\rangle$ is continuous. A *topological algebra over a topological ring* $\mathscr{R} = \langle R, \sigma', \mu', v\rangle$ is a multiplet $\mathscr{L} = \langle L, \sigma, \mu, u, \varrho, \sigma', \mu', v\rangle$ such that $\langle L, \sigma, u, \varrho, \sigma', \mu', v\rangle$ is a topological module over $\mathscr{R}$ (called the *underlying topological module over* $\mathscr{R}$), $\langle L, \sigma, \mu, \varrho, \sigma', \mu'\rangle$ is an algebra over $\langle R, \sigma', \mu'\rangle$ (called the *underlying algebra of* $\mathscr{L}$ *over the underlying ring of* $\mathscr{R}$) and $\langle L, \sigma, \mu, u\rangle$ is a topological ring (called the *underlying topological ring of* $\mathscr{L}$).

19 D.2. A topological module $\mathscr{L}$ over a topological field $\mathscr{R}$ is usually called a *topological linear space*, and if $\mathscr{R}$ is the topological field of reals or complex numbers, then $\mathscr{L}$ is called a *real* topological linear space or a *complex* topological linear space.

19 D.3. Convention. In what follows, unless the contrary is explicitly stated, the external multiplication will be denoted by $\cdot$, and the compositions, currently called addition or multiplication, will usually be denoted by $+$ or $\cdot$, respectively. Thus, if $\mathscr{L}$ is a module over a ring $\mathscr{R}$ then there is the addition of $\mathscr{L}$ (addition of vectors), the addition of $\mathscr{R}$ (addition of scalars), the multiplication of $\mathscr{R}$ and the multiplication associated with the external composition (multiplication of scalars and vectors). If $\mathscr{L}$ is an algebra then there is yet another multiplication, namely the multiplication of the ring $\mathscr{L}$ (vector multiplication).

It may be in place to list all the topological conditions required of a topological linear space $\mathscr{L}$ over a topological ring $\mathscr{R}$ (the underlying closure space of $\mathscr{L}$ is denoted by $\mathscr{L}'$ and the underlying closure space of $\mathscr{R}$ is denoted by $\mathscr{R}'$). All the following mappings are to be continuous:

$$\begin{gathered} \cdot : \mathscr{R}' \times \mathscr{R}' \to \mathscr{R}' , \ + : \mathscr{R}' \times \mathscr{R}' \to \mathscr{R}' , \ \{r \to -r\} : \mathscr{R}' \to \mathscr{R}' \\ + : \mathscr{L}' \times \mathscr{L}' \to \mathscr{L}' , \ \{x \to -x\} : \mathscr{L}' \to \mathscr{L}' , \\ \cdot : \mathscr{R}' \times \mathscr{L}' \to \mathscr{L}' . \end{gathered}$$

For a topological algebra, there is yet another requirement:

$$\cdot : \mathscr{L}' \times \mathscr{L}' \to \mathscr{L}' \text{ is continuous.}$$

19 D.4. Examples. (a) If $\langle L, \alpha, \varrho, \beta\rangle$ is a module (algebra) over a ring $\langle R, \beta\rangle$ and u and v are either accrete closures for L and R or discrete closures for L and R, then $\langle L, \alpha, u, \varrho, \beta, v\rangle$ is a topological module (algebra) over the topological ring $\langle R, \beta, v\rangle$. (b) If $\langle L, \alpha, \varrho, \beta\rangle$ is a module (algebra) over a ring $\langle R, \beta\rangle$ and if v is any admissible closure for the ring $\langle R, \beta\rangle$ and u is the accrete closure for L, then $\langle L, \alpha, u, \varrho, \beta, v\rangle$ is a topological module (algebra) over the topological ring $\langle R, \beta, v\rangle$. (c) Let $\mathscr{R} = \langle R, +, \cdot, u\rangle$ be a topological ring and let $\varrho = \{x \to \langle y, x \cdot y\rangle \mid x \in R, y \in R\}$. Then ϱ is an external composition on R over R, $\mathscr{L}_1 = \langle R, +, u, \varrho, +, \cdot, u\rangle$ is a topological module over the topological ring $\langle R, +, \cdot, u\rangle$ and $\mathscr{L}_2 = \langle R, +, \cdot, u, \varrho, +, \cdot, u\rangle$ is a topological algebra over the topological ring $\mathscr{R}$. If we say that we consider a topological ring $\mathscr{R}$ as a topological module or a topological algebra over itself it is to be understood that we consider the topological module $\mathscr{L}_1$ or the topological algebra $\mathscr{L}_2$ over $\mathscr{R}$.

Further examples can be obtained by two basic operations, of taking subspaces or forming products, which will now be introduced.

19 D.5. Definition. A *topological submodule of a topological module* $\mathscr{L}$ over a topological ring $\mathscr{R}$ is a topological module $\mathscr{L}'$ over $\mathscr{R}$ such that the underlying topological group of $\mathscr{L}'$ is a subgroup of the underlying topological group of $\mathscr{L}$ and the external composition of $\mathscr{L}'$ is a restriction of the external composition of $\mathscr{L}$. A *topological subalgebra of a topological algebra* is defined in a similar way (it is sufficient to replace the expressions "module" and "group" by "algebra" and "ring").

Of course we must show that a topological submodule is actually a topological module. This follows, however, from the fact that the restriction of a continuous external composition is a continuous external composition (19 C.9), and the fact that a topological subgroup (subring) of a topological group (ring) is a topological group (ring).

If $\mathcal{R}$ is a topological ring, module or algebra, and if $X \subset |\mathcal{R}|$ is stable under the corresponding structure, then the closure of X is stable.

19 D.6. Remark. By definition 19 D.5 a topological submodule of an $\mathcal{R}$-module is an $\mathcal{R}$-module. It is easily seen that, more generally, the notion of a topological submodule can also be introduced by restricting the scalars. Indeed, if $\mathcal{L}$ is a topological module over a topological ring $\mathcal{R}$ and if $\mathcal{R}'$ is a topological subring of $\mathcal{R}$, then we can construct a topological module $\mathcal{L}'$ over $\mathcal{R}'$ by requiring the underlying topological group of $\mathcal{L}'$ to coincide with that of $\mathcal{L}$ and the external composition of $\mathcal{L}'$ to be a restriction of the external composition of $\mathcal{L}$ to an external composition over $\mathcal{R}'$; we refer to $\mathcal{L}'$ as the topological module obtained from $\mathcal{L}$ by restricting the scalars to $\mathcal{R}'$. Now, given a topological submodule $\mathcal{L}'$ of a topological module $\mathcal{L}$ over a topological ring $\mathcal{R}$ and given a topological subring $\mathcal{R}'$ of $\mathcal{R}$, we can consider the topological module $\mathcal{L}''$ over $\mathcal{R}'$ obtained from $\mathcal{L}'$ by restricting the scalars to $\mathcal{R}'$, and the topological $\mathcal{R}'$-module $\mathcal{L}''$ is the above mentioned submodule of $\mathcal{L}$ in a generalized sense.

19 D.7. Theorem. *Let $\mathcal{R}$ be a topological ring and let $\{\mathcal{L}_a \mid a \in A\}$ be a non-void family of topological $\mathcal{R}$-modules ($\mathcal{R}$-algebras). Then the struct $\mathcal{L} = \langle L, \alpha; u, \varrho, \beta; v\rangle$ is a topological $\mathcal{R}$-module ($\mathcal{R}$-algebra) where $\langle L, \alpha; u\rangle$ is the product of the family $\{\mathcal{G}_a \mid a \in A\}$, each $\mathcal{G}_a$ being the underlying topological group (ring) of $\mathcal{L}_a$, ϱ is the product of $\{\varrho_a \mid a \in A\}$, each ϱ_a being the external composition of $\mathcal{L}_a$, and $\langle \beta, v\rangle$ is the topological ring structure of $\mathcal{R}$.*

19 D.8. Definition. The topological $\mathcal{R}$-module ($\mathcal{R}$-algebra) $\mathcal{L}$ from 19 D.7 will be termed the *product of the family $\{\mathcal{L}_a \mid a \in A\}$ of topological $\mathcal{R}$-modules ($\mathcal{R}$-algebras)* and will be denoted by $\Pi\{\mathcal{L}_a \mid a \in A\}$. As usual, the product of the family $\{\mathcal{L} \mid a \in A\}$ is denoted by $\mathcal{L}^A$.

Proof of 19 D.7. By definition 19 D.1 it is sufficient to show that $\langle L, \alpha, u\rangle$ is a topological group (ring) and this has already been proved (19 B.8, 19 B.21)); and that $\langle u, \varrho, v\rangle$ is a continuous external composition and this was proved in 19 C.11.

19 D.9. Remark. Let $\mathcal{L}$ be the product of a family $\{\mathcal{L}_a \mid a \in A\}$ of topological $\mathcal{R}$-modules. Then the underlying closure space of $\mathcal{L}$ is the product of underlying closure space of the $\mathcal{L}_a$, the underlying topological group of $\mathcal{L}$ is the product of underlying topological groups of the $\mathcal{L}_a$, the external composition of $\mathcal{L}$ is the product of the external compositions of the $\mathcal{L}_a$ and, finally, $\mathcal{L}$ is over the same ring as each $\mathcal{L}_a$, namely over $\mathcal{R}$. It is to be noted that we can define the product of a family of topological modules $\{\mathcal{L}_a\}$, each $\mathcal{L}_a$ being a topological module over $\mathcal{R}_a$, as a topological module over the product topological ring $\Pi\{\mathcal{R}_a\}$; nevertheles this definition is not appropriate because the product $\Pi\{\mathcal{R}_a\}$ need not be a field even if all $\mathcal{R}_a$ are fields. Modules over fields are of principal importance, and in fact very little is known about topological modules over topological rings.

19 D.10. Examples. (a) Let us consider the field R of reals as a topological algebra over R (see example 19 D.4 (c)), and let A be a non-void set. By 19 D.8 and

19 D.7 we obtain a topological algebra R^A over R. Unless otherwise stated, if R^A is considered as a topological algebra or a module it is to be understood that R^A is the R-algebra or R-module described above. (b) Let $\mathscr{L}$ be a topological $\mathscr{R}$-module and let $\mathscr{S}$ be any struct with a non-void underlying set. The mapping $\{f \to \operatorname{gr} f\}$ of the set $\mathbf{F}(\mathscr{S}, \mathscr{L})$ (of all mappings of $\mathscr{S}$ into $\mathscr{L}$) into the $\mathscr{R}$-module $\mathscr{L}^{|\mathscr{S}|}$ is bijective and therefore we can endow the set $\mathbf{F}(\mathscr{S}, \mathscr{L})$ with the structure of a topological $\mathscr{R}$-module so that the mapping mentioned above becomes an isomorphism (we hope that the meaning of the word isomorphism is clear here although the definition will not be given until the next subsection). If $\mathbf{F}(\mathscr{S}, \mathscr{L})$ is considered as a topological $\mathscr{R}$-module then it is to be understood that we mean the topological $\mathscr{R}$-module just defined. We leave to the reader the simple task of defining the topological $\mathscr{R}$-algebra $\mathbf{F}(\mathscr{S}, \mathscr{L})$ where $\mathscr{L}$ is a given topological $\mathscr{R}$-algebra. (c) In particular, given a closure space $\mathscr{P}$, $\mathbf{F}(\mathscr{P}, \mathsf{R})$ is a topological algebra and $\mathbf{C}(\mathscr{P}, \mathsf{R})$ can be considered as a topological subalgebra of $\mathbf{F}(\mathscr{P}, \mathsf{R})$.

19 D.11. If $\mathscr{L}$ is the product of a family $\{\mathscr{L}_a\}$ of topological modules or algebras, and if $X_a \subset |\mathscr{L}_a|$ is stable under the structure of $\mathscr{L}_a$, then the closure of $\Pi\{X_a\}$ is stable.

In conclusion we shall introduce the concept of a normed module or algebra over a normed ring.

19 D.12. Definition. A *norm for a module (algebra)* $\mathscr{L} = \langle L, \alpha; \varrho, \beta; \varphi\rangle$ *over a normed ring* $\mathscr{R} = \langle R, \beta; \varphi\rangle$ ($\langle R, \beta\rangle$ is the underlying ring of $\mathscr{R}$ and φ is the norm) is a norm ψ for the underlying group (ring) $\langle L, \alpha\rangle$ of $\mathscr{L}$ satisfying the following condition:

$$x \in L,\ r \in R \ \Rightarrow\ \psi(r\varrho x) = \varphi r \,.\, \psi x\,.$$

A *normed module (algebra) over a normed ring* $\mathscr{R}$ or simply a *normed* $\mathscr{R}$*-module* ($\mathscr{R}$*-algebra*) is a struct $\langle L, \alpha; \psi, \varrho, \beta; \varphi\rangle$ such that ψ is a norm for the module (algebra) $\langle L, \alpha; \varrho, \beta; \varphi\rangle$ over the normed ring $\mathscr{R}$. We shall say a *real* or *complex normed module (algebra)* instead of normed R-modul or normed C-algebra, where R is the normed field of reals and C is the normed field of complex numbers. The norm ψ will usually be denoted by $\|\cdot\|$.

19 D.13. Theorem. *Let* $\mathscr{L} = \langle L, \alpha; \|\cdot\|, \varrho, \beta; \varphi\rangle$ *be a normed module (algebra) over a normed ring* $\mathscr{R} = \langle R, \beta; \varphi\rangle$. *By* 19 B.26 *the norm* $\|\cdot\|$ *induces a closure operation* u *admissible for the group (ring)* $\langle L, \alpha\rangle$, *and* φ *induces an admissible closure operation* v *for the ring* $\langle R, \beta\rangle$. *The topologized external composition* $\langle u, \varrho, v\rangle$ *is continuous and hence* $\langle L, \alpha; u, \varrho, \beta; v\rangle$ *is a topological module (algebra) over* $\langle R, \beta; v\rangle$.

Convention. A normed $\mathscr{R}$-module ($\mathscr{R}$-algebra) will be considered as a topological module (algebra) with the closure structures just introduced.

Proof. Let $+$ denote addition in $\langle L, \alpha\rangle$ as well as in $\langle R, \beta\rangle$. The closure u is induced by the pseudometric $\{\langle x, y\rangle \to \|x - y\|\}$ and the closure v is induced by the pseudometric $\{\langle r, s\rangle \to \varphi(r - s)\}$. Now, continuity is implied by the following inequality which holds for each $r, s \in R$ and $x, y \in L$:

$$\|r\varrho x - s\varrho y\| \leqq \varphi r\|x - y\| + \varphi(r - s)\,\|y\|\,.$$

19 D.14. Normed modules (algebras) of bounded mappings into a given normed module (algebra) over a normed ring. Let $\mathscr{L}$ be a normed module (algebra) over a normed ring $\mathscr{R}$ and let $\mathscr{S}$ be any non-void struct. For each mapping f of $\mathscr{S}$ into $\mathscr{L}$ put

$$\|f\| = \sup\{\|fx\| \mid x \in |\mathscr{S}|\}\,,$$

and let $\mathbf{F}^*(\mathscr{S}, \mathscr{L})$ denote the set of all f such that $\|f\|$ is finite. Clearly $f \in \mathbf{F}^*(\mathscr{S}, \mathscr{L})$ if and only if $\mathbf{E}\{x \to \|fx\| \mid x \in |\mathscr{S}|\}$ is a bounded subset of R, or equivalently, $\mathbf{E}f$ is a bounded subset of the pseudometric space $\langle|\mathscr{L}|, \{\langle x, y\rangle \to \|x - y\|\}\rangle$. We shall say that such f are bounded; thus $\mathbf{F}^*(\mathscr{S}, \mathscr{L})$ is the set of all bounded mappings of $\mathscr{S}$ into $\mathscr{L}$. For each f and g in $\mathbf{F}(\mathscr{S}, \mathscr{L})$ and r in $\mathscr{R}$ let $f + g = \{x \to (fx + gx)\}$: : $\mathscr{S} \to \mathscr{L}$ (if $\mathscr{L}$ is an algebra, let, in addition, $f\,.\,g = \{x \to fx\,.\,gx\} : \mathscr{S} \to \mathscr{L}$) and $r\,.\,f = \{x \to r\,.\,fx\} : \mathscr{S} \to \mathscr{L}$. Now $\mathbf{F}(\mathscr{S}, \mathscr{L})$ endowed with the addition $\{\langle f, g\rangle \to$ $\to f + g\}$ (and multiplication $\{\langle f, g\rangle \to f\,.\,g\}$) and the external composition corresponding to the external multiplication $\{\langle r, f\rangle \to r\,.\,f\}$ over $\mathscr{R}$ is a module (algebra) over $\mathscr{R}$ and the mapping $\{f \to \operatorname{gr} f\} : \mathbf{F}(\mathscr{S}, \mathscr{L}) \to \mathscr{L}_1{}^{|\mathscr{S}|}$ is an isomorphism, where $\mathscr{L}_1$ is the underlying algebraic module of $\mathscr{L}$ (i.e. $\mathscr{L}_1$ is obtained by omitting norms). If $f, g \in \mathbf{F}^*(\mathscr{S}, \mathscr{L})$ and $r \in \mathscr{R}$ then clearly

$$\|f + g\| \leqq \|f\| + \|g\|\,, \ (\|f\,.\,g\| \leqq \|f\|\,.\,\|g\|)\,, \ \|r\,.\,f\| = |r|\,.\,\|f\|\,, \ \|f\| = \|-f\|\,,$$

and consequently $\mathbf{F}^*(\mathscr{S}, \mathscr{L})$ is stable under the module structure (algebra structure) of $\mathbf{F}(\mathscr{S}, \mathscr{L})$. In what follows $\mathbf{F}^*(\mathscr{S}, \mathscr{L})$ will be considered as a module (algebra) under the relativization of the module structure (algebra structure) of $\mathbf{F}(\mathscr{S}, \mathscr{L})$. The formulae mentioned above show that $\{f \to \|f\| \mid f \in \mathbf{F}^*(\mathscr{S}, \mathscr{L})\}$ is a norm for the module $\mathbf{F}^*(\mathscr{S}, \mathscr{L})$ over the normed ring $\mathscr{R}$. In what follows, $\mathbf{F}^*(\mathscr{S}, \mathscr{L})$ will denote this normed module (algebra) over $\mathscr{R}$ and we shall refer to $\mathbf{F}^*(\mathscr{S}, \mathscr{L})$ as to the *normed module (algebra) of all bounded mappings of the struct $\mathscr{S}$ into the normed module (algebra) $\mathscr{L}$.*

19 D.15. Theorem. *The closure of the normed module $\mathbf{F}^*(\mathscr{S}, \mathscr{L})$ of bounded mappings of a non-void struct $\mathscr{S}$ into a normed module $\mathscr{L}$ over $\mathscr{R}$ is the closure of uniform convergence, more precisely, if $\mathscr{G}$ is the underlying topological group of the topological module associated with $\mathscr{L}$, then the mapping*

$(*)$ $\{f \to f : \mathscr{S} \to \mathscr{G}\} : \mathbf{F}^*(\mathscr{S}, \mathscr{L}) \to \operatorname{unif} \mathbf{F}(\mathscr{S}, \mathscr{G})$

is a topological embedding. In addition, the range of $(*)$ is closed in unif $\mathbf{F}(\mathscr{S}, \mathscr{G})$.

Proof. If $r > 0$ and U is the closed r-sphere about the zero of $\mathscr{R}$, then $\|f\| \leqq r$ if and only if $fx \in U$ for each x, and therefore the mapping $(*)$ is an embedding and if g is not bounded then no h such that $\|h - g\| \leqq r$ is bounded.

We are interested in the case where $\mathcal{L}$ is the normed topological field R of reals and $\mathcal{R} = \mathsf{R}$ as well. For this case we shall need the following theorem which is the analytic background of the famous Stone-Weierstrass theorem.

19 D.16. Theorem. *Let $\mathcal{S}$ be a struct and A a closed subalgebra of* $\mathbf{F}^*(\mathcal{S}, \mathsf{R})$ *containing all constant functions. If $f \in A$, then* $|f| = \{x \to |fx|\} : \mathcal{S} \to \mathsf{R}$ *also belongs to A and consequently, if $f, g \in A$, then* $\max(f, g) = \{x \to \max(fx, gx)\} : \mathcal{S} \to \mathsf{R}$ *and* $\min(f, g) = \{x \to \min(fx, gx)\} : \mathcal{S} \to \mathsf{R}$ *also belong to A. Thus A is lattice-stable in* $\langle \mathbf{F}^*(\mathcal{S}, \mathsf{R}), \leqq \rangle$ *where $\leqq$ is the product-order.* – For proof see ex. 6.

E. CONTINUOUS HOMOMORPHISMS

It turns out that closures admissible for algebraic structs possess rather special properties, e.g. a closure admissible for a group is topological and we shall see later that then it is necessarily uniformizable. Similarly, if f is a mapping of a topologized algebraic struct into another one and if f possesses some algebraic properties (e.g. f is a homomorphism) and some topological properties (e.g. f is continuous at a point), then f possesses some further topological or algebraic properties. There are many profound results which under some weaker algebraic and topological assumptions assert that a closure or a mapping has very striking topological or algebraic properties. Here we shall prove two very simple results with which the reader is surely familiar. The main purpose is to introduce an appropriate terminology.

19 E.1. Theorem. *Let $\mathcal{G} = \langle G, \sigma, u \rangle$ and $\mathcal{H} = \langle H, \mu, v \rangle$ be topologized groups and let f be a homomorphism of $\langle G, \sigma \rangle$ into $\langle H, \mu \rangle$. If both $\mathcal{G}$ and $\mathcal{H}$ are inductively continuous and $f: \langle G, u \rangle \to \langle H, v \rangle$ is continuous at a point, then $f: \langle G, u \rangle \to \langle H, v \rangle$ is continuous.*

Proof. Suppose that f is continuous at a point x and let y be any point of G. We must show that f is continuous at y. It is sufficient to find continuous mappings h of $\langle G, u \rangle$ into itself and k of $\langle H, u \rangle$ into itself such that $f = k \circ f \circ h$ and $hy = x$. Since the topologized groups in question are inductively continuous we can take the mappings

$$h = \{z \to x\sigma z\sigma y^{-1}\}, \quad k = \{w \to (fx)^{-1}\, \mu w \mu (fy)\}.$$

Indeed, clearly $x = hy$, $kfhz = kf(x\sigma z\sigma y^{-1}) = k(fx\mu fz\mu fy^{-1}) = fz$ and h and k are continuous because of the inductive continuity of $\langle \sigma, u \rangle$ and $\langle \mu, v \rangle$.

It is evident that the preceding theorem may be applied to continuous groups, topological groups and also to richer structs as for instance topological rings, modules and algebras.

19 E.2. Theorem. *Let f be a homomorphism of the underlying module $\mathcal{L}_1$ of a normed real module $\mathcal{L}$ into the underlying module $\mathcal{K}_1$ of a normed real module $\mathcal{K}$.*

The following conditions are equivalent:

(a) $f : \mathscr{L} \to \mathscr{K}$ *is continuous;*

(b) $f : \mathscr{L} \to \mathscr{K}$ *is continuous at a point of* $\mathscr{L}$;

(c) $f : \mathscr{L} \to \mathscr{K}$ *is continuous at the zero of* $\mathscr{L}$;

(d) *the image of a sphere about the zero of* $\mathscr{L}$ *is bounded in* $\mathscr{K}$;

(e) $f : \mathscr{L} \to \mathscr{K}$ *is Lipschitz continuous, i.e.* $\|fx\|_{\mathscr{K}} \leqq r\|x\|_{\mathscr{L}}$ *for some* r.

Proof. Conditions (a), (b) and (c) are equivalent by 19 E.1. Clearly (c) implies (*d*) and (e) implies (c). It remains to show that (d) implies (e). Suppose that the image of a sphere S about the zero of $\mathscr{L}$, say with radius $r > 0$, is bounded in $\mathscr{K}$, say $\|fx\|_{\mathscr{K}} \leqq s$ for each $x \in S$. It is easily seen that $\|fx\|_{\mathscr{K}} \leqq s \cdot r^{-1}\|z\|_{\mathscr{L}}$ for each x.

In Section 8 we defined the concept of a homomorphism of an algebraic struct into an algebraic struct of the same type. In the preceding two theorems we were concerned with a mapping of an algebraic struct endowed with further structures, in the former case with closure operations and with norms in the latter case. This situation occurs frequently and therefore we shall introduce further terminology.

19 E.3. Convention. Consider a mapping f of a topologized algebraic struct $\langle P, \alpha; u\rangle$ into another one $\langle Q, \beta; v\rangle$. If $\mathfrak{P}$ is a property of mappings of closure spaces and if the transposed mapping $f : \langle P, u\rangle \to \langle Q, v\rangle$ has property $\mathfrak{P}$, then we shall say that f has the property $\mathfrak{P}$. E.g., f is continuous means that $f : \langle P, u\rangle \to \langle Q, v\rangle$ is continuous. If $\mathfrak{P}$ is a property of mappings of algebraic structs and if the transposed mapping $f : \langle P, \alpha\rangle \to \langle Q, \beta\rangle$ has the property $\mathfrak{P}$, then we shall say that the mapping f has the property $\mathfrak{P}$. E.g., f is a homomorphism means that $f : \langle P, \alpha\rangle \to \langle Q, \beta\rangle$ is a homomorphism.

For example, the result of theorem 19 E.1 can be stated as follows: if a homomorphism of an inductively continuous group is continuous at a point, then f is continuous.

In accordance with 19 E.3, if we say that a mapping f is an embedding, then it is to be understood that $f : \langle P, u\rangle \to \langle Q, v\rangle$ is an embedding and also $f : \langle P, \alpha\rangle \to \to \langle Q, \beta\rangle$ is an embedding. Indeed, to be an embedding is defined for mappings of closure spaces as well as for mappings of algebraic structs. If we want to say that $f : \langle P, u\rangle \to \langle Q, v\rangle$ is an embedding, then we must say that f is an embedding of closure spaces, or simply a topological embedding. Similarly, if we want to say that $f : \langle P, \alpha\rangle \to \langle Q, \beta\rangle$ is an embedding then we must say that f is an embedding of algebraic structs, or simply an algebraic embedding.

19 E.4. Examples. (a) Consider the group $\mathscr{K}$ of all mappings of a topological group $\mathscr{G}$ into a topological commutative group $\mathscr{H}$. The set L of all homomorphisms of $\mathscr{G}$ into $\mathscr{H}$ is a subgroup of $\mathscr{K}$, and the set L_c of all continuous $f \in L$ is also a subgroup of $\mathscr{K}$. Relative to the closure of uniform convergence, L_c is closed in L, but neither L_c nor L need be closed (e.g. consider an accrete $\mathscr{H}$); on the other hand, if the neutral element of $\mathscr{K}$ is closed, then both L and L_c are necessarily closed.

(b) Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be normed linear spaces and $\mathscr{K}$ the submodule of $\mathbf{F}(\mathscr{L}_1, \mathscr{L}_2)$ consisting of all continuous linear mappings. If

$$\varphi f = \sup \{\|fx\|\mathscr{L}_2 | \, \|x\|\mathscr{L}_1 \leqq 1\}$$

for all f in $\mathscr{K}$, then φ is a norm for $\mathscr{K}$. If $\mathscr{L}_1 = \mathscr{L}_2$ then $\varphi(g \circ f) \leqq \varphi g \cdot \varphi f$ for all f and g, and hence φ is a norm for the corresponding algebra.

F. SERIES

Without doubt the reader is familiar with the definition and basic properties of sums of series of real numbers and related concepts. For the sake of completeness and also as an example of application of earlier results we shall introduce the concept of a sum of a family in a topologized commutative semi-group.

It should be remarked that all the results will be needed for real numbers only.

Let $\mathscr{G} = \langle G, \sigma \rangle$ be a commutative semi-group. For each non-void finite family $\{x_a \mid a \in A\}$ ranging in $\mathscr{G}$ there is defined the so-called composite of $\{x_a\}$ in $\mathscr{G}$, denoted by $\sigma\{x_a\}$, to be the element $(\ldots((x_{a_0} \sigma x_{a_1}) \sigma x_{a_2}) \ldots) \sigma x_{a_n}$ of G where $\{a_i\}$ is any finite one-to-one sequence ranging on A. If $\mathscr{G}$ is endowed with a closure operation then we can define composites of infinite families.

19 F.1. Definition. Let $\mathscr{G} = \langle G, \sigma, u \rangle$ be a topologized commutative semi-group and let $\{x_a \mid a \in A\}$ be a non-void family ranging in $\mathscr{G}$. Let $\mathscr{A}$ be the collection of all non-void finite subsets of A. Clearly $\langle \mathscr{A}, \subset \rangle$ is a directed set and hence $N = \langle \{\sigma\{x_a \mid a \in F\} \mid F \in \mathscr{A}\}, \subset \rangle$ is a net in $\mathscr{G}$. Any limit point of N in $\langle G, u \rangle$ is called a *composite* of the family $\{x_a \mid a \in A\}$ in the topologized semi-group $\mathscr{G}$ and denoted by $\sigma_u\{x_a\}$. If x is a composite of $\{x_a\}$, then we shall sometimes say that $\{x_a\}$ composes to x. If $\{x_a\}$ has at least one composite in $\mathscr{G}$ then we shall say that $\{x_a\}$ is *composable* in $\mathscr{G}$. If $\mathscr{G}$ is written additively, then we shall speak about a *sum* instead of a composite and we shall say *summable* instead of composable.

Often we shall need the following direct description of composites: An $x \in G$ is a composite of a family $\{x_a \mid a \in A\}$ in a topologized commutative semi-group $\mathscr{G} = \langle G, \sigma, u \rangle$ if and only if for each neighborhood U of x there exists a finite non-void subset A_0 of A such that the composite of $\{x_a \mid a \in F\}$ in $\langle G, \sigma \rangle$ lies in U for each finite $F \supset A_0$, $F \subset A$.

If u is an accrete closure then N converges to each point of G and therefore each point of G is a composite of $\{x_a\}$ in $\mathscr{G}$. If $\{x_a\}$ is a finite family then the composite $\sigma\{x_a\}$ in $\langle G, \sigma \rangle$ is a composite in $\langle G, \sigma, u \rangle$ for each u.

19 F.2. *Let $\mathscr{G}$ be a topologized commutative semi-group, $\{x_a \mid a \in A\}$ be a family in $\mathscr{G}$ and ϱ be a permuting relation for A (i.e. ϱ is a one-to-one relation on A ranging on A). Then a point x is a composite of $\{x_a\}$ in $\mathscr{G}$ if and only if x is a composite* *of* $\{x_{\varrho a} \mid a \in A\}$ *in* $\mathcal{G}$. *Roughly speaking, the operation of forming composites is commutative.* – Evident.

19 F.3. *Let* $\{x_a \mid a \in A\}$ *and* $\{y_a \mid a \in A\}$ *be families ranging in a continuous commutative semi-group* $\mathcal{G}$. *If* x *is a composite of* $\{x_a\}$ *in* $\mathcal{G}$ *and* y *is a composite of* $\{y_a\}$ *in* $\mathcal{G}$, *then* $x\sigma y$ *is a composite of* $\{x_a \sigma y_a\}$ *in* $\mathcal{G}$.

Proof. The net $\{\sigma\{x_a \mid a \in F\}\}$ converges to x, the net $\{\sigma\{y_a \mid a \in F\}\}$ converges to y and we must show that the net $\{\sigma\{x_a \sigma y_a \mid a \in F\}\}$ converges to $x\sigma y$. This is evident because $(\sigma\{x_a \mid a \in F\})\,\sigma(\{y_a \mid a \in F\}) = \sigma\{x_a \sigma y_a \mid a \in F\}$ for each F and $\mathcal{G}$ is continuous.

19 F.4. *Let* $\{x_a \mid a \in A\}$ *be a family ranging in a continuous commutative semi-group* $\mathcal{G}$, $B_1 \cap B_2 = \emptyset$, $B_1 \cup B_2 = A$, $B_1 \neq \emptyset \neq B_2$. *If* y_1 *is a composite of* $\{x_a \mid a \in B_1\}$, *and* y_2 *is a composite of* $\{x_a \mid a \in B_2\}$ *then* $y_1 \sigma y_2$ *is a composite of* $\{x_a \mid a \in A\}$.

Proof. Let U be a neighborhood of $y_1 \sigma y_2$. Since $\mathcal{G}$ is continuous we can choose a neighborhood V_i of y_i, $i = 1, 2$, such that $[V_1]\,\sigma[V_2] \subset U$. Let B'_i, $i = 1, 2$, be a non-void finite subset of B_i such that $\sigma\{x_a \mid a \in F\} \in V_i$ for each finite subset F of B_i containing B'_i. If $A' = B'_1 \cup B'_2$, then $\sigma\{x_a \mid a \in F\} \in U$ for each finite subset F of A containing A' which shows that $y_1 \sigma y_2$ is a composite of $\{x_a \mid a \in A\}$ in $\mathcal{G}$.

Remark. By induction the result of 19 A.4 can be proved for every finite decomposition of A.

19 F.5. *Let* $\mathcal{G} = \langle G, \sigma, u \rangle$ *be a topological commutative group and* $\{x_a \mid a \in A\}$ *be a composable family in* $\mathcal{G}$. *For each neighborhood* U *of the neutral element* e *of* $\mathcal{G}$ *there exists a finite subset* A' *of* A *such that* $\sigma\{x_a \mid a \in F\} \in U$ *for each non-void finite subset* F *of* $A - A'$.

Proof. Let x be a composite of $\{x_a \mid a \in A\}$ in $\mathcal{G}$. Choose a symmetric neighborhood V of the neutral element such that $[V]\,\sigma\,[V] \subset U$ and then a non-void finite subset A' of A such that $\sigma\{x_a \mid a \in F\} \in x\,\sigma[V]$ for each finite subset F of A containing A'. If F is a non-void finite subset of $A - A'$, then $\sigma\{x_a \mid a \in (A' \cup F)\} \in x\sigma[V]$ and $\sigma\{x_a \mid a \in A'\} \in x\sigma[V]$ and therefore the element $\sigma\{x_a \mid a \in F\}$, which is the difference in $\mathcal{G}$ of these two elements, belongs to $[V]\,\sigma\,[V] \subset U$.

19 F.6. Corollary. *If* $\{x_a \mid a \in A\}$ *is composable in a topological commutative group* $\mathcal{G}$, *then each neighborhood of the neutral element contains all* x_a *except for a finite number.*

19 F.7. Corollary. *If* $\{x_a\}$ *is a composable family in a topological commutative group* $\mathcal{G}$ *with a countable local character, then all the* x_a *lie in the closure of the neutral element, except for a countable number of indices.*

Proof. Notice that the closure of the neutral element is the intersection of neighborhoods of the neutral element.

19 F.8. *Let* f *be a continuous homomorphism of a commutative topological semi-group* $\mathcal{G}_1$ *into another one* $\mathcal{G}_2$. *If* x *is a composite of* $\{x_a\}$ *in* $\mathcal{G}_1$ *then* fx *is a composite of* $\{fx_a\}$ *in* $\mathcal{G}_2$.

Proof. If $\{x_a\}$ is finite then the algebraic composite of $\{fx_a\}$ in $\mathscr{G}_2$ is the image under f of the algebraic composite of $\{x_a\}$ in $\mathscr{G}_1$ (f is a homomorphism). If x is a limit point of a net N, then fx is a limit point of $f \circ N$.

A family in a commutative topological group may have many composites, e.g. if the closure structure is an accrete closure then each point is a composite of each non-void family.

19 F.9. *If e is the neutral element of a topological commutative group $\mathscr{G} = \langle G, \sigma, u \rangle$ and E is the closure of (e), then the set E is a closed subgroup of $\mathscr{G}$, the closure of a point x is $x\sigma[E]$, and if x is a composite of a family $\{x_a\}$ then $x\sigma[E]$ is the set of all composites of $\{x_a\}$.*

Proof. E is closed because u is topological, and E is an invariant subgroup because (e) has these properties (19 B.9). The set $x\sigma[E]$ is the closure of (x) because $\{y \to x . y\} : \mathscr{G} \to \mathscr{G}$ is a homeomorphism which carries e into x and E onto $x\sigma[E]$. To prove the last statement we shall prove that if x is a limit point of a net N, then a point y is a limit point of N if and only if $y \in x\sigma[E]$. The set of all limit points of a net in a topological space is closed and hence each point of $x\sigma[E]$ is a limit point of N. If $y \notin x\sigma[E]$ then $G - x\sigma[E]$ is a neighborhood of y and hence there exists a neighborhood U of e such that $y\sigma U$ is contained in $G - x\sigma[E]$. Choose a symmetric neighborhood V of e such that $[V]\,\sigma[V] \subset U$. It is easily seen that $(x\sigma[V]) \cap (y\sigma[V]) = \emptyset$. The net N is eventually in $x\sigma[V]$ and hence N is not eventually in $y\sigma[V]$.

Remark. We have proved that if $x \notin u(y)$, then x and y have disjoint neighborhoods in $\mathscr{G}$.

19 F.10. Corollary. *If the neutral element of a topological commutative group $\mathscr{G}$ is closed then any family in $\mathscr{G}$ has at most one composite.*

Remark. Each point of R is closed and therefore 19 F.10 applies to the additive group R as well as the multiplicative group $\mathsf{R} - (0)$.

19 F.11. Definition. Let $\mathscr{G}$ be a commutative topological group and let $\mathscr{S}$ be any struct. We shall say that a family $\{f_a\}$ of mappings of $\mathscr{S}$ into $\mathscr{G}$ composes to f *pointwise* or *uniformly* if f is a composite of $\{f_a\}$ in the topological group $\mathsf{F}(\mathscr{S}, \mathscr{G})$ or unif $\mathsf{F}(\mathscr{S}, \mathscr{G})$, respectively. If $\{f_a\}$ composes to f pointwise or uniformly then f is termed a *pointwise* or *uniform composite* of $\{f_a\}$. The *support* of a mapping f of a closure space $\mathscr{S}$ into a topological group $\mathscr{G}$ is the set of all $x \in |\mathscr{S}|$ such that fx is distinct from the neutral element of $\mathscr{G}$; the smallest closed set containing the support of f is termed the *closed support of f*. A family $\{f_a\}$ of mappings of a closure space $\mathscr{S}$ into $\mathscr{G}$ is said to be *locally finite* if {support f_a} is a locally finite family, where support f_a denotes the support of f_a. The definitions just stated are carried over to mappings into a topological ring, field, module or algebra $\mathscr{R}$ by replacing $\mathscr{R}$ by the underlying topological group $\mathscr{G}$, e.g. if R is the field of reals then $\{f_a\}$ composes pointwise to f if and only if $\{g_a\}$ composes pointwise to g where g_a or g is f_a or f regarded as a mapping into the additive topological group of reals.

The following result is simple but very important.

19 F.12. Theorem. *Let $\{f_a\}$ be a family of mappings of a closure space $\mathscr{P}$ into a commutative topological group $\mathscr{G}$. Then*

(a) *Any uniform composite of $\{f_a\}$ is a pointwise composite.*

(b) *If $\{f_a\}$ is finite then any pointwise composite is a uniform composite.*

(c) *If all f_a are continuous then any uniform composite of $\{f_a\}$ is continuous.*

(d) *If $\{f_a\}$ is a locally finite family then f has a pointwise composite, and if, in addition, all $\{f_a\}$ are continuous, then any composite of $\{f_a\}$ is continuous.*

Proof. Statement (a) follows from the fact that the closure structure of unif $\mathbf{F}(\mathscr{P}, \mathscr{G})$ is finer than the closure structure of $\mathbf{F}(\mathscr{P}, \mathscr{G})$. Statement (b) follows from the fact that the closures of a singleton in unif $\mathbf{F}(\mathscr{P}, \mathscr{G})$, and $\mathbf{F}(\mathscr{P}, \mathscr{G})$ coincide (cf. 17 ex. 2). Statement (c) follows from Theorem 19 B.16 which states that the set of all continuous mappings of $\mathscr{P}$ into $\mathscr{G}$ is a closed sub-group of unif $\mathbf{F}(\mathscr{P}, \mathscr{G})$. We shall prove (d). If $\{f_a\}$ is locally finite and $x \in |\mathscr{P}|$, then the set A_x of all a such that $f_a x \neq e$ is finite and therefore the algebraic composite fx or $\{f_a x \mid a \in A_x\}$ is a composite of $\{f_a x\}$ in $\mathscr{G}$. The mapping $f = \{x \to fx\} : \mathscr{P} \to \mathscr{G}$ is a pointwise composite of $\{f_a\}$. (Notice that we have only used the fact that the supports form a point-finite family.) Suppose that f is a pointwise composite of $\{f_a\}$, and all f_a are continuous. We shall prove that f is continuous by showing that any point x of $\mathscr{P}$ has a neighborhood U such that the domain-restriction of f to the subspace U of $\mathscr{P}$ is continuous. Given any x, choose a neighborhood U of x a such that $B = \mathbf{E}\{a \mid$ the support of f_a intersects $U\}$ is finite, and let $g = f : U \to \mathscr{G}$, $g_a = f_a : U \to \mathscr{G}$. Evidently g is a pointwise composite of $\{g_a\}$. It is clear that g is a pointwise composite of $\{g_a \mid a \in B\}$. The set B is finite, and hence g is a uniform composite of $\{g_a \mid a \in B\}$. Since each g_a is continuous, g is continuous by (c). The proof is complete.

20. SEPARATION AND CONNECTEDNESS

In the first subsection two important relations for the set of all subsets of a closure space will be introduced and examined, namely the relations $\mathsf{E}\{\langle X, Y\rangle \mid (\overline{X} \cap Y) \cup \cup (X \cap \overline{Y}) = \emptyset\}$ and $\mathsf{E}\{\langle X, Y\rangle \mid U \cap V = \emptyset$ for some neighborhoods U of X and V of $Y\}$. Both relations will occur frequently in the next chapter. Here the importance of the first relation might be guessed from Theorem 20 A.9 and its corollary, Theorem 20 A.10. Furthermore, the concept of the boundary of a set in a space is introduced. Subsection B, which is closely related to subsection A, is concerned with defining and developing the properties of connected spaces.

A. SEPARATION AND SEMI-SEPARATION

20 A.1. Definition. Let P be a closure space. Two subsets X_1 and X_2 of P are said to be *semi-separated* if there exist neighborhoods U_1 of X_1 and U_2 of X_2 such that $U_1 \cap X_2 = \emptyset = U_2 \cap X_1$. Two subsets X_1 and X_2 of P are said to be *separated* if there exist neighborhoods U_1 of X_1 and U_2 of X_2 such that $U_1 \cap U_2 = \emptyset$.

From the definition we immediately obtain the following properties for the relation "to be separated" and "to be semi-separated".

20 A.2. Let P be a closure space. Both relations for $\exp P$ $\mathsf{E}\{\langle X, Y\rangle \mid X$ and Y are semi-separated$\}$ and $\mathsf{E}\{\langle X, Y\rangle \mid X$ and Y are separated$\}$ are symmetric. If X_1 and X_2 are semi-separated (separated) in P and $Y_i \subset X_i$, $i = 1, 2$, then Y_1 and Y_2 are also semi-separated (separated). Any two separated sets are semi-separated and any two semi-separated sets are disjoint.

Of course, for discrete spaces the relation $\{X$ and Y are separated$\}$, $\{X$ and Y are semi-separated$\}$ and $\{X$ and Y are disjoint$\}$ coincide. In an accrete space no two non-void sets are semi-separated. If P is an infinite topological space such that only all finite sets and P are closed, then no two non-void sets are separated, and two sets are semi-separated if and only if they are disjoint and finite. The following self-evident proposition will often be needed.

20 A.3. *Let P be a space. Two closed subsets are semi-separated if and only if they are disjoint, and two open subsets are separated if and only if they are disjoint.*

It is to be noted that two disjoint closed sets need not be separated (see the example preceding 20 A.3). The class of all topological spaces in which every two disjoint closed sets are separated is very important but rather small as will be seen in Section 29. Nevertheless, it contains all pseudometrizable spaces as stated in the next proposition.

20 A.4. Example. In a pseudometrizable space every two semi-separated sets, and hence every two disjoint closed sets, are separated.

Proof. Let X_1 and X_2 be non-void semi-separated subsets of a pseudometrizable space P. Choose a pseudometric d inducing the closure of P and consider the sets U_1 and U_2 defined as follows, where $i \neq j$:

$$U_i = \mathbf{E}\{x \mid x \in P, \operatorname{dist}(x, X_i) < \operatorname{dist}(x, X_j)\}.$$

The sets U_i are obviously disjoint. If $x \in X_i$, then $\operatorname{dist}(x, X_i) = 0$ and $\operatorname{dist}(x, X_j) > 0$, where $i \neq j$, because $X_i \cap \overline{X}_j = \emptyset$ by our assumption and consequently, $x \in U_i$. Thus $U_i \supset X_i$. Finally, both sets U_i are open because the functions

$$f_i = \{x \rightarrow (\operatorname{dist}(x, X_i) - \operatorname{dist}(x, X_j))\}$$

are continuous (by 18 A.12) and U_i is the inverse image under f_i of an open subset of R, e.g. of the open interval $]\leftarrow, 0[$.

20 A.5. *In order that two subsets X_1 and X_2 of a closure space P be separated it is necessary and sufficient that there exist a neighborhood U_1 of X_1 such that $\overline{U}_1 \cap X_2 = \emptyset$.*

Proof. If U_1 and U_2 are disjoint neighborhoods of X_1 and X_2 then $\overline{U}_1 \cap X_2 \subset \subset \overline{P - U_2} \cap X_2 = \emptyset$. Conversely, if U_1 is a neighborhood of X_1 such that $\overline{U}_1 \cap \cap X_2 = \emptyset$, then $U_2 = P - U_1$ is clearly a neighborhood of X_2 and $U_1 \cap U_2 = \emptyset$.

20 A.6. *Each of the following conditions is necessary and sufficient for two subsets X_1 and X_2 of a given closure space P to be semi-separated:*

(a) $(\overline{X}_1 \cap X_2) \cup (X_1 \cap \overline{X}_2) = \emptyset$

(b) *$X_1 \cap X_2 = \emptyset$ and both X_1 and X_2 are relatively closed in $X_1 \cup X_2$*

(c) *$X_1 \cap X_2 = \emptyset$ and both X_1 and X_2 are relatively open in $X_1 \cup X_2$.*

Proof. It will be shown that (a) is necessary, (c) is sufficient, (a) implies (b) and (b) implies (c). Let X_1 and X_2 be semi-separated and U_i be neighborhoods of X_i such that $U_i \cap X_j = \emptyset$ for $i \neq j$. If $i \neq j$ then evidently $\overline{X}_i \cap X_j \subset \overline{P - U_j} \cap X_j = \emptyset$, which establishes (a). Assume (a) and consider the subspace $Q = X_1 \cup X_2$ of P. We have $\overline{X}_i^Q = \overline{X}_i \cap Q = X_i$ which means that both X_i are closed in Q. Obviously (b) implies (c). Finally suppose (c) and let Q stand for the subspace $X_1 \cup X_2$ of P. Since X_i, $i = 1, 2$, is open in Q, X_i is a neighborhood of itself in Q and by 17 A.9 there exists a neighborhood U_i of X_i in P such that $U_i \cap Q = X_i$. Obviously $X_1 \cap U_2 = = \emptyset = X_2 \cap U_1$ which, by definition, means that X_1 and X_2 are semi-separated in P.

Remark. From the foregoing theorem if follows that two sets semi-separated in a subspace of a space P, are also semi-separated in P. The analogous result for the relation $\{X \to Y \mid X \text{ and } Y \text{ are separated}\}$ is not true. For example, if P is an infinite topological space such that only all finite sets and P are closed, then no two points are separated in P and every finite subspace of P is discrete.

20 A.7. Theorem. *Let X and Y be the unions of the families $\{X_a \mid a \in A\}$ and $\{Y_b \mid b \in B\}$ of subsets of a closure space P. If both families are closure-preserving (for instance, finite) and X_a and Y_b are semi-separated for each a in A and b in B, then X and Y are also semi-separated. If both families are finite and X_a and Y_b are separated for each a in A and b in B, then X and Y are also separated.*

Proof. If both families $\{X_a\}$ and $\{Y_b\}$ are closure-preserving, then $\overline{X} = \bigcup\{\overline{X}_a\}$ and $\overline{Y} = \bigcup\{\overline{Y}_a\}$. It follows that $(\overline{X}_a \cap Y_b) \cup (X_a \cap \overline{Y}_b) = \emptyset$ for each a and b, so that $(\overline{X} \cap Y) \cup (X \cup \overline{Y}) \subset \bigcup\{(\overline{X}_a \cap Y_b) \mid a \in A, b \in B\} \cup \bigcup\{X_a \cap \overline{Y}_b \mid a \in A, b \in B\} = \emptyset$. By 20 A.6 we have the first statement. Now suppose that $\{X_a\}$ and $\{Y_b\}$ are finite and for each a in A and b in B the sets X_a and Y_b are separated. By 20 A.5 a family $\{U_{ab} \mid a \in A, b \in B\}$ can be chosen such that U_{ab} is a neighborhood of X_a and $\overline{U}_{ab} \cap \cap Y_b = \emptyset$. Put $V_a = \bigcap\{U_{ab} \mid b \in B\}$ and $V = \bigcup\{V_a \mid a \in A\}$. Since B is finite, each V_a is necessarily a neighborhood of X_a. Thus V is a neighborhood of X and $\{V_a\}$ being finite, $\overline{V} = \bigcup\{\overline{V}_a\}$. Finally, $\overline{V} \cap Y = (\bigcup\{\overline{V}_a\}) \cap (\bigcup\{Y_b\}) \subset \bigcup\{\overline{V}_a \cap Y_b \mid a \in A, b \in B\} = \emptyset$ which implies that X and Y are separated (again by 20 A.5).

Remark. Apparently the assumption that the $\{X_a\}$ and $\{Y_b\}$ are closure-preserving is essential ((x) and (y) are separated for each y in $\mathsf{R} - (x)$ but x and $\mathsf{R} - (x)$ are not semi-separated). In the second statement the assumption of finiteness cannot be replaced by the weaker assumption that $\{X_a\}$ and $\{Y_b\}$ are locally finite.

20 A.8. *Let f be a continuous mapping of a closure space P into another space Q. If X and Y are semi-separated or separated in Q, then $f^{-1}[X]$ and $f^{-1}[Y]$ possess the corresponding property in P.*

Proof. If $\overline{f^{-1}[X]} \cap f^{-1}[Y] \neq \emptyset$, then also $\overline{X} \cap Y \neq \emptyset$ by the continuity of f which establishes (by 20 A.6) the assertion concerning semi-separated sets. If U and V are disjoint neighborhoods of X and Y in Q, then $f^{-1}[U]$ and $f^{-1}[V]$ are disjoint neighborhoods of $f^{-1}[X]$ and $f^{-1}[Y]$ in P which establishes the second statement.

Let $\mathscr{X}$ be a collection of subsets of a set P and let u be a closure operation for P. It is sometimes important to know under what condition the subspaces X of $\langle P, u\rangle$, $X \in \mathscr{X}$, entirely determine the closure u or, more precisely, under what conditions is the following statement true: if v is a closure for P such that the relativizations of v and u to each X in $\mathscr{X}$ coincide, then $v = u$. For the case of a collection consisting of two elements the question is answered by the following theorem.

20 A.9. Theorem. *Let $\langle P, u\rangle$ be a closure space, X_1 and X_2 be subsets of P, and u^i be the relativization of u to X_i, $i = 1, 2$. In order that*

$$(*) \qquad uX = u_1(X \cap X_1) \cup u_2(X \cap X_2)$$

for each $X \subset P$ it is necessary and sufficient that the sets $P - X_1$ and $P - X_2$ be semi-separated in $\langle P, u \rangle$.

Proof. Remember that $u_i(X \cap X_i) = X_i \cap u(X \cap X_i)$ and therefore we always have $uX \supset (u_1(X \cap X_1) \cup u_2(X \cap X_2))$. — I. First suppose that $(*)$ is true for each $X \subset P$. For $X = P - X_1$ we obtain $u(P - X_1) = u_2((P - X_1) \cap X_2) \subset X_2$ and hence $(P - X_2) \cap u(P - X_1) = \emptyset$. Similarly $(P - X_1) \cap u(P - X_2) = \emptyset$. — II. Now suppose that the sets $P - X_1$ and $P - X_2$ are semi-separated in $\langle P, u \rangle$ and $X \subset P$. We have $uX = u((P - X_1) \cap X) \cup u(X_1 \cap X_2 \cap X) \cup u((P - X_2) \cap X)$. But $u(P - X_1) \cap (P - X_2) = \emptyset$, hence $u((P - X_1) \cap X) = X_2 \cap u((P - X_1) \cap \cap X) = u_2((P - X_1) \cap X)$ and similarly $u((P - X_2) \cap X) = u_1((P - X_2) \cap X)$. Next, we obviously always have $u(X_1 \cap X_2 \cap X) = u_1(X_1 \cap X_2 \cap X) \cup \cup u_2(X_1 \cap X_2 \cap X)$. As a consequence, $uX = u_2((P - X_1) \cap X) \cup u_2(X_1 \cap X_2 \cap X) \cup \cup u_1((P - X_2) \cap X) \cup u_1(X_1 \cap X_2 \cap X) = u_2(X_2 \cap X) \cup u_1(X_1 \cap X)$ (because $(P - X_1) \subset X_2$ and $(P - X_2) \subset X_1$).

20 A.10. Theorem. *Let f be a mapping of a closure space $\langle P, u \rangle$ into another one Q and let X_1 and X_2 be subsets of P such that the sets $P - X_1$ and $P - X_2$ are semi-separated. If the domain-restriction of f to each subspace X_i of P is continuous then f is continuous.*

Proof. By 20 A.9, for each $X \subset P$, $f[uX]$ is contained in the set $f[u_1(X \cap X_1)] \cup \cup f[u_2(X \cap X_2)]$ which is contained in the set $\overline{f[X \cap X_1]}^Q \cup \overline{f[X \cap X_2]}^Q = \overline{f[X]}^Q$ by the continuity of the restrictions.

20 A.11. Definition. The *boundary* of a subset X of a closure space P is defined to be the set $\operatorname{bd} X = \overline{X} \cap \overline{P - X}$.

According to 14 B.6 the boundary of a set can be described as follows:

20 A.12. *A point $x \in P$ belongs to the boundary of a subset X of a space P if and only if each neighborhood of x intersects both X and $P - X$.*

20 A.13. *Let X be a subset of a closure space P. Then $\operatorname{bd} X = \operatorname{bd}(P - X)$, $\overline{X} = X \cup \operatorname{bd} X$ and $\operatorname{int} X = X - \operatorname{bd} X$.*

Proof. The first formula is self-evident and the last two are clearly equivalent. It will be shown that $\overline{X} = X \cup \operatorname{bd} X$. The inclusion $\overline{X} \supset X \cup \operatorname{bd} X$ is obvious, and to prove the converse inclusion it is sufficient to show that $\overline{X} - X \subset \operatorname{bd} X$. We have $\overline{X} - X \subset \overline{X} \cap (P - X) \subset \overline{X} \cap \overline{P - X} = \operatorname{bd} X$.

Corollary. *Let P be a closure space. A subset X of P is closed if and only if $\operatorname{bd} X \subset X$, and X is open if and only if $\operatorname{bd} X \cap X = \emptyset$, i.e. $\operatorname{bd} X = \overline{X} - X$. In particular, $X \subset P$ is simultaneously open and closed if and only if $\operatorname{bd} X = \emptyset$.*

According to 20 A.13 the closure operation of a closure space is completely determined by the boundary operation $\{X \to \operatorname{bd} X\}$; indeed $\overline{X} = X \cup \operatorname{bd} X$. It follows that every topological property can be described in terms of the boundary operation. For example we shall prove the following characterization of a topological closure space.

20 A.14. *A closure space P is topological if and only if the boundary of each subset of P is closed.*

Proof. If P is topological, then the boundary of $X \subset P$ is closed as the intersection of two closed sets, namely $\overline{X}$ and $\overline{P - X}$. Conversely, if $X \subset P$ and $\operatorname{bd} X$ is closed, then $\overline{\overline{X}} = \overline{X \cup \operatorname{bd} X} = \overline{X} \cup \operatorname{bd} X = X \cup \operatorname{bd} X = \overline{X}$ and hence $\overline{\overline{X}} = \overline{X}$.

20 A.15. *If X is a subset of a closure space P, then the sets* $\operatorname{int} X = X - \operatorname{bd} X$ *and* $\operatorname{int}(P - X) = P - \overline{X} = (P - X) - \operatorname{bd}(P - X)$ *are semi-separated.*

Proof. It will be shown that $(\overline{\operatorname{int} X} \cap \operatorname{int}(P - X)) \cup (\operatorname{int} X \cap \overline{\operatorname{int}(P - X)}) = \emptyset$. According to the symmetry it is sufficient to show that $\overline{\operatorname{int} X} \cap \operatorname{int}(P - X) = \emptyset$, but this is obvious because $\overline{\operatorname{int} X} \subset \overline{X}$ and $\operatorname{int}(P - X) = P - \overline{X}$.

Further properties of the boundary operation can be found in the exercises. Now we shall turn to connected sets and spaces.

B. CONNECTEDNESS

20 B.1. Definition. A subset X of a closure space $\mathscr{P}$ is said to be *connected* in $\mathscr{P}$ if X is not the union of two non-void semi-separated subsets of $\mathscr{P}$, that is, $X = X_1 \cup X_2$, $(\overline{X} \cup X_2) \cup (X_1 \cap \overline{X}_2) = \emptyset$ implies that $X_1 = \emptyset$ or $X_2 = \emptyset$. A space $\mathscr{P}$ is said to be *connected* if the underlying set $|\mathscr{P}|$ of $\mathscr{P}$ is connected in $\mathscr{P}$.

Every accrete space is connected, and a non-void discrete space is connected if and only if its underlying set is a singleton.

20 B.2. Theorem. *A closure space $\mathscr{P}$ is connected if and only if $\mathscr{P}$ is not the union of two disjoint non-void open subsets, that is, $\mathscr{P}$ contains no proper non-void subset simultaneously open and closed. A subset X of a closure space $\mathscr{P}$ is connected if and only if the subspace X of $\mathscr{P}$ is connected.* – A straightforward consequence of the definitions.

As an example we shall describe all connected subsets of a boundedly order-complete ordered space. A pair of *consecutive elements* of a monotone ordered set is defined to be a pair $\langle x, y \rangle$ such that $x < y$ and the order-open interval $]\!] x, y [\![$ is empty, that is, y immediately follows x.

20 B.3. Theorem. *Let $\langle P, \leqq, u \rangle$ be an ordered space (thus $\leqq$ is monotone). Then the following statements hold:*

(a) *If X is a connected subset of $\langle P, u \rangle$ then X contains each interval $[\![x, y]\!]$ with x and y in X, i.e. X is an interval-like subset of $\langle P, \leqq \rangle$ if $X \neq \emptyset$.*

(b) *The space $\langle P, u \rangle$ is connected if and only if the ordered set $\langle P, \leqq \rangle$ is boundedly order-complete and there exists no pair of consecutive elements in $\langle P, \leqq \rangle$.*

(c) *If $\langle P, u \rangle$ is connected, then $X \subset P$ is a connected subset of $\langle P, u \rangle$ if and only if X is an interval in $\langle P, \leqq \rangle$.*

Proof. I. Statement (a) is almost evident. Indeed, if $x \in X$, $y \in X$ and $z \in$ $\in (\llbracket x, y \rrbracket - X)$, then the non-void and disjoint sets $U_1 = X \cap \rrbracket \leftarrow, z \llbracket$ and $U_2 =$ $= X \cap \rrbracket z, \rightarrow \llbracket$ cover X, and they are open in X because they are intersections of X with open sets in $\langle P, u \rangle$, namely with the sets $\rrbracket \leftarrow, z \llbracket$ and $\rrbracket z, \rightarrow \llbracket$; thus X is not connected (20 B.2). – II. Now we shall prove that $\langle P, u \rangle$ is not connected whenever $\langle P, \leqq \rangle$ is not boundedly order-complete or there exists a pair of consecutive elements in $\langle P, u \rangle$. If $\langle x, y \rangle$ is a pair of consecutive elements, then $(\rrbracket \leftarrow, y \llbracket, \rrbracket x, \rightarrow \llbracket)$ is an open disjoint cover of $\langle P, u \rangle$ consisting of non-void sets and hence $\langle P, u \rangle$ is not connected (by 20 B.2). Assuming that $\langle P, \leqq \rangle$ is not boundedly order-complete, let us choose a bounded non-void subset X of $\langle P, \leqq \rangle$ such that the least upper bound of X does not exist; now let us consider the set X_1 of all y in P such that $y < x$ for some x in X, and the set X_2 of all upper bounds of X. Clearly (X_1, X_2) is a disjoint cover of P and both sets X_1 and X_2 are non-void. To prove that the space $\langle P, u \rangle$ is not connected it remains to show that both sets X_1 and X_2 are open (20 B.2). The set X_1 is open because it is the union of all open intervals $\llbracket \leftarrow, x \llbracket$, $x \in X$. The set X_2 is the union of all intervals $\rrbracket y, \rightarrow \rrbracket$, $y \in X_2$; indeed, if $z \in X_2$, then z is not the least element of X_2 because sup X does not exist, and hence $z \in \rrbracket y, \rightarrow \rrbracket$ for some $y \in X_2$. – III. Assuming that $\langle P, \leqq \rangle$ is boundedly order-complete and there exists no pair of consecutive elements, we shall prove that $\langle P, u \rangle$ is connected. By 20 B.2 it is sufficient to show that no non-void open proper subset U of $\langle P, \leqq, u \rangle$ is closed. Since $\langle P, \leqq \rangle$ is boundedly order-complete, the open set U is the union of a disjoint collection $\mathscr{V}$ of non-void order-open intervals. If $V \in \mathscr{V}$ and $z \in (uV - V)$, then $z \in (uU - U)$; in fact, clearly $z \in uU$ and, if $z \in U$, then $z \in V_1$ for some V_1 in $\mathscr{V}$ and hence $V_1 \neq V$ and $V_1 \cap V \neq \emptyset$; this contradicts the fact that $\mathscr{V}$ is disjoint. Thus it is enough to show that $uV - V \neq \emptyset$ for some V in $\mathscr{V}$. Let $V \in \mathscr{V}$. Since $U \neq P$, also $V \neq P$, and hence either $V = \llbracket \leftarrow, x \llbracket$ or $V = \rrbracket x, \rightarrow \rrbracket$ or $V = \rrbracket x, y \llbracket$. Clearly $V \neq \emptyset$. It follows from our assumption on the non-existence of consecutive elements that $x \in uV$ in all three cases. E.g. if $V = \llbracket \leftarrow, x \llbracket$ and $x \notin V$, then $\rrbracket y, x \llbracket \cap V = \emptyset$ for some $y, y < x$ (remember that $V \neq 0$), and hence $\langle y, x \rangle$ is a pair of consecutive elements. The proof of (b) is complete. – IV. Statement (c) is an immediate consequence of statements (a) and (b) and the trivial fact that each interval in a boundedly order-complete ordered set is boundedly order-complete.

Corollary a. *A subset of the space* R *of reals is connected if and only if X is an interval in* R. *In particular, the space* R *is connected.*

Corollary b. *Suppose that the closure structure of a connected space $\mathscr{P}$ is induced by orders $\prec_1$ and $\prec_2$. Then a subset X of $|\mathscr{P}|$ is an interval in $\langle |\mathscr{P}|, \prec_1 \rangle$ if and only if X is an interval in $\langle |\mathscr{P}|, \prec_2 \rangle$* (20 B.3 (c)). *Moreover, if X is an interval in $\langle |\mathscr{P}|, \prec_1 \rangle$ with end points x and y then X is an interval in $\langle |\mathscr{P}|, \prec_2 \rangle$ with the same end points. Stated in other words, in a connected ordered space the concepts of an interval and its end points do not depend on the order, i.e., in a connected ordered space an interval and its end points are topological concepts.*

If $\langle P, u, \leqq\rangle$ is an ordered space, then $\langle P, u, \leqq^{-1}\rangle$ is an ordered space. In general, there are many other orders such that $\langle P, u, \prec\rangle$ is an ordered space; e.g. each monotone order for a finite set induces the discrete closure. On the other hand, if u is connected then there are only two orders.

Corollary c. *If $\langle P, u, \leqq\rangle$ is a connected ordered space and an order $\prec$ induces u, then either $\prec\, = \,\leqq$ or $\prec\, = \,\leqq^{-1}$.*

Proof. Suppose that $\leqq$ and $\prec$ induce a connected closure u for a set P. If P is void or a singleton, then clearly $\leqq\, = \,\prec$. In the other case let x and y be any two distinct points of P such that $x \leqq y$. It is sufficient to prove that if $x \prec y$, then $\leqq\, = \,\prec$. But this follows without difficulty from Corollary b.

The following restatement of the definition of connected sets will often be convenient in proofs of results which follow.

20 B.4. *A subset C of a closure space is connected if and only if the following condition is fulfilled: if C is contained in the union of two semi-separated sets X_1 and X_2, then $C \subset X_1$ or $C \subset X_2$.*

Proof. If C is not connected then there exist semi-separated sets X_1 and X_2 such that $X_1 \cup X_2 = C$ and $X_1 \neq \emptyset \neq X_2$. Clearly C is contained neither in X_1 nor in X_2. Conversely, suppose C is connected. If $C \subset X_1 \cup X_2$ and if the sets X_1 and X_2 are semi-separated, then the sets $C \cap X_1$ and $C \cap X_2$ are also semi-separated, and consequently $C \cap X_1 = \emptyset$ or $C \cap X_2 = \emptyset$, that is, either $C \subset X_2$ or $C \subset X_1$.

20 B.5. Theorem. *If $\{X_a \mid a \in A\}$ is a family of connected non-void subsets of a closure space $\mathscr{P}$ and if X is a connected subset of $\mathscr{P}$ such that the sets X and X_a are not semi-separated for any $a \in A$, then the set $Y = X \cup \bigcup\{X_a \mid a \in A\}$ is connected.*

Proof. Let Y_1 and Y_2 be two semi-separated subsets of $\mathscr{P}$ such that $Y \subset Y_1 \cup Y_2$. It is to be shown that $Y \subset Y_1$ or $Y \subset Y_2$. Since $X \subset Y_1 \cup Y_2$, X is connected and the sets Y_1 and Y_2 are semi-separated, the set X is contained in Y_1 or in Y_2, say in Y_1. Again from 20 B.4 we find that each X_a is contained in Y_1 and hence $Y \subset Y_1$. Indeed, the set X_a being connected, it is contained in Y_1 or Y_2; but it is not contained in Y_2 because $X \subset Y_1$, the sets Y_1 and Y_2 are semi-separated and the sets X and X_a are not semi-separated.

Corollary. *Let $\mathscr{P}$ be a closure space. Then the closure of each connected subset of $\mathscr{P}$ is connected, and the union of a family $\{X_a\}$ of connected subsets of $\mathscr{P}$ is connected provided the intersection of $\{X_a\}$ is non-void.*

Proof. Notice that the sets (x) and X are semi-separated for no $x \in \overline{X}$ and apply the theorem to the set X and the family $\{(x) \mid x \in \overline{X}\}$. The second statement is obtained by application of the theorem to the set (x) and the family $\{X_a\}$ where x is a point of the intersection of $\{X_a\}$.

Now we are prepared to introduce the notion of a component of a space.

20 B.6. Definition. A *component of a closure space* $\mathscr{P}$ is a connected subset X of $\mathscr{P}$ with the following property: If $X \subset C \subset \mathscr{P}$ and C is connected, then $C = X$. Thus components are maximal connected subsets. The *component of a point* x in a space $\mathscr{P}$ is the component of $\mathscr{P}$ containing x.

The definition of the component of a point in a space requires proof of the fact that every point is contained in exactly one component of the space. But this is obvious, and for convenience will be stated in the following theorem. First we recall Definition 12 A.1 of the decomposition of a struct $\langle X, \xi\rangle$ as a disjoint cover of X whose elements or members are non-void provided that $X \neq \emptyset$, and $\emptyset$ if $X = \emptyset$.

20 B.7. Theorem. *Let $\mathscr{P}$ be a closure space. Each component of $\mathscr{P}$ is closed, and distinct components of $\mathscr{P}$ are disjoint and hence semi-separated. Every non-void connected subset of $\mathscr{P}$ is contained in exactly one component of $\mathscr{P}$. The collection of all components of a non-void $\mathscr{P}$ is a decomposition of $\mathscr{P}$. If $\mathscr{P} = \emptyset$, then $\emptyset$ is the unique component of $\mathscr{P}$.*

Proof. Each component is closed because the closure of a connected set is a connected set by the Corollary of 20 B.5. Two distinct components are disjoint because the union of two connected sets is connected provided their intersection is non-void by the Corollary of 20 B.5. Since disjoint closed sets are semi-separated, distinct components are necessarily semi-separated. If X is a non-void connected subset of $\mathscr{P}$ then the union of all connected sets intersecting X is the component of $\mathscr{P}$ (by 20 B.5) which obviously contains X; the uniqueness is clear. The last statement follows from the fact that every singleton is connected.

20 B.8. Definition. A space $\mathscr{P}$ is said to be *connected between its points x and y* if each simultaneously open and closed subset of $\mathscr{P}$ containing x contains y as well. The *quasi-component of a point* $x \in \mathscr{P}$ in $\mathscr{P}$ is the set of all $y \in \mathscr{P}$ such that $\mathscr{P}$ is connected between x and y; stated in other words, the quasi-component of x is the intersection of all simultaneously closed and open subsets of $\mathscr{P}$ containing x. A *quasi-component of a space* $\mathscr{P}$ is a quasi-component of some point of $\mathscr{P}$ or $\emptyset$ if $|\mathscr{P}| = \emptyset$.

20 B.9. Theorem. *Let $\mathscr{P}$ be a closure space. The relation $\mathbf{E}\{\langle x, y\rangle \mid$ the space $\mathscr{P}$ is connected between x and $y\}$ is an equivalence on $\mathscr{P}$. The equivalence classes are the quasi-components if $\mathscr{P} \neq \emptyset$. Each component is contained in a unique quasi-component, and consequently each quasi-component is the union of all components contained in it. A quasi-component is a component if and only if it is connected. The quasi-components of $\mathscr{P}$ are closed and distinct quasi-components are separated. A space $\mathscr{P}$ is connected if and only if $\mathscr{P}$ is the only quasi-component of $\mathscr{P}$.*

Proof. Let $\mathscr{O}$ be the collection of all simultaneously closed and open subsets of $\mathscr{P}$. It is easy to see that $\mathscr{O}$ is additive and multiplicative, i.e. closed under finite unions and finite intersections. Let ϱ be the relation under question, i.e. $\langle x, y\rangle \in \varrho$ if and only if $\mathscr{P}$ is connected between x and y. Obviously ϱ is reflexive, i.e. $\langle x, x\rangle \in \varrho$ for each $x \in \mathscr{P}$. If $\langle x, y\rangle \notin \varrho$, $x \in \mathscr{P}$, $y \in \mathscr{P}$, then there exists an O in $\mathscr{O}$ such that $x \in O$,

$y \in (\mathscr{P} - O)$. But $(\mathscr{P} - O) \in \mathcal{O}$ and hence $\langle y, x \rangle \notin \varrho$ which establishes the symmetry of $P \times P - \varrho$, and hence of ϱ. Only transitivity remains to be proved. Let $\langle x, y \rangle \in \varrho$ and $\langle y, x \rangle \in \varrho$; it is to be shown that $\langle x, z \rangle \in \varrho$. If $O \in \mathcal{O}$ and $x \in O$, then $y \in O$ because $\langle x, y \rangle \in \varrho$, and hence $z \in O$ because $\langle y, z \rangle \in \varrho$. Thus each $O \in \mathcal{O}$ containing x also contains z, so that $\langle x, z \rangle \in \varrho$. If $O \in \mathcal{O}$ and C is connected, then $C \subset O$ or $C \subset P - O$ because of 20 B.4. It follows that $C \times C \subset \varrho$ for each connected subset of $\mathscr{P}$. Hence each quasi-component is the union of all components contained in it. If a quasi-component X is a component, then X is connected because each component is connected. Conversely, if a quasi-component X is connected, then X is contained in the component of any of its points; on the other hand, it contains the component of all its points. It follows that X is a component. The quasi-components are closed as intersections of closed sets. Let X_1 and X_2 be two distinct quasi-components. There exists an O in $\mathcal{O}$ such that $X_1 \subset O$, $X_2 \subset (P - O)$. Now O and $\mathscr{P} - O$ are disjoint neighborhoods of X_1 and X_2. The last statement is almost self-evident.

A quasi-component need not be connected, in other words, a quasi-component need not be a component and a component need not be a quasi-component. A very simple example will be given. Let us define a closure operation u for $P = \mathsf{N} \cup (x_1) \cup \cup (x_2)$, where $x_1 \neq x_2$ and $x_1, x_2 \notin \mathsf{N}$, so that the subspaces N and $(x_1) \cup (x_2)$ are discrete, N is open and x_i belongs to the closure of a subset N_1 of N if and only if N_1 is infinite. It is easy to see that every one-point set is a component and every one-point set (n) with n in N is a quasi-component. On the other hand, neither (x_1) nor (x_2) are quasi-components. Indeed, if O is a simultaneously open and closed set containing x_1, then $O \cap \mathsf{N}$ is infinite because O is open and $x_2 \in O$ because O is closed and $x_2 \in u(O \cap \mathsf{N})$. In 20 ex. 6 we give an example of a subspace of $\mathsf{R} \times \mathsf{R}$ in which these concepts do not coincide.

Nevertheless in some extensive classes of spaces the concepts of a component and of a quasi-component do coincide. Here we shall state only the following trivial result, the proof of which is left to the reader.

20 B.10. *If a component X of a space $\mathscr{P}$ is open, then X is a quasi-component.*

Corollary. *If a space $\mathscr{P}$ possesses a finite number of components, then each component is open and hence each component is a quasi-component.*

The concluding part of the section is devoted to an investigation of connected spaces and the class of all connected spaces.

20 B.11. *Let C be a connected subset of a connected space $\mathscr{P}$. If $|\mathscr{P}| - C = X_1 \cup X_2$ and the sets X_1 and X_2 are semi-separated, then the sets $C \cup X_1$ and $C \cup X_2$ are connected. In other words, if X is simultaneously closed and open in $|\mathscr{P}| - C$, then $C \cup X$ is connected.*

Proof. Let us suppose that $C \cup X_1 = Y_1 \cup Y_2$ where Y_1 and Y_2 are non-void semi-separated sets. Since C is connected, C must be contained either in Y_1 or in Y_2 (by 20 B.4). Without loss of generality we may assume $C \subset Y_1$. It follows that $Y_2 \subset X_1$, and consequently the sets Y_2 and X_2 are semi-separated. Thus the pairs of sets Y_1, Y_2 and X_2, Y_2 are semi-separated. By 20 A.7 the sets $Y_1 \cup X_2$ and Y_2 are also semi-separated. Since $(Y_1 \cup X_2) \cup Y_2 = |\mathcal{P}|$ and $\mathcal{P}$ is connected, we have $Y_2 = \emptyset$ or $Y_1 \cup \cup X_2 = \emptyset$. But this contradicts our assumption that $Y_1 \neq \emptyset \neq Y_2$.

Corollary a. *If Y_1 and Y_2 are closed subsets of a space $\mathcal{Q}$ such that the sets $Y_1 \cup Y_2$ and $Y_1 \cap Y_2$ are connected, then the sets Y_1 and Y_2 are both connected.*

Proof. Consider the subspace $P = Y_1 \cup Y_2$ of $\mathcal{Q}$ and the sets $X_1 = Y_1 - Y_2$, $X_2 = Y_2 - Y_1$, $C = Y_1 \cap Y_2$. By our assumption the space P is connected, C is a connected subset of P, $C \cap X_i = \emptyset$ and $C \cup X_1 \cup X_2 = P$. Since Y_i are closed in $\mathcal{Q}$, the sets Y_i are closed in P as well. It follows that the sets X_1 and X_2 are semi-separated. According to the theorem the sets $Y_1 = X_1 \cup C$ and $Y_2 = X_2 \cup C$ are connected.

Corollary b. *Let C be a connected subset of a connected space $\mathcal{P}$. If K is a component of $|\mathcal{P}| - C$, then $|\mathcal{P}| - K$ is a connected set.*

Proof. If $X_1 \cup X_2 = |\mathcal{P}| - K$ and the sets X_1 and X_2 are semi-separated, then $X_i \cap C = \emptyset$ for some $i = 1, 2$, because C is connected; on the other hand, $K \cup X_i$ is connected by 20 B.11. Since K is a component of $|\mathcal{P}| - C$ and the set $K \cup X_i \subset |\mathcal{P}| - C$ is connected, $K \cup X_i = K$, that is $X_i = \emptyset$, which establishes the connectedness of $|\mathcal{P}| - K$.

20 B.12. Theorem. *If $\mathcal{U}$ is an open cover of a connected space $\mathcal{P}$ then each two points x and y of $\mathcal{P}$ can be joined by a finite chain in $\mathcal{U}$, i.e. for each x and y in $\mathcal{P}$ there exists a finite sequence $\{U_i \mid i \leqq n\}$ in $\mathcal{U}$ such that $x \in U_0$, $y \in U_n$ and $U_{i-1} \cap \cap U_i \neq \emptyset$ for each $i = 1, \ldots, n$ (such a finite sequence is called a* chain *from x to y).* (Compare with ex. 10(d)).

Proof. If $|\mathcal{P}| = \emptyset$ then the assertion is trivial. In the remaining case fix a point x in $\mathcal{P}$ and let us consider the set X of all $y \in \mathcal{P}$ which can be joined to x by a finite chain in $\mathcal{U}$. We must show that $X = |\mathcal{P}|$. Since $\mathcal{P}$ is connected and X is non-void, it will be sufficient to show that X is simultaneously open and closed in $\mathcal{P}$. If $y \in X$ then there exists a finite chain $\{U_i \mid i \leqq n\}$ from x to y. Now clearly $\{U_0, \ldots, U_n\}$ is a chain in $\mathcal{U}$, joining x to each point of U_n. It follows that $U_n \subset X$. Since y was arbitrarily chosen in X, X is a union of open sets and hence X is open. Now let $y \in \overline{X}$. It is to be proved that $y \in X$. Choose $U \in \mathcal{U}$ such that $y \in U$. Since $y \in \overline{X}$ and U is a neighborhood of y, we can choose a point z in $X \cap U$. By the definition of X there exists a chain $\{U_i \mid i \leqq n\}$ from x to z, and clearly $\{U_0, \ldots, U_n, U\}$ is a chain from x to y. It follows that $y \in X$ which concludes the proof.

Corollary. *If G is a connected topological group and U is a neighborhood of the neutral element, then $\bigcup\{U^n \mid n \in \mathrm{N}\} = G$, where $U^1 = U$ and $U^{n+1} = [U^n] \cdot [U]$.*

Proof. We may assume that U is open. Consider the open cover $\mathcal{U} = \{[U] \cdot x \mid x \in G\}$. By induction it is easy to show that U^n is the set of all $y \in G$

which can be joined to the neutral element by a chain in $\mathscr{U}$ of a length at most n (of course, the length of a chain $\{U_i \mid i \leqq n\}$ is defined as n).

Remark. The Corollary can be proved directly. Obviously $H = \bigcup\{U^n \mid n \in \mathrm{N}\}$ is an open subset of G. It is easy to see that H is a group. But an open subgroup of a group is closed by 19 B.12. Since G is connected, $G = H$.

The sum $\mathscr{P}$ of a family $\{\mathscr{P}_a\}$ of connected closure spaces is not connected provided at least two spaces from the family are non-void. Indeed, the subspace $(a) \times \mathscr{P}_a$ of $\mathscr{P}$ is both open and closed in $\mathscr{P}$ for each index a.

20 B.13. The class of all connected spaces is closed under surjective continuous mappings; that is, if f is a continuous mapping of a space $\mathscr{P}$ onto a space $\mathscr{Q}$ and if $\mathscr{P}$ is connected, then $\mathscr{Q}$ is connected.

Corollary. *If $f: \mathscr{P} \to \mathscr{Q}$ is a continuous mapping and $X \subset |\mathscr{P}|$ is connected, then $f[X]$ is connected.*

Proof. Let f be a continuous mapping of $\mathscr{P}$ onto $\mathscr{Q}$. If $\mathscr{Q}$ is not connected then $\mathscr{P}$ is not connected. Indeed, if $|\mathscr{Q}| = Y_1 \cup Y_2$ and Y_1 and Y_2 are semi-separated, then the sets $X_1 = f^{-1}[Y_1]$ and $X_2 = f^{-1}[Y_2]$ are also semi-separated by 20 A.8 and $X_1 \cup X_2 = |\mathscr{P}|$. Since f is surjective, if $Y_1 \neq \emptyset \neq Y_2$, then also $X_1 \neq \emptyset \neq X_2$.

If $f: \mathscr{P} \to \mathscr{Q}$ is a surjective continuous mapping and $\mathscr{Q}$ is connected, then $\mathscr{P}$ need not be connected even if f is one-to-one. In fact, every space is the image under a one-to-one continuous mapping of a discrete space, and a discrete space is connected if and only if it has at most one point. On the other hand

20 B.14. *A closure space $\mathscr{P}$ is connected if and only if its topological modification $\tau\mathscr{P}$ is connected.*

Proof. If $\mathscr{P}$ is connected, then $\tau\mathscr{P}$ is connected by the foregoing result because the identity mapping of $\mathscr{P}$ onto $\tau\mathscr{P}$ is continuous. If $\mathscr{P}$ is not connected, then there exists a non-void proper subset X of $\mathscr{P}$ which is simultaneously closed and open in $\mathscr{P}$. The set X possesses the same properties in $\tau\mathscr{P}$ and hence $\tau\mathscr{P}$ is not connected.

20 B.15. *The product $\mathscr{P}$ of a family $\{\mathscr{P}_a \mid a \in A\}$ of connected closure spaces is a connected space.*

Proof. I. If $A = (\alpha, \beta)$, $|\mathscr{P}| \neq \emptyset$ and $y \in \mathscr{P}_\alpha$, then $\mathscr{P}$ is the union of a family $\{Y_x \mid x \in \mathscr{P}_a\}$ of connected sets $Y_x = \mathbf{E}\{w \mid w \in \mathscr{P}, \operatorname{pr}_a w = x\}$ each of which meets the connected set $\mathbf{E}\{w \mid \operatorname{pr}_\beta w = y\}$. By 20 B.5 $\mathscr{P}$ is connected.

II. By induction one can deduce from I that the theorem is true for every finite A.

III. Now let A be an arbitrary set. The empty space is connected and therefore we may assume that $\mathscr{P} \neq \emptyset$. Choose a point $x = \{x_a\}$ in $\mathscr{P}$. Let C be the component of x. We shall prove that $C = |\mathscr{P}|$. For each finite subset F of A let

$$P(F) = \mathbf{E}\{y = \{y_a\} \mid y_a \in \mathscr{P}_a, a \in (A - F) \Rightarrow y_a = x_a\}.$$

Clearly $P(F)$ is homeomorphic with the product space $\Pi\{\mathscr{P}_a \mid a \in F\}$. According to the second part of the proof, $P(F)$ is connected. Thus the union Q of all $P(F)$ where

F runs over finite subsets of A, is contained in C. The proof will be complete if we show that $\bar{Q} = |\mathscr{P}|$, because components are closed. But this is obvious (see also 22 A.7).

20 B.16. Corollary. *For each cardinal* $\aleph$ *the product spaces* $\mathsf{R}^{\aleph}$ *and* $I^{\aleph}$ *are connected, where* I *is an interval in* R.

Proof. According to 20 B.3 the spaces R and I are connected. The assertion follows from the preceding result.

The following more general result will be needed later.

20 B.17. *Let* $\mathscr{P}$ *be the product of a family* $\{\mathscr{P}_a \mid a \in A\}$ *of closure spaces. Each component* C *of* $\mathscr{P}$ *is of the form* $\Pi\{C_a \mid a \in A\}$ *where* C_a *is a component of* $\mathscr{P}_a$ *for each* a *in* A.

Proof. Let C be a component of $\mathscr{P}$ and let $C_a = \mathrm{pr}_a [C]$ for each $a \in A$. The sets C_a are connected as images under continuous mappings of a connected set C. For each $a \in A$ we consider the component C'_a of P containing C_a. The set $C' = \Pi\{C'_a \mid a \in \in A\}$ is connected by 20 B.15. But C is a component and $C \subset C'$. Thus $C = C'$ and C is the product of components of coordinate spaces.

21. LOCALIZATION OF PROPERTIES

We now intend to explain various possibilities for the localization of the properties of sets in a closure space.

First (subsection A) we give all definitions and some illustrative examples, and explain some logical difficulties connected with the idea of the localization of an arbitrary property. Then, a non-trivial and rather important case is considered, namely the localization of connectedness. Further important cases will be treated in the following section. It is to be noted that no general theorems about localization are proved; various general results, however, are stated in the exercises. As a matter of fact, it seems that no non-trivial theorem on localization (in arbitrary spaces) is known; on the other hand, there are many profound results concerning localization of special properties in special cases, mainly in paracompact spaces (30 E).

A. LOCALIZATION

Given a property $\mathfrak{P}$ of sets in a closure space P, we want to define what is meant by the statement that a set X locally possesses the property in question at a point x of the space P. The following possibilities arise naturally:

(a) there exists a neighborhood U of x such that the set $U \cap X$ has the property $\mathfrak{P}$ in the space P;

(b) $U \cap X$ has the property $\mathfrak{P}$ in P for each neighborhood U of x;

(c) there exist arbitrarily small neighborhoods U of x such that $U \cap X$ possesses the property $\mathfrak{P}$ in the space P.

More intuitively, condition (a) requires that there exists a set (in P) which is "large" relative to x (that is, which is a neighborhood of x) and intersects X in a set with property $\mathfrak{P}$. Condition (c) requires the existence of large (relative to x) but also arbitrarily small sets intersecting X in a set with property $\mathfrak{P}$. Finally, (b) requires that each large set intersects X in a set with property $\mathfrak{P}$.

Conditions (a), (b), (c) occur, explicitly or merely implicitly, in many theorems and definitions and, in fact, they have already occurred in our exposition. For instance, if P is a space, $x \in P$, $X \subset P$, then $x \notin \overline{X}$ if and only if there exists a neighborhood U of x such that $X \cap U = \emptyset$ (condition (a)), $x \in \overline{X}$ if and only if $U \cap X \neq \emptyset$ for each neighborhood of x (condition (b)), $x \in \overline{X}$ if and only if there exist arbitrarily small neighborhoods U of x such that $U \cap X \neq \emptyset$ (condition (c)). It turns out that condition (b) is usually too strong and often leads to uninteresting properties; on the other hand, if it leads to an interesting property, then it coincides with condition (c). Therefore we restrict our attention to conditions (a) and (c).

Apparently, we can now define e.g. the "feeble localization" as follows: "If $\mathfrak{P}$ is a property of sets in a space (that is, if $\mathfrak{P}$ is a logical relation between sets and closure spaces or, in other words, if $\mathfrak{P}$ is a property of pairs $\langle X, \mathscr{P} \rangle$, X being a set and $\mathscr{P}$ a space with $X \subset |\mathscr{P}|$), if $\mathscr{P}$ is a space, $X \subset |\mathscr{P}|$, $x \in |\mathscr{P}|$, and there exists a neighborhood U of x in $\mathscr{P}$ such that $U \cap X$ possesses property $\mathfrak{P}$ in $\mathscr{P}$, then we shall say that X locally posesses property $\mathfrak{P}$ at x in $\mathscr{P}$". This statement may be surely considered as a correct definition. However, its logical character is not in line with the approach adopted in this book where we avoid considering properties of properties or operations with properties (and even, as far as possible, statements referring to "every property"). For this reason, we formulate definitions of localization for arbitrary relations (between sets and spaces) and consider the usual expressions, such as indicated above, as conventional substitutes, more intuitive and more adapted for "practical" use.

21 A.1. Definition. Let α be a relation for the class of all sets and the class of all closure spaces such that $\langle X, \mathscr{P} \rangle \in \alpha$ implies that X is a subset of the underlying set $|\mathscr{P}|$ of $\mathscr{P}$; we shall say that X is an *α-set* in $\mathscr{P}$ if $\langle X, \mathscr{P} \rangle \in \alpha$.

Let $\mathscr{P}$ be a closure space, $X \subset |\mathscr{P}|$, $x \in |\mathscr{P}|$. If there exists a neighborhood U of x in $\mathscr{P}$ such that $U \cap X$ is an α-set in $\mathscr{P}$ (i.e. such that $\langle U \cap X, \mathscr{P} \rangle \in \alpha$), then we shall say that X is *feebly locally an α-set at x in $\mathscr{P}$*. If there exist arbitrarily small neighborhoods U of x in $\mathscr{P}$ such that $U \cap X$ is an α-set in $\mathscr{P}$ (i.e. if for any neighborhood V of x in $\mathscr{P}$ there exists a neighborhood U of x in $\mathscr{P}$ such that $U \subset V$, $\langle U \cap X, \mathscr{P} \rangle \in \alpha$), then we shall say that X is *locally an* α-set at x in $\mathscr{P}$.

Thus, X is locally an α-set at x in $\mathscr{P}$ if and only if there exists a local base $\mathscr{U}$ at x in $\mathscr{P}$ such that $U \cap X$ is an α-set in $\mathscr{P}$ for each $U \in \mathscr{U}$. Obviously

21 A.2. *If X is locally an α-set at x in $\mathscr{P}$, then X is feebly locally an α-set at x in $\mathscr{P}$.*

21 A.3. Convention. Consider a property $\mathfrak{P}$ of sets in closure spaces, i.e. a logical relation between sets and closure spaces. Let α be a relation such that $\langle X, \mathscr{P} \rangle \in \alpha$ if and only if X possesses property $\mathfrak{P}$ in $\mathscr{P}$; suppose that $\langle X, \mathscr{P} \rangle \in \alpha$ implies $X \subset |\mathscr{P}|$. If X is locally (feebly locally) an α-set at x in $\mathscr{P}$, then we shall also say that X *locally (feebly locally) possesses property $\mathfrak{P}$ at x in $\mathscr{P}$.*

It turns out that the above convention leads to the current terminology. For example, if α consists of all pairs $\langle X, \mathscr{P} \rangle$ such that $\mathscr{P}$ is a closure space and X is a closed set in $\mathscr{P}$, then the expression "X is locally closed at x in $\mathscr{P}$" means the same as "X is locally an α-set at x in $\mathscr{P}$".

21 A.4. Examples. (a) Let α consist of all pairs $\langle X, \mathscr{P} \rangle$ such that $\mathscr{P}$ is a closure space and X is a non-void subset of $\mathscr{P}$. Thus X is an α-set in a space $\mathscr{P}$ if and only if X is a non-void subset of $|\mathscr{P}|$. A subset X of a space $\mathscr{P}$ is feebly locally an α-set at a point $x \in |\mathscr{P}|$ in $\mathscr{P}$, i.e. X is feebly locally non-void at x in $\mathscr{P}$, if and only if $X \neq \emptyset$. Indeed, if $X \neq \emptyset$ then $|\mathscr{P}|$ is a neighborhood of x intersecting X in a non-void set, and conversely if $U \cap X \neq \emptyset$ for some set U, then clearly $X \neq \emptyset$ as well. On the other hand, X is locally non-void at x in $\mathscr{P}$ if and only if x belongs to the closure of X in $\mathscr{P}$. Indeed, by definition, X is locally non-void at x in $\mathscr{P}$ if and only if $x \in |\mathscr{P}|$ and there exists a local base $\mathscr{U}$ at x in $\mathscr{P}$ such that $U \cap X \neq \emptyset$ for each $U \in \mathscr{U}$; the statement now follows from 14 B.6. This example shows that the converse of 21 A.2 is not true. Nevertheless the converse is true for relations α of a certain important kind.

(b) A relation α whose domain is a class of sets will be called *hereditary* if $\langle X, \mathscr{P} \rangle \in \alpha$ and $Y \subset X$ imply $\langle Y, \mathscr{P} \rangle \in \alpha$. It is evident that the converse of 21 A.2 holds for hereditary α. It is to be noted that the notion of a hereditary relation corresponds to the currently utilized concept of a hereditary property. A property $\mathfrak{P}$ of sets in a space is often said to be hereditary if each subset of a set possessing $\mathfrak{P}$ in a space $\mathscr{P}$ possesses $\mathfrak{P}$ in $\mathscr{P}$. In this book, we avoid considering "properties of properties"; nevertheless, we shall use the above expression for convenience; thus when we speak of a hereditary property, we shall mean that the corresponding relation α is hereditary.

(c) Let α consist of all pairs $\langle \emptyset, \mathscr{P} \rangle$ such that $\mathscr{P}$ is a closure space. Thus X is an α-set in a space $\mathscr{P}$ if and only if $X = \emptyset$. Evidently the following three statements are equivalent, where $\mathscr{P}$ is a space, $x \in |\mathscr{P}|, X \subset |\mathscr{P}|$: X is feebly locally empty at x in $\mathscr{P}$; X is locally empty at x in $\mathscr{P}$; x does not belong to the closure of X in $\mathscr{P}$.

Let α be a relation as in 21 A.1, $\mathscr{P}$ be a space and X be a subset of $|\mathscr{P}|$. Let X_φ be the set of all points x such that X is locally (feebly locally) an α-set at x in $\mathscr{P}$. There are two important cases, namely $X \subset X_\varphi$ and $X_\varphi = |\mathscr{P}|$.

21 A.5. Definition. Let α be a relation for classes of all sets and for all closure spaces such that $\langle X, \mathscr{P} \rangle \in \alpha$ implies $X \subset |\mathscr{P}|$. We shall say that a set X is *locally an α-set* in a closure space $\mathscr{P}$ if X is locally an α-set at each point $x \in |\mathscr{P}|$ in $\mathscr{P}$. We shall say that a set X is *relatively locally an α-set* in a closure space $\mathscr{P}$ if X is locally an α-set at each point $x \in X$ in $\mathscr{P}$. The corresponding definitions for feeble localization are obtained by replacing "locally" by "feebly locally".

Notice that for any α, the empty set is relatively locally an α-set in every closure space.

21 A.6. *If X is (relatively) locally an α-set in a space $\mathscr{P}$, then X is (relatively) feebly locally an α-set in $\mathscr{P}$. If X is (feebly) locally an α-set in $\mathscr{P}$, then X is relatively (feebly) locally an α-set in $\mathscr{P}$.* — Obvious.

21 A.7. Convention. Consider a property $\mathfrak{P}$ of sets in closure spaces. Let α be a relation such that $\langle X, \mathscr{P} \rangle \in \alpha$ if and only if X possesses property $\mathfrak{P}$ in $\mathscr{P}$; suppose that $\langle X, \mathscr{P} \rangle \in \alpha$ implies $X \subset |\mathscr{P}|$. If X is locally (relatively locally, feebly locally, relatively feebly locally) an α-set in $\mathscr{P}$, then we shall also say that X *locally* (*relatively locally*, etc) *possesses property* $\mathfrak{P}$ *in* $\mathscr{P}$.

21 A.8. Examples. (a) For the property "to be empty in $\mathscr{P}$" the corresponding α is equal to $(\emptyset) \times \mathbf{C}$. If $\mathscr{P}$ is a space, then the empty set is the only one which is locally (feebly locally, relatively feebly locally, relatively locally) an α-set in $\mathscr{P}$.

(b) The property "to be non-void". Every subset of a space $\mathscr{P}$ is relatively locally non-void in $\mathscr{P}$. On the other hand, X is locally non-void in $\mathscr{P}$ if and only if the closure of X is $|\mathscr{P}|$ (such sets are said to be dense in $\mathscr{P}$ and will be studied in 22 A). Every subset of a space $\mathscr{P}$ is relatively feebly locally non-void in $\mathscr{P}$, but a subset X of $\mathscr{P}$ is feebly locally non-void in $\mathscr{P}$ if and only if $X \neq \emptyset$.

(c) The property "to be closed in $\mathscr{P}$". Every closed subset of a space $\mathscr{P}$ is feebly locally closed in $\mathscr{P}$, and hence, relatively feebly locally closed in $\mathscr{P}$. Conversely, if X is feebly locally closed in $\mathscr{P}$, then X is closed in $\mathscr{P}$. Indeed, if $x \in \overline{X}$ and if U is a neighborhood of x such that the set $U \cap X$ is closed, then $x \in \overline{U \cap X}$ because U is a neighborhood of x, and hence $x \in U \cap X = \overline{U \cap X}$, in particular, $x \in X$. On the other hand, a relatively feebly locally closed set need not be closed, e.g. the set X of all n^{-1}, $n = 1, 2, \ldots$ is relatively feebly locally closed in the space R of reals but X is not closed in R because $0 \in \overline{X} - X$. Locally closed and relatively locally closed sets will be considered in Section 27.

(d) The property "to be open". It is easily seen that the following three statements are equivalent, where $\mathscr{P}$ is a space: X is relatively feebly locally open in $\mathscr{P}$; X is feebly locally open in $\mathscr{P}$; and X is open in $\mathscr{P}$. We shall prove that X is relatively locally open in $\mathscr{P}$ if and only if X is open in $\mathscr{P}$ and the subspace X of $\mathscr{P}$ is topological. First suppose that X is open in $\mathscr{P}$ and the subspace X of $\mathscr{P}$ is topological. Let U be any neighborhood of $x \in X$ in $\mathscr{P}$. Clearly $U \cap X$ is a neighborhood of x in $\mathscr{P}$ and hence in X, and X being topological, the interior V of $U \cap X$ in X is open in X; X being open in $\mathscr{P}$, V is open in $\mathscr{P}$. Thus V is a neighborhood of x contained in a given neighborhood U such that $V \cap X = U$ is open in $\mathscr{P}$; by definition, X is relatively locally open. Conversely. suppose that X is relatively locally open in $\mathscr{P}$ and x is a point of X. If U is a neighborhood of x, then, by our assumption, there exists a neighborhood V of x contained in U such that $V \cap X$ is open in $\mathscr{P}$; but $x \in V \cap X$ and hence $V \cap X$ is an open set in $\mathscr{P}$ containing x and contained in X. It follows that X is open in $\mathscr{P}$ and the subspace X of $\mathscr{P}$ is topological. It remains to describe locally open sets and this will be performed in the exercises.

21 A.9. Definition. Let α be a relation for the class of all sets and the class of all closure spaces such that $\langle X, \mathscr{P}\rangle \in \alpha$ implies $X \subset |\mathscr{P}|$. A closure space $\mathscr{P}$ is said to be *locally an* α*-set at* x, *feebly locally an* α*-set at* x, *locally an* α*-set* or *feebly locally an* α*-set* is the underlying set $|\mathscr{P}|$ of $\mathscr{P}$ has the corresponding property in $\mathscr{P}$.

21 A.10. Convention. If $\mathfrak{P}$ is a property of sets in a closure space, then we shall say that a space $\mathscr{P}$ locally possesses $\mathfrak{P}$ at x, feebly locally possesses $\mathfrak{P}$ at x, locally possesses $\mathfrak{P}$, feebly locally possesses $\mathfrak{P}$ if the underlying set of $|\mathscr{P}|$ of $\mathscr{P}$ has the corresponding property in $\mathscr{P}$ (see conventions 21 A.3 and 21 A.7).

Notice that a space $\mathscr{P}$ is locally empty if and only if $|\mathscr{P}| = \emptyset$, and every space is locally non-void. Next, a space $\mathscr{P}$ is locally a singleton (= is locally one-point) if and only if $\mathscr{P}$ is discrete. Finally, every space is feebly locally open and feebly locally closed. Locally closed spaces will be considered in Section 27, and locally open spaces are described in the following theorem.

21 A.11. Theorem. *A space $\mathscr{P}$ is topological if and only if $\mathscr{P}$ is locally open.* — A particular case of Example 21 A.8 (d).

In the next subsection a non-trivial example of the localization of connectedness will be exhibited, and in Section 22 further non-trivial properties will be localized (locally non-meager spaces). In closing we shall introduce a terminology which is sometimes utilized in the literature.

21 A.12. Let $\mathscr{P}$ be a space and let $\mathscr{X}$ be a collection of subsets of $\mathscr{P}$. Put $\alpha = \mathscr{X} \times (|\mathscr{P}|)$. We shall say that X *locally belongs at* x *to* $\mathscr{X}$ *in* $\mathscr{P}$ if X is locally an α-set at x in $\mathscr{P}$. The following expressions are treated in a similar way: X *(relatively) locally belongs to* $\mathscr{X}$ *in* $\mathscr{P}$, *feebly locally belongs to* $\mathscr{X}$ *at* x *in* $\mathscr{P}$ and X *(relatively) feebly, locally belongs to* $\mathscr{X}$. Let $\mathscr{X}_1$ ($\mathscr{X}_{1r}$) be the collection of all subsets of $\mathscr{P}$ which locally (relatively locally) belong to $\mathscr{X}$ in $\mathscr{P}$, and let $\mathscr{X}_2$ ($\mathscr{X}_{2r}$) be the collection of all subsets of $\mathscr{P}$ which feebly locally (relatively feebly locally) belong to $\mathscr{X}$. The collection $\mathscr{X}$ is said to be *locally (relatively locally, feebly locally, relatively feebly locally) determined in* $\mathscr{P}$ if $\mathscr{X} = \mathscr{X}_1$ ($\mathscr{X} = \mathscr{X}_{1r}, \mathscr{X} = \mathscr{X}_2, \mathscr{X} = \mathscr{X}_{2r}$). For example, by Example 21 A.8 (c) and (d), the collection of closed subsets of a space $\mathscr{P}$ is feebly locally determined in $\mathscr{P}$ but not relatively feebly locally determined in $\mathscr{P}$, and the collection of all open subsets of a space $\mathscr{P}$ is feebly locally as well as relatively feebly locally determined in $\mathscr{P}$.

B. LOCALLY CONNECTED SPACES

In accordance with preceding definitions, a subset X of a closure space $\mathscr{P}$ is locally connected at a point x if every neighborhood of x contains a neighborhood V of x such that $V \cap X$ is connected, and a space $\mathscr{P}$ is locally connected at $x \in \mathscr{P}$ if every neighborhood of x contains a connected neighborhood of x, that is, if connected neighborhoods of x form a local base at x; stated in other words, x possesses arbitrarily small connected neighborhoods. A subset X of a space $\mathscr{P}$ is feebly locally connected at $x \in \mathscr{P}$ if there exists a neighborhood V of x such that $V \cap X$ is connected, and a space $\mathscr{P}$ is feebly locally connected at $x \in \mathscr{P}$ if there exists a connected neighborhood of x. A subset X of $\mathscr{P}$ is locally connected if every neighborhood of each point x of $\mathscr{P}$ contains a neighborhood U of x such that $U \cap X$ is connected, and a space $\mathscr{P}$ is locally connected if each point of $\mathscr{P}$ possesses arbitrarily small connected neighborhoods. A subset X of a space is relatively locally connected if the subspace X

of $\mathscr{P}$ is locally connected. Similarly, a subset X of $\mathscr{P}$ is feebly locally connected if every point of $\mathscr{P}$ possesses a neighborhood U such that $U \cap X$ is connected, and a space $\mathscr{P}$ is feebly locally connected if every point of $\mathscr{P}$ possesses at least one connected neighborhood. Finally, a subset X of $\mathscr{P}$ is relatively feebly locally connected if the subspace X of $\mathscr{P}$ is feebly locally connected.

21 B.1. Examples. A) (a) Every connected ordered space is locally connected. Indeed, by 20 B.3 every interval in an ordered connected space is connected; since open intervals form a base for the ordered space, the statement follows. In particular, the space R of reals is locally connected. On the other hand, a locally connected ordered space need not be connected. Obviously, the union X of any finite family of order-closed connected intervals of an ordered space is a relatively locally connected subset of the space, and hence a locally connected subspace which need not be connected. On the other hand X is induced by an order, i.e. X is an ordered space (because the intervals were assumed order-closed).

(b) It is easily seen that a connected space need not be locally connected. The following example is typical. Let X be a subspace of $\mathsf{R} \times \mathsf{R}$, the underlying set of which is $(\mathsf{R} \times 0)) \cup \bigcup \{(x) \times \mathsf{R} \mid x$ is a rational number$\}$. The space X is connected as the union of connected sets each of which intersects a connected subset of $\mathsf{R} \times \mathsf{R}$. On the other hand, X is locally connected in precisely those points which have zero second coordinate (as may be proved directly).

B) Let X be the subset of the space R of reals (which is locally connected by A(a)) consisting of all points $1/n$, $n = 1, 2, \ldots$ Then:

(a) X is relatively locally connected in R because X is a discrete subspace of R and every discrete space is locally connected (a one-point set is always connected).

(b) X is not locally connected at the point 0. Indeed, if U is any neighborhood of 0, then $U \cap X$ is an infinite discrete subspace of R.

(c) $\bar{X} = (0) \cup X$, both X and (0) are relatively locally connected but $\bar{X}$ is not relatively locally connected; moreover, $\bar{X}$ is not relatively feebly locally connected. It follows that the closure of a relatively locally connected set need not be relatively locally connected, and the union of two relatively locally connected sets need not be relatively locally connected.

C) Let $X =]\!]-1, 0[\![\cup]\!] 0, 1 [\![$ be a subset of R. Clearly X is not locally connected (not even feebly locally connected) at 0. On the other hand, each of the intervals is obviously a locally connected subset of $\mathscr{P}$. Thus the (disjoint) union of two locally connected subsets need not be locally connected.

Any subset X of a space $\mathscr{P}$ is locally connected at each point $x \in (\mathscr{P} - \bar{X})$ because each neighborhood U of x contains a neighborhood V such that $V \cap X$ is empty and hence connected, since V can be taken as the neighborhood $U \cap (\mathscr{P} - X)$.

21 B.2. *Every closed relatively (feebly) locally connected subset of a space $\mathscr{P}$ is (feebly) locally connected.*

The proposition need not be true if the set is not closed (see 21 B.1, B (a) and (b)).

A closed subset of a locally connected space need not be relatively locally connected (see 21 B.1, B (b), $\overline{X}$ is closed), in other words, a closed subspace of a locally connected space need not be locally connected. On the other hand, open subsets of locally connected spaces obviously are relatively locally connected.

21 B.3. Theorem. *If a subset X of a topological space $\mathscr{P}$ is (feebly) locally connected, then the closure of X is also (feebly) locally connected.*

Proof. Since $\overline{X}$ is closed, it is sufficient to prove that $\overline{X}$ is locally connected at each point $x \in \overline{X}$ (see 21 B.2). Suppose U is a neighborhood of a point $x \in \overline{X}$ such that $U \cap X$ is connected. Consider the set $V = (U \cap \overline{U \cap X}) \cup (U - \overline{X})$. First, it is easy to see that $V \cap \overline{X}$ is a connected set. Indeed, $V \cap \overline{X} = U \cap \overline{U \cap X} \cap \overline{X} = U \cap \overline{U \cap X} = \overline{U \cap X}^U$, that is, $V \cap \overline{X}$ is the closure of a connected set in U, namely of $U \cap X$. On the other hand, V is a neighborhood of x. Here the assumption "$\mathscr{P}$ is topological" is essential. Choose an open neighborhood W of x such that $W \subset U$. By 14 B.26 we have $W \cap \overline{X} = W \cap \overline{W \cap X} \subset \overline{U \cap X} \cap U$ and hence $W \subset V$. We have proved that each neighborhood U of a point $x \in \overline{X}$ contains a neighborhood V of x such that $V \cap \overline{X}$ is connected. Both statements of the theorem follow.

Remark. The assumption "$\mathscr{P}$ is topological" cannot be omitted from 21 B.3 (see 21 ex. 2).

21 B.4. *The union of a locally finite family of closed locally connected (feebly locally connected) subsets of a space $\mathscr{P}$ is a locally connected (feebly locally connected) subset of $\mathscr{P}$.*

Proof. Let X be the union of a locally finite family $\{X_a \mid a \in A\}$ of closed locally connected subsets of a space $\mathscr{P}$. Let x be any point of $\mathscr{P}$; then there exists a neighborhood U of x and a finite subset A_x of A such that $a \in (A - A_x)$ implies $U \cap X_a = \emptyset$ and $x \in \bigcap\{X_a \mid a \in A_x\}$. Let V be a neighborhood of x. For each $a \in A_x$ there exists a neighborhood W_a of x such that $W_a \subset V \cap U$ and $W_a \cap X_a$ is connected. Put $W' = \bigcup\{W_a \cap X_a \mid a \in A_x\}$, $W = W' \cup ((U \cap V) - X)$. First, W is a neighborhood of x because it contains a neighborhood $\bigcap\{W_a \mid a \in A_x\}$ of x. Clearly $W \cap X = W'$ and finally, W' is connected as the union of connected sets $W_a \cap X_a$ containing a common point, namely x. Thus W is a neighborhood of x contained in a given neighborhood V of x and $W \cap X$ is connected. The proof for feebly locally connectedness is left to the reader.

Now we proceed to an investigation of properties of locally connected and feebly locally connected spaces. We begin with a description of feebly locally connected spaces.

21 B.5. Theorem. *Each of the following two conditions is necessary and sufficient for a closure space $\mathscr{P}$ to be feebly locally connected:*

(a) *The components of $\mathscr{P}$ are open.*

(b) *The space $\mathscr{P}$ is homeomorphic with the sum of a family of connected spaces.*

Proof. It will be shown that (a) is necessary, (a) implies (b) and (b) is sufficient. Let x be a point of a component C of a feebly locally connected space $\mathscr{P}$. There exists a connected neighborhood V of x. Since both V and C are connected and $V \cap \cap C \neq \emptyset$, the set $V \cup C$ is connected and consequently $V \cup C = C$, i.e. $V \subset C$. Thus x is an interior point of C. It follows that (a) is necessary. Let each component C of a space $\mathscr{P}$ be open and let $\mathscr{C}$ be the collection of all components of $\mathscr{P}$. The sum of the family $\{C \mid C \in \mathscr{C}\}$ of subspaces of $\mathscr{P}$ is clearly homeomorphic with $\mathscr{P}$. Finally, if $\mathscr{P}$ is the sum of a family $\{P_a\}$ of connected spaces, then $i_a[P_a]$ is a connected neighborhood of each of its points for all a, where i_a is the canonical embedding of P_a into $\mathscr{P}$. Since $\bigcup\{i_a[P_a]\} = |\mathscr{P}|$, $\mathscr{P}$ is feebly locally connected.

Corollary a. *The components and quasi-components of a feebly locally connected (in particular, of a locally connected) space coincide.*

Indeed, by 20 B.10 every open component is a quasi-component (use the condition (a)).

Corollary b. *If a space $\mathscr{P}$ possesses only a finite number of components, then $\mathscr{P}$ is feebly locally connected*; indeed, each component is open as the complement of a finite union of closed sets, namely the union of the remaining components.

Obviously, the open subsets of a locally connected space are relatively locally connected; stated in other words, open subspaces of locally connected spaces are locally connected. According to Theorem 21 B.5, the components of an open subspace U of a locally connected space are necessarily open in U and hence in the space, because U is open. If the space is topological, then this property characterizes local connectedness as is stated in the following theorem.

21 B.6. Theorem. *The components of open subspaces of a locally connected space $\mathscr{P}$ are open in $\mathscr{P}$. A topological space $\mathscr{P}$ is locally connected if and only if the components of open subspaces of $\mathscr{P}$ are open.*

Proof. The first statement, and hence the necessity in the second statement, has been already proved. Conversely, suppose that the components of open sets are open. If U is a neighborhood of a point x of $\mathscr{P}$, then we can choose an open neighborhood V of x contained in U. Now the component of the point x in V is open, by our assumption, and consequently it is a neighborhood of x, which is connected and contained in U.

If $C \neq \emptyset$ is a connected subset of a subset X of a space $\mathscr{P}$ such that $\operatorname{bd} C \cap X = \emptyset$, then C is simultaneously relatively open and closed in X, and consequently C contains a quasi-component of X. On the other hand, C being connected, C is contained in a component of X. But each component is contained in a quasi-component and consequently C is simultaneously a component and a quasi-component of X. If C is a component of a subspace X of a space $\mathscr{P}$, then $X \cap \operatorname{bd} C$ need not be empty because $X \cap \operatorname{bd} C = \emptyset$ if and only if C is simultaneously open and closed in X. Nevertheless, if X is an open subspace of a locally connected space, then each component of X is open in $\mathcal{P}$ and closed in X and hence bd $C \cap X = \emptyset$. We have thus proved the following proposition.

21 B.7. *Let $C \neq \emptyset$ be a connected subset of a subset X of a space $\mathcal{P}$. If $X \cap$ bd $C = \emptyset$, then C is a component of X. If $\mathcal{P}$ is locally connected, X is open in $\mathcal{P}$ and C is a component of X, then $X \cap$ bd $C = \emptyset$.*

21 B.8. Theorem. *Let C be a component or a quasi-component of a subset X of a locally connected space P. Then* bd $C \subset$ bd X.

Proof. Let us suppose that there exists a point x in bd C − bd X. Since bd $C \subset \subset \overline{C} \subset \overline{X}$, clearly $x \in \overline{X}$. By our assumption $x \notin \operatorname{bd} X = \overline{X} \cap \overline{P - X}$, and hence $x \in (P - \overline{P - X}) = \operatorname{int} X$ because $x \in \overline{X}$. The space being locally connected, there exists a connected neighborhood U of x contained in the neighborhood X of x. Since $x \in \operatorname{bd} C \subset \overline{C}$ and U is a neighborhood of x, the set $U \cap C$ is non-void. Thus $x \in \operatorname{bd} C$, U is a connected neighborhood of x, $U \subset X$, and $U \cap C \neq \emptyset$. If C is a component, then $U \cup C$ is a connected set as the union of two non-disjoint connected sets, and hence $U \cup C \subset C$, i.e. $U \subset C$ which implies that x is an interior point of C; this contradicts $x \in \operatorname{bd} C$. If C is a quasi-component, then the component of X containing U intersects C and hence it is contained in C. It follows that $U \subset C$ which implies $x \in \operatorname{int} C$, and this contradicts $x \in \operatorname{bd} C$.

The concluding part is devoted to an examination of the classes of all locally connected and of all feebly locally connected spaces.

Every discrete space and also every accrete space is locally connected. Since every space is the image under a one-to-one continuous mapping of a discrete space and every space admits a one-to-one continuous mapping onto an accrete space, the images and the inverse images under one-to-one continuous mappings of locally connected spaces need not be locally connected. Invariance under certain special types of mappings will be proved later. The following simple result will be verified here.

21 B.9. *If a closure space $\mathcal{P}$ is locally connected or feebly locally connected, then the topological modification of $\mathcal{P}$ possesses the corresponding property.*

Proof. It has already been shown that if $X \subset \mathcal{P}$ is open or connected in $\mathcal{P}$, then X possesses the corresponding property in the topological modification $\tau\mathcal{P}$ of $\mathcal{P}$. Now if $\mathcal{P}$ is locally connected, $x \in \tau\mathcal{P}$ and U is a neighborhood of x in $\tau\mathcal{P}$, then there exists an open neighborhood V of x with $V \subset U$. By 21 B.6 the component C of V in $\mathcal{P}$ containing x is open. It follows that C is open and connected in $\tau\mathcal{P}$ and $x \in C \subset U$. Thus the arbitrarily chosen point x of $\tau\mathcal{P}$ possesses arbitrarily small connected neighborhoods. If $\mathcal{P}$ is feebly locally connected and C is a component of $\mathcal{P}$, then C is open in $\mathcal{P}$ (by 21 B.5) and hence C is open in $\tau\mathcal{P}$. Since C is connected in $\mathcal{P}$, C is connected in $\tau\mathcal{P}$ (the mapping $\mathsf{J} : \mathcal{P} \to \tau\mathcal{P}$ is continuous). On the other hand C is closed in $\mathcal{P}$ and thus in τP. Consequently, C is a component of $\tau\mathcal{P}$. Thus each component of $\tau\mathcal{P}$ is open and consequently the space $\tau\mathcal{P}$ is feebly locally connected by 21 B.5.

21 B.10. *The sum of any family of locally connected (feebly locally connected) spaces is a locally connected (feebly locally connected) space.* — Evident.

21 B.11. Theorem. *A non-void topological product $\mathscr{P}$ of a family $\{\mathscr{P}_a \mid a \in A\}$ of closure spaces is locally connected if and only if each coordinate space $\mathscr{P}_a$ is locally connected and there exists a finite subset A_0 of A such that all $\mathscr{P}_a$, $a \in A - A_0$, are connected.*

Proof. First suppose that the conditions are fulfilled, $x \in \mathscr{P}$ and U is a neighborhood of x. Choose a canonical neighborhood V of x so that $V \subset U$. Thus $V = \bigcap\{\pi_a^{-1}[V_a] \mid a \in A_1\}$ where V_a is a neighborhood of $x_a = \mathrm{pr}_a x$ in $\mathscr{P}$ and A_1 is a finite subset of A. Since each $\mathscr{P}_a$, $a \in A$, is a locally connected space, we can choose a family $\{W_a \mid a \in A_0 \cup A_1\}$ such that W_a is a connected neighborhood of x_a for each $a \in A_0 \cup A_1$ and $W_a \subset V_a$ if $a \in A_1$. Clearly $W = \mathbf{E}\{y \mid y \in \mathscr{P},\ a \in A_0 \cup A_1 \Rightarrow \mathrm{pr}_a y \in W_a\}$ is a neighborhood of x in $\mathscr{P}$ which is contained in V and hence in U. On the other hand W is a connected subset of $\mathscr{P}$ as the product of connected sets (20 B.15); indeed, $W = \Pi\{X_a \mid a \in A\}$ where $X_a = W_a$ for a $a \in (A_0 \cup A_1)$ and $X_a = |\mathscr{P}_a|$ otherwise. The sets W_a were chosen connected and each $\mathscr{P}_a$, $a \in A - A_0$ was assumed to be connected.

Conversely, let $\mathscr{P}$ be locally connected, $a \in A$, $x_a \in \mathscr{P}_a$ and U_a a neighborhood of x_a in $\mathscr{P}_a$. Choose a point x of $\mathscr{P}$ so that $\mathrm{pr}_a x = x_a$, and a connected neighborhood V of x contained in the neighborhood $\mathbf{E}\{y \mid y \in \mathscr{P}, \mathrm{pr}_a y \in U_a\}$ of x in $\mathscr{P}$. The set $\mathrm{pr}_a[V]$ is obviously a connected neighborhood of x_a and $\mathrm{pr}_a[V] \subset U$. It follows that each $\mathscr{P}_a$ is locally connected.

To prove that each $\mathscr{P}_a$ except for finite number of a's in A is connected, by 21 B.6 it is sufficient to prove that if an infinite number of $\mathscr{P}_a$ are not connected, then there exists a component C of $\mathscr{P}$ which is not open in $\mathscr{P}$. But this follows from the fact that each component of the product is the product of components of coordinate spaces (20 B.17). Indeed, if there exists an infinite $A_0 \subset A$ such that no $\mathscr{P}_a$, $a \in A_0$, is connected, then we can choose a component $C = \Pi\{C_a \mid a \in A\}$ of $\mathscr{P}$ so that $C_a \neq \mathscr{P}_a$ for each $a \in A_0$. It is obvious that C contains no non-void canonical neighborhood.

Corollary a. Each product space $\mathsf{R}^{\aleph}$ is locally connected; in particular R^n, $n \in \mathsf{N}$, is always locally connected. More generally, the topological product of any family of connected ordered spaces is a locally connected space.

Corollary b. The spaces $\mathsf{N}^{\aleph_0}$ and $2^{\aleph_0}$ are not locally connected.

22. DESCRIPTIVE PROPERTIES OF SETS

In subsection A dense sets (i.e. locally non-void) and nowhere dense sets having dense closures of their complements are introduced and their properties studied. The most profound result is Theorem 22 A.10 which asserts that the density character (i.e. the smallest cardinal of a dense set) of a cube $[\![0, 1]\!]^{\exp \aleph}$, $\aleph \geqq \aleph_0$, is at most $\aleph$. Given a space $\langle P, u \rangle$, the relation $\varrho = \{X \to Y \mid X \subset Y \subset P,\ Y \subset uX\}$ entirely determines u; in fact, uX is the union of the fibre of ϱ over X. Thus every topological concept can be described by means of ϱ; e.g. $\langle P, u \rangle$ is topological if and only if ϱ is transitive, X is dense in $\langle P, u \rangle$ if and only if $X \in \varrho^{-1}[P]$.

In a certain sense the nowhere dense sets are very small. Countable unions of nowhere dense sets, called meager sets, often appear as small sets, e.g. the set of all continuous functions f on $[\![0, 1]\!]$ which have a derivative in some point is meager in the space $C^* = \mathbf{C}^*([\![0, 1]\!], \mathsf{R})$ (ex. 8); on the other hand this fact can be used to prove that there exists a continuous function which has a derivative in no point; it is sufficient to show that C^* is not meager. Another example: if f is a lower or an upper semi-continuous function on a space $\mathscr{P}$, then the set of all points at which f is not continuous is meager. Meager sets and non-meager sets (the sets which are not meager) are studied in subsection B. The importance of non-meager spaces is seen from Theorems 22 B.3, 22 B.4 and 22 B.6 which state that the non-meager spaces $\mathscr{P}$ are characterized among all topological spaces by either of the following two conditions:

(a) if a collection $\mathscr{F}$ of lower semi-continuous functions on $\mathscr{P}$ is bounded from above at each point, then $\mathscr{F}$ is bounded from above on a non-void open subset of $\mathscr{P}$.

(b) each lower (upper) semi-continuous function on $\mathscr{P}$ is continuous at some point.

The main result is Theorem 22 B.11, which states that a feebly locally meager set is meager, and its consequence, the Decomposition Theorem 22 B.12, which gives a description of points at which a given space is locally non-meager.

In the last subsection the so-called Baire sets and Baire measurable mappings will be introduced and studied. A Baire set is the symmetric difference of an open set and a meager set; thus Baire sets are "almost" open sets. It turns out that the collection of all Baire sets of a topological space is closed under countable unions, countable products and complementation, and hence, contains all Borel sets, i.e. the elements of the smallest collection which contains all open sets and is closed under countable unions and complementation. Recall that a mapping into a topological space is continuous if (an only if) the inverse images of open sets are open. A mapping of topological spaces is said to be Baire (Borel) measurable if the inverse image of each open set is a Baire set (a Borel set). Let us mention Theorem 22 C.14 which states that a Baire measurable mapping of a topological space into a topological space with a countable open base is continuous on some subspace with meager complement. Theorem 22 C.17 is the starting point of many applications in function theory.

A great deal of the results of this section apply only to topological spaces. It is to be noted that the results of subsections B and C will not be needed in what follows, except in the exercises.

A. DENSE AND NOWHERE DENSE SETS

22 A.1. Definition. Let $\mathscr{P}$ be a closure space. A subset X of $\mathscr{P}$ is said to be *dense in a subset Y of $\mathscr{P}$* if $X \subset Y$ and $\overline{X} \supset Y$. A subset X of $\mathscr{P}$ is said to be *dense in $\mathscr{P}$* if X is dense in the underlying set of $\mathscr{P}$, that is, if $\overline{X} = |\mathscr{P}|$. The *density character of a space $\mathscr{P}$* is the least cardinal $\aleph$ such that there exists a dense set in $\mathscr{P}$ of cardinal $\aleph$.

Since a subset X of a space $\mathscr{P}$ is locally non-void at a point $x \in \mathscr{P}$ if and only if $x \in \overline{X}$, the definition can be restated as follows.

22 A.2. *A subset X of a space $\mathscr{P}$ is dense in $\mathscr{P}$ if and only if X is locally non-void. A subset X of a space $\mathscr{P}$ is dense in $Y \subset \mathscr{P}$ if and only if $X \subset Y$ and X is locally non-void at each point of Y.*

It is to be noted that a subset X is dense in a subset Y if and only if X is dense in the subspace Y.

22 A.3. *If U is an open subset of a space $\mathscr{P}$, then for any subset X of $\mathscr{P}$ the set $X \cap U$ is dense in $\overline{X} \cap U$. In particular, if X is dense in $\mathscr{P}$ and U is open in $\mathscr{P}$, then $X \cap U$ is dense in U.*

Proof. The first statement is an immediate consequence of the formula $U \cap \overline{X} = U \cap \overline{X \cap U}$ which obtains (by 14 B.20) for each X and each open U.

Remark. The relation $\mathsf{E}\{\langle X, Y\rangle \mid X \text{ is dense in } Y\}$ completely determines the closure structure of the space. Indeed, the closure of X is the union of all sets in which X is dense.

22 A.4. *A closure space $\mathscr{P}$ is topological if and only if the relation $\mathsf{E}\{\langle X, Y\rangle \mid X$ is dense in $Y\}$* for exp $|\mathscr{P}|$ *is transitive, that is, X dense in Y and Y dense in Z imply X is dense in Z.*

Proof. If the relation is transitive, then $\overline{X} = \overline{\overline{X}}$ for each $X \subset \mathscr{P}$ because X is dense in $\overline{X}$, $\overline{X}$ is dense in $\overline{\overline{X}}$ and the transitivity implies that X is dense in $\overline{\overline{X}}$, that is, $\overline{\overline{X}} \subset \overline{X}$. Conversely, if $\mathscr{P}$ is topological and $X \subset Y \subset Z$, $Y \subset \overline{X}$ and $Z \subset \overline{Y}$, then $\overline{X} = \overline{\overline{X}} \supset Z$, which establishes transitivity.

Corollary a. *If $\mathscr{P}$ is a topological space and X, Y are subsets of $\mathscr{P}$ such that X is dense in Y and Y is dense in $\mathscr{P}$, then X is dense in $\mathscr{P}$.* – Obvious.

Corollary b. *If $\mathscr{P}$ is a topological space, X and U are dense subsets of $\mathscr{P}$ and U is open, then $X \cap U$ is a dense subset of $\mathscr{P}$.*

Proof. By 22 A.3 the set $X \cap U$ is dense in U and, U being dense, $X \cap U$ is dense because of transitivity (Corollary (a) of 22 A.4).

Since the intersection of two open subsets is an open subset, the following very simple but useful result follows by induction from Corollary (b).

22 A.5. Theorem. *The intersection of a finite number of open dense subsets of a topological space $\mathscr{P}$ is a dense subset of $\mathscr{P}$.*

22 A.6. *If $\{\mathscr{P}_a\}$ is a family of closure spaces and $\{X_a\}$ is a family such that each X_a is dense in $\mathscr{P}_a$, then the set $\Sigma\{X_a\}$ is dense in the sum space $\Sigma\{\mathscr{P}_a\}$, and the product set $\Pi\{X_a\}$ is dense in the product space $\Pi\{\mathscr{P}_a\}$. If f is a continuous mapping of a space $\mathscr{P}$ onto another one $\mathscr{Q}$ and if X is dense in $\mathscr{P}$, then $f[X]$ is dense in $\mathscr{Q}$.*

Proof. The first statement is a consequence of the fact that the closure of $\Sigma\{X_a\}$ is $\Sigma\{\overline{X}_a\}$ (by 17 B.1) and the closure of $\Pi\{X_a\}$ is $\Pi\{\overline{X}_a\}$ (by 17 C.2). The second statement is obvious.

The assertion concerning products can be strenghtened as follows. (It is to be noted that the fact to be proved was declared to be obvious in the proof of 20 B.15.)

22 A.7. *Let $\mathscr{P}$ be the product of a family $\{\mathscr{P}_a \mid a \in A\}$ and let $\{X_a\}$ be a family such that X_a is dense in $\mathscr{P}$ for each a in A. Let $y = \{y_a\} \in \Pi\{X_a\}$ and let X be the set of all $x = \{x_a\} \in \mathscr{P}$ such that $x_a \in X_a$ for each $a \in A$ and $x_a \neq y_a$ for a finite number of $a \in A$ only. Then X is dense in $\mathscr{P}$.*

Proof. Let $z = \{z_a\}$ be any point of $\mathscr{P}$. It will be shown that $z \in \overline{X}$. It is sufficient to show that every canonical neighborhood $V = \bigcap\{\pi_a^{-1}[V_a] \mid a \in A_1\}$ of z intersects X, where $\pi_a = \mathrm{pr}_a : \mathscr{P} \to \mathscr{P}_a$. Since each X_a is dense in $\mathscr{P}_a$ and each V_a is a neighborhood of z_a, we can choose x'_a in $V_a \cap X_a$ for each a in A_1. Let $x_a = x'_a$ if $a \in A_1$ and $x_a = y_a$ otherwise. Since A_1 is a finite set, x belongs to X. Obviously x belongs to V and hence $x \in V \cap X$, which shows that $V \cap X \neq \emptyset$.

Corollary. *The density character of the product space is less than or equal to the sum of the density characters provided the density characters of the coordinate spaces are infinite.*

Obviously the density character of a space is always less than or equal to the cardinal of the underlying set. If a topological space $\mathscr{P}$ possesses an open base $\mathscr{B}$ of cardinal $\mathfrak{m}$, then the density character of $\mathscr{P}$ is at most $\mathfrak{m}$, because if $\{x_B \mid B \in \mathscr{B}\}$ is any family such that $x_B \in B$ for each B in $\mathscr{B}$, then the set of all x_B, $B \in \mathscr{B}$, is dense in $\mathscr{P}$ by 22 A.2. It follows that the density character of a topological space is less than or equal to the total character. If a space $\mathscr{P}$ is pseudometrizable with an infinite density character, then the total character and the density character coincide. Indeed, if an infinite set X is dense in $\mathscr{P}$ and d is a pseudometric inducing the closure of $\mathscr{P}$, then the collection of all open r-spheres about x, $x \in X$, r being a positive rational number, is obviously an open base of the same cardinal as X.

In general, the density character is less than the total character (see example 22 A.9 (d)). More will be proved in the exercises and in the following chapters. The facts which will be needed in the sequel are listed in the following proposition.

22 A.8. *If $\mathscr{P}$ is a topological space, then the total character is equal to or greater than the density character. The density character of a space is less than or equal to the cardinal of the underlying set. For infinite metrizable spaces the total character and the density character coincide (an infinite metrizable space has an infinite density character).*

22 A.9. Examples. (a) The space R of reals has a countably infinite character. Indeed, the set Q of rational numbers is countable and dense in R. On the other hand, each finite subset of R is a closed proper subset of R, and hence no finite subset is dense in R.

(b) The density character of a subspace of a pseudometrizable space is less than or equal to the density character of the space. Indeed, this is true for total characters which coincide with density characters if the density character of the space is $\geqq \aleph_0$. If the density character is finite, then the statement is evident. In particular, each subspace of R has a countable density character (not necessarily infinite).

(c) Each subspace of R^n or $[\![0, 1]\!]^n$ has a countable density character provided that $n \leqq \aleph_0$. Apply Corollary 22 A.7 and example (b) above.

(d) Let us define a new closure operation u for the set R of all real numbers so that $uX = X$ if X consists of irrational numbers and $uX = \overline{X}$ if X consists of rational numbers. Clearly Q is dense in $\langle \mathsf{R}, u \rangle$; thus $\langle \mathsf{R}, u \rangle$ has a countably infinite density character. On the other hand the subspace $\mathsf{R} - \mathsf{Q}$ is discrete, and consequently the density character of $\mathsf{R} - \mathsf{Q}$ is equal to the cardinal of $\mathsf{R} - \mathsf{Q}$, that is, $\exp \aleph_0$. Next, $\langle \mathsf{R}, u \rangle$ is topological and the total character of $\langle \mathsf{R}, u \rangle$ is $\exp \aleph_0$ (each $(x) \cup \mathsf{Q}$ is open).

The first part, devoted to dense sets, concludes with the following useful and interesting result which will be used in the sequel and which gives a more precise estimate of the density character of a cube.

22 A.10. Theorem. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be infinite cardinals and $\mathfrak{m} \leqq \exp \mathfrak{n}$. Then the density character of the cube $[\![0, 1]\!]^{\mathfrak{m}}$ is at most $\mathfrak{n}$.*

Proof. It is sufficient to show that if $\mathfrak{n}$ is an infinite cardinal, then the cube $[\![0, 1]\!]^{\exp \mathfrak{n}}$ contains a dense subset of cardinal $\mathfrak{n}$. Let N be a set of cardinal $\mathfrak{n}$ and let $\mathscr{P} = [\![0, 1]\!]^{\exp N}$. For brevity call a point x of $\mathscr{P}$ rational if each coordinate x_X of x is a rational number. For each finite subset F of N let Φ_F be the set of all rational points of $\mathscr{P}$ such that $X_1 \in \exp N$, $X_2 \in \exp N$, $X_1 \cap F = X_2 \cap F$ imply $x_{X_1} = x_{X_2}$. It follows that each member of Φ_F is completely determined by its coordinates x_X,

$X \subset F$. But $\exp F$ is finite and the x_X are elements of a countable set $Q \cap [\![0, 1]\!]$. It follows that each Φ_F, $F \subset N$, F finite, is countable. The union Φ of all Φ_F, $F \subset N$, F finite, has the cardinal $\aleph_0 \cdot \mathfrak{n}$, i.e. $\mathfrak{n}$ because $\mathfrak{n}$ is infinite. It will be shown that Φ is dense in $\mathscr{P}$. It will suffice to show that each canonical open subset of $\mathscr{P}$ intersects Φ. Actually, $\mathscr{P}$ is a topological space and hence the canonical open subsets of $\mathscr{P}$ form an open base for $\mathscr{P}$. Let V be any non-void canonical open set, that is, there exist distinct elements $X_1, \ldots, X_k$ of $\exp N$, and V is the set of all $x \in \mathscr{P}$ such that $x_{X_i} \in U_i$ for each i where U_i are appropriate open subsets of $[\![0, 1]\!]$. For each $i, j = 1, \ldots, k$, $i \neq j$, choose $n_{ij} \in (X_i - X_j)$ if $(X_i - X_j) \neq \emptyset$, and consider the set F of all these n_{ij}. If $i \neq j$, then $X_i \neq X_j$ by our assumption, and clearly $F \cap X_i \neq F \cap X_j$ by construction. Choose rational numbers r_i in U_i, $i = 1, \ldots, k$; this is possible because the set of all rationals of $[\![0, 1]\!]$ is obviously dense in $[\![0, 1]\!]$ and $U_i \neq \emptyset$. Now define an $x \in \Phi$ as follows: $x_X = r_i$ if $X \cap F = X_i \cap F$ and $x_X = 0$ if $X \cap F \neq X_i \cap F$ for each $i = 1, \ldots, k$. The point x is well defined because $F \cap X_i \neq F \cap X_j$ for $i \neq j$. Finally, by definition of x, x belongs to $\Phi_F \subset \Phi$ and also to V. It follows that $V \cap \Phi \neq$ $\neq \emptyset$ which completes the proof.

Remark. The proof of the foregoing theorem can be used to obtain more general results. See ex. 1.

22 A.11. Definition. A subset X of a closure space $\mathscr{P}$ is said to be *nowhere dense* if the set $|\mathscr{P}| - \overline{X}$ is dense, that is, if $\operatorname{int} \overline{X} = \emptyset$. A subset X of $\mathscr{P}$ is said to be *nowhere dense in a subset* Y of $\mathscr{P}$ if X is nowhere dense in the subspace Y of $\mathscr{P}$.

It is easy to verify that X is nowhere dense in Y if and only if $X \subset Y \subset \overline{Y - \overline{X}}$.

22 A.12. *A subset X of a space $\mathscr{P}$ is nowhere dense if and only if the closure of X is contained in its boundary, i.e.,* $\overline{X} \subset \operatorname{bd} \overline{X}$.

Proof. From the formula $\operatorname{bd} \overline{X} = \overline{\overline{X}} \cap \overline{|\mathscr{P}| - \overline{X}}$ it follows that $\operatorname{bd} \overline{X} \supset \overline{X}$ if and only if $\overline{|\mathscr{P}| - \overline{X}} \supset \overline{X}$. Since obviously $\overline{|\mathscr{P}| - \overline{X}} \supset |\mathscr{P}| - \overline{X}$, we have $\overline{|\mathscr{P}| - \overline{X}} = |\mathscr{P}|$ if and only if $\overline{|\mathscr{P}| - \overline{X}} \supset \overline{X}$, which means that $\operatorname{bd} \overline{X} \supset \overline{X}$ if and only if $\overline{|\mathscr{P}| - \overline{X}} =$ $= |\mathscr{P}|$; this is precisely the statement.

Corollary. *A closed subset X of a space is nowhere dense if and only if* $X =$ $= \operatorname{bd} X$. (Remember that always $\operatorname{bd} X \subset \overline{X}$.)

22 A.13. *If X is nowhere dense in a space $\mathscr{P}$ and U is open in $\mathscr{P}$, then $X \cap U$ is nowhere dense in U.*

Proof. According to 22 A.3, if $|\mathscr{P}| - \overline{X}$ is dense and U is open, then the set $(|\mathscr{P}| - \overline{X}) \cap U = U - (U \cap \overline{X})$ is dense in U, and consequently, the larger set $U -$ $- (U \cap \overline{X \cap U})$ is also dense in U, which means that $X \cap U$ is nowhere dense in U.

In the following propositions 22 A.14–22 A.18 the assumption that the space is topological cannot be omitted; the corresponding examples are given in 22 A.19.

22 A.14. Theorem. *Let $\mathscr{P}$ be a topological space. Each of the following conditions is necessary and sufficient for a subset X of $\mathscr{P}$ to be nowhere dense*:

(a) *The closure of X contains no non-void open subset;*

(b) *The closure of X is nowhere dense;*

(c) *Every non-void open subset U of $\mathscr{P}$ contains a non-void open subset V of $\mathscr{P}$ such that $V \cap X = \emptyset$.*

Proof. Since in any topological space the interior of a set X is the largest open set contained in X, (a) is equivalent to the assertion $\operatorname{int} \overline{X} = \emptyset$. Thus condition (a) is both necessary and sufficient. Since $\mathscr{P}$ is a T-space, we have $\overline{\overline{X}} = \overline{X}$ and hence $\operatorname{int} \overline{\overline{X}} = \operatorname{int} \overline{X}$ which shows that (b) is both necessary and sufficient. Condition (b) is sufficient in any space, because the property "to be nowhere dense" is hereditary. If X is not nowhere dense, then $\operatorname{int} X$ is a non-void open subset of $\mathscr{P}$ every non-void open subset of which meets X. Thus (c) is a sufficient condition. If X is nowhere dense and U is an open non-void subset of $\mathscr{P}$, then $V = U \cap (|\mathscr{P}| - \overline{X})$ is dense in U since $|\mathscr{P}| - \overline{X}$ is dense. In particular, $V \neq \emptyset$. Clearly $V \cap X = \emptyset$. Thus (c) is a necessary condition.

22 A.15. *Let $\mathscr{P}$ be a topological space and $X \subset Y \subset |\mathscr{P}|$. If X is nowhere dense in Y, then X is nowhere dense in $\mathscr{P}$.*

Proof. Assuming that X is not nowhere dense in $\mathscr{P}$, i.e. $U = \operatorname{int} \overline{X} \neq \emptyset$, we shall prove that X is not nowhere dense in Y. Clearly the set $U \cap Y$ is open in Y and is contained in the closure of X in Y, i.e. in $\overline{X} \cap Y$. Thus it is sufficient to show that $U \cap Y$ is non-void. The set $U \cap \overline{Y}$ is non-void because $U \cap \overline{Y} \supset U \cap \overline{X} = U$, and the set $U \cap Y$ is dense in $U \cap \overline{Y}$ by 22 A.3 because U is open; consequently, $U \cap Y$ is non-void.

22 A.16. *Let $\mathscr{P}$ be a topological space and $X \subset \mathscr{P}$. Suppose that for each x in X there exists a neighborhood U of x such that $U \cap X$ is nowhere dense. Then the set X is nowhere dense. In other words, if $X \subset \mathscr{P}$ is relatively feebly locally nowhere dense, then X is nowhere dense.*

Proof. Suppose X is not nowhere dense. Hence $V = \operatorname{int} \overline{X} \neq \emptyset$. Choose a point x in $V \cap X$ and a neighborhood U of x such that $U \cap X$ is nowhere dense. Since $\mathscr{P}$ is a topological space, we can choose an open neighborhood W of x with $W \subset (U \cap V)$. Since W is open, we have (by 14 B.20) $W \cap \overline{W \cap X} = (W \cap \overline{X}) \supset W \cap V = W$, that is, $W \subset \overline{W \cap X} \subset \overline{U \cap X}$; this contradicts the fact that $U \cap X$ is nowhere dense.

22 A.17. Theorem. *Let $\mathscr{P}$ be a topological space. The union X of a locally finite family $\{X_a \mid a \in A\}$ of nowhere dense sets is a nowhere dense set.*

Proof. First suppose that A is finite. Since the intersection of a finite number of open dense sets is a dense set, we have that

$$|\mathscr{P}| - \overline{\bigcup\{X_a\}} = |\mathscr{P}| - \bigcup\{\overline{X}_a\} = \bigcap\{|\mathscr{P}| - \overline{X}_a\}$$

is dense, and consequently $\bigcup\{X_a\}$ is nowhere dense. The general case follows immediately from 22 A.16. Indeed, the union of a locally finite family of nowhere dense sets $\{X_a\}$ is feebly locally, and hence relatively feebly locally nowhere dense by the first part of the proof.

22 A.18 *Let $\mathscr{P}$ be a topological space. If a subset X of $\mathscr{P}$ is either open or closed, then the boundary of X is a nowhere dense set.*

22 A.19 Example. Let P be a set such that, for some element x of P, $P - (x) = (\mathsf{N} \times \mathsf{N}) \cup \mathsf{N}$; we shall assume that $\mathsf{N} \cap (\mathsf{N} \times \mathsf{N}) = \emptyset$. Let us define a closure operation u for P such that $\mathsf{N} \times \mathsf{N}$ is an open discrete subspace, N is a discrete subspace, (x) is closed, $x \in uX - X$ if and only if $X \cap \mathsf{N}$ is infinite, and $n \in uX - X$ for $n \in \mathsf{N}$ if and only if $X \cap ((n) \times \mathsf{N})$ is infinite. It can be proved that

(a) The subspace $(\mathsf{N} \times \mathsf{N}) \cup \mathsf{N}$ of $\langle P, u\rangle$ is topological, $(x) \cup \mathsf{N}$ is a neighborhood of x which contains no non-void open set. Thus $\langle P, u\rangle$ is not topological.

(b) The set $\mathsf{N} \times \mathsf{N}$ is not dense but every non-void open set intersects $\mathsf{N} \times \mathsf{N}$.

(c) $\mathsf{N} \times \mathsf{N}$ is dense in $P - (x) = (\mathsf{N} \times \mathsf{N}) \cup \mathsf{N}$, $P - (x)$ is open dense, $\mathsf{N} \times \mathsf{N}$ is open, but $(\mathsf{N} \times \mathsf{N}) \cap (P - (x)) = \mathsf{N} \times \mathsf{N}$ is not dense.

(d) Let $\mathsf{N} = N_1 \cup N_2$, $N_1 \cap N_2 = \emptyset$, N_i infinite. Then $G_i = (\mathsf{N} \times \mathsf{N}) \cup N_i$ are dense open subsets, but $G_1 \cap G_2 = \mathsf{N} \times \mathsf{N}$ is not dense. Thus the intersection of two open dense sets need not be dense. The sets $P - G_i$ are closed nowhere dense sets and their union is not nowhere dense.

(e) The set N is nowhere dense in $P - (x)$ but not in P (int $u\mathsf{N} = (x) \neq \emptyset$).

(f) The set N is not nowhere dense in P but $u\mathsf{N}$ contains no non-void open set, and every non-void open set contains a non-void open set disjoint with N.

(g) The set N is the boundary of the open set $\mathsf{N} \times \mathsf{N}$; on the other hand N is not nowhere dense.

B. MEAGER AND NON-MEAGER SETS

22 B.1. Definition. A subset X of a space $\mathscr{P}$ is said to be *meager* (or of the *first category*) in a subset Y of $\mathscr{P}$, if X is the union of a countable number of nowhere dense subsets in Y. A subset X is *non-meager* (or of the *second category*) in a subset Y, if $X \subset Y$ and X is not meager in Y. A subset X will be called *residual* in a subset Y, if $X \subset Y$ and $Y - X$ is meager in Y. A subset X is *meager*, *non-meager* or *residual* in the space $\mathscr{P}$ if X has the corresponding property in the underlying set of $\mathscr{P}$. A space $\mathscr{P}$ is *meager* or *non-meager* if the underlying set has the corresponding property.

For example, every non-void discrete or non-void accrete space is non-meager because such a space contains no non-void nowhere dense set. Next, every space possessing only a finite number of accumulation points is non-meager because it contains only a finite number of nowhere dense sets. On the other hand, the space Q of all rational numbers is meager, because each one-point set (x), $x \in \mathsf{Q}$, is nowhere dense in Q ($\overline{(x)} = (x)$ and $x \in \overline{\mathsf{Q} - (x)}$) and Q is countable. As a less trivial example we shall prove that the space R of reals in non-meager. A similar argument yields the Baire theorem asserting that every complete pseudometric space is non-meager. Complete pseudometric spaces will not be considered until Chapter 7 and even there in a more general situation; in ex. 6 the notion of a complete pseudometric will be introduced, and a proof of the Baire theorem will be given.

22 B.2. *Every non-void boundedly order-complete ordered space is non-meager. In particular, the space* R *of reals is non-meager.*

Proof. Let the closure of a space $\mathscr{P}$ be induced by a boundedly order-complete monotone order $\leqq$. Suppose the contrary, that $\mathscr{P}$ is meager. Let $\{X_n\}$ be a sequence of nowhere dense subsets of $\mathscr{P}$ which have union $|\mathscr{P}|$. Since the space $\mathscr{P}$ is topological. according to 22 A.14 we can construct, by induction, a decreasing sequence $\{I_n\}$ of bounded open intervals such that $\bar{I}_n \subset P - X_n$ for each n. Indeed, if a finite sequence $\{I_n \mid n \leqq k\}$ possesses the required properties then there exists a non-void open interval $J =]\!] x, y [\![$ in $\mathscr{P}$ such that $J \subset I_k \cap (P - X_{k+1})$ by 22 A.14 (c). Of course, the end points x and y may belong to X_{k+1}. It is easy to find a non-void open interval $I_{k+1} \subset J$ such that $(\bar{I}_{k+1} \cap X_{k+1}) = \emptyset$. Now if $\{I_n\}$ is such an infinite sequence, then $\bigcap\{\bar{I}_n\} \neq \emptyset$ because, for example, the least upper bound of left end points of the intervals I_k belongs to this intersection. On the other hand, obviously

$$\bigcap\{\bar{I}_n\} \subset \bigcap\{|\mathscr{P}| - X_n\} = |\mathscr{P}| - \bigcup\{X_n\} = \emptyset .$$

This contradiction establishes the result.

Clearly a space $\mathscr{P}$ is non-meager if and only if each residual subset of $\mathscr{P}$ is non-void. The following theorem shows that a "point-bounded" from above collection of lower semi-continuous functions on a non-meager space is bounded from above on a non-void open set. Various applications can be found in the concluding part of the section and in the exercises.

22 B.3. Theorem. *Let $\mathscr{P}$ be a non-meager topological space. Let Φ be a collection of lower semi-continuous functions on $\mathscr{P}$ such that*

(1) $x \in P,\ f \in \Phi$ *imply* $fx \leqq gx$,

where g is a given function on $\mathscr{P}$. Then there exists an open non-void subset U of $\mathscr{P}$ and a real number M such that

(2) $x \in U,\ f \in \Phi$ *implies* $fx \leqq M$.

Proof. For each n in N let us consider the set

(3) $C_n = \mathbf{E}\{x \mid x \in \mathscr{P}, f \in \Phi \Rightarrow fx \leqq n\}$.

According to (1), the union of all C_n, $n \in$ N, is $|\mathscr{P}|$. Moreover, the sets C_n are closed. because each C_n is the intersection of all sets $C_n(f) = \mathbf{E}\{x \mid fx \leqq n\}$, $f \in \Phi$, and each $C_n(f)$ is closed, because f is lower semi-continuous. Since P is not meager, at least one of the sets C_n, $n \in$ N, say C_k, is not nowhere dense. Thus $U = \operatorname{int} C_k$ is a non-void open set. According to (3), assertion (2) holds with $M = k$.

In the converse direction we shall prove the following result.

22 B.4. *If $\mathscr{P}$ is a meager topological space, then there exists a lower semi-continuous function f on $\mathscr{P}$ which is not bounded from above on any non-void open subset of $\mathscr{P}$, i.e., if U is a non-void open subset of $\mathscr{P}$, then the set $f[U]$ has no upper bound.*

Proof. If $\mathscr{P}$ is meager, then there exists a sequence X_n of nowhere dense subsets of $\mathscr{P}$ such that $\bigcup\{X_n\} = |\mathscr{P}|$. Since $\mathscr{P}$ is a topological space, the sets $\overline{X}_n$ and $F_n = \bigcup\{\overline{X}_k \mid k \leqq n\}$ are also nowhere dense (by 22 A.14 and 22 A.17). Clearly $\bigcup\{F_n\} = |\mathscr{P}|$.

Let us consider the function f on $\mathscr{P}$ which assigns to each $x \in |\mathscr{P}|$ the least integer n with $x \in F_n$, i.e. $fx = \inf\{n \mid x \in F_n\}$. The function f is lower semi-continuous, because for every real c we have $f^{-1}[\rrbracket \leftarrow, c \rrbracket] = F_n$ where n fulfils the inequalities $n \leqq c < n + 1$, and F_n is closed. If U is a non-void open subset of $\mathscr{P}$ and $n \in \mathrm{N}$, then $F_n \cap U$ is nowhere dense in U (by 22 A.13), in particular $U - F_n \neq \emptyset$ which shows that n is not an upper bound of $f[U]$. Thus no natural number is an upper bound of $f[U]$ and consequently $f[U]$ has no upper bound. The proof is complete.

Observe also that the function f of 22 B.4 is continuous at no point. If $\mathscr{P}$ is non-meager then each semi-continuous function is continuous at one point at least. This follows from the following simple result.

22 B.5. *Let f be a semi-continuous function on a topological space $\mathscr{P}$. The set $D = \mathbf{E}\{x \mid f \text{ is not continuous at } x\}$ is meager in $\mathscr{P}$.*

Proof. Suppose that f is lower semi-continuous and $U_r = \mathbf{E}\{x \mid fx > r\}$. It is easy to show that $D \subset \bigcup\{\overline{U}_r - U_r \mid r \in \mathrm{Q}\}$. The sets U_r are open and hence the sets $\overline{U}_r - U_r$ are nowhere dense (22 A.18). If f is upper semi-continuous then $-f$ is lower semi-continuous, and f is continuous at a point x if and only if $-f$ is continuous at x. Thus the statement for upper semi-continuous functions follows from that for lower semi-continuous functions.

According to the remark following 22 B.4 we obtain the following characterization of non-meager spaces.

22 B.6. Theorem. *A topological space $\mathscr{P}$ is non-meager if and only if the following condition is fulfilled: each semi-continuous function on $\mathscr{P}$ is continuous at some point.*

Now we proceed to the investigation of meager and non-meager sets. First we shall prove some almost immediate consequences of the preceding results concerning nowhere dense sets.

22 B.7. *Let $\mathscr{P}$ be a topological space. The union of a countable number of meager subsets of Y is a meager set in Y. If $X \subset Y \subset Z$ and Y is a meager subset in Z, then X is also meager in Z. If X is meager in the space $\mathscr{P}$, then $X \cap U$ is meager in U for every open subset U of $\mathscr{P}$.*

Proof. The first assertion is an immediate consequence of the fact that a countable union of countable unions is a countable union. The property "to be meager in Z" is hereditary, since the property "to be nowhere dense" is hereditary. Finally, the last assertion is a corollary of the corresponding result 22 A.13 about nowhere dense sets.

22 B.8. *Let $\mathcal{P}$ be a topological space, $X \subset Y \subset |\mathcal{P}|$. If X is meager in Y, then X is meager in $\mathcal{P}$.* (From 22 A.15).

22 B.9. Definition. A family $\{X_a \mid a \in A\}$ of subsets of a space $\mathcal{P}$ is said to be *σ-locally finite* if there exists a sequence $\{A_n\}$ so that $A = \bigcup\{A_n \mid n \in \mathsf{N}\}$ and all families $\{X_a \mid a \in A_n\}$, $n \in \mathsf{N}$, are locally finite.

Now we can prove, for topological spaces, the following generalization of the first assertion of 22 B.7.

22 B.10. Theorem. *Let $\mathcal{P}$ be a topological space. The union X of a σ-locally finite family $\{X_a \mid a \in A\}$ of meager subsets of $\mathcal{P}$ is a meager subset of $\mathcal{P}$.*

Proof. By definition there exists a sequence $\{A_n\}$ such that $\bigcup\{A_n\} = A$ and every family $\{X_a \mid a \in A_n\}$, $n \in \mathsf{N}$, is locally finite. For each a in A there exists a sequence $\{X(a, k) \mid k \in \mathsf{N}\}$ of nowhere dense sets such that $\bigcup\{X(a, k) \mid k \in \mathsf{N}\} = X_a$. Clearly $\{X(a, k) \mid a \in A_n\}$ is a locally finite family of uowhere dense sets. According to 22 A.17 the sets $Y(n, k) = \bigcup\{X(a, k) \mid a \in A_n\}$ are nowhere dense. Obviously $X = \bigcup\{Y(n, k) \mid n \in \mathsf{N}, k \in \mathsf{N}\}$. Since the set $\mathsf{N} \times \mathsf{N}$ is countable, X is a meager set.

Now we proceed to a more difficult part of the section. The following two theorems are fundamental for further results.

22 B.11. Theorem. *Let X be a subset of a topological space $\mathcal{P}$. If every point of X possesses a neighborhood U such that $U \cap X$ is meager in $\mathcal{P}$, then X is meager in $\mathcal{P}$. In other words, every relatively feebly locally meager set is meager.*

Proof. Let $\mathcal{U}$ be the collection of all open subsets U of $\mathcal{P}$ such that $U \cap X$ is meager; by the "maximality principle" there exists a maximal disjoint subcollection $\mathcal{V}$ of $\mathcal{U}$. Let V be the union of $\mathcal{V}$. First we shall prove that $X \subset \overline{V}$. If $x \in X$ then there exists a neighborhood U of x such that $U \cap X$ is meager. If moreover $x \notin \overline{V}$, then $U_1 = (|\mathcal{P}| - \overline{V}) \cap \operatorname{int} U$ is a non-void open set (for $\mathcal{P}$ is a topological space) and $U_1 \cap X$ is meager since the property "to be meager" is hereditary. But this contradicts the maximality of $\mathcal{V}$ and proves $X \subset \overline{V}$. Since $X \subset \overline{V}$ we have $X \subset ((X \cap V) \cup (\overline{V} - V))$. The set $\overline{V} - V$ is nowhere dense (by 22 A.18). As a consequence, to prove X is meager, it is sufficient to show $X \cap V$ is meager. The family $\{W \mid W \in \mathcal{V}\}$ is locally finite in the subspace V of $\mathcal{P}$ (for W is a neighborhood of each of its points meeting only one member of $\{W\}$). Since the sets $W \cap X$ are meager in $\mathcal{P}$ and V is open, they are meager in V as well (22 B.7). According to 22 B.10 $V \cap X = \bigcup\{W \cup X \mid W \in \mathcal{V}\}$ is meager in V and by 22 B.8 in $\mathcal{P}$.

22 B.12. Decomposition theorem. *Let X be a subset of a topological space $\mathcal{P}$. Let Y_1 be the set of all points $x \in \mathcal{P}$ in which X is feebly locally meager, and let Y_2 be the set of all points $x \in \mathcal{P}$, in which the set X is locally non-meager. Put $X_i = X \cap Y_i$, $(i = 1, 2)$. Then $Y_1 \cap Y_2 = \emptyset$, $Y_1 \cup Y_2 = |\mathcal{P}|$, X_1 is meager, X_2 and Y_2 are relatively locally non-meager and $Y_2 = \overline{X_2 \cap \operatorname{int} Y_2}$.*

Proof. Obviously, $Y_2 = |\mathcal{P}| - Y_1$ and Y_1 is the union of the collection $\mathcal{U}$ of all open subsets U of $\mathcal{P}$ for which the set $U \cap X$ is meager. In particular, Y_1 is open,

Y_2 is closed. Clearly, the set $X_1 = Y_1 \cap X$ is relatively feebly locally meager. According to the preceding Theorem 22 B.11 the set X_1 is meager. The remaining two statements follow from the following assertion:

(*) if V is open and $V \cap Y_2 \neq \emptyset$, then $V \cap X_2$ is non-meager, in particular, $V \cap X_2 \neq \emptyset$ and $V \cap \operatorname{int} Y_2 \neq \emptyset$.

Indeed, the first statement implies that X_2 is locally non-meager at each point of Y_2, and hence both sets X_2 and Y_2 are relatively locally non-meager (remember that $X_2 \subset Y_2$). The formula $Y_2 = \overline{X_2 \cap \operatorname{int} Y_2}$ is derived as follows. First notice that (*) implies that $\overline{Y}_2 \subset \overline{X}_2$ and $\overline{Y}_2 \subset \overline{\operatorname{int} Y_2}$. Since Y_2 is closed and $X_2 \subset Y_2$ we obtain that $Y_2 = \overline{X}_2$ and $Y_2 = \overline{\operatorname{int} Y_2}$. Thus both sets X_2 and $\operatorname{int} Y_2$ are dense in Y_2, and hence this intersection is dense in Y_2 (by 22 A.3 because $\operatorname{int} Y_2$ is open), i.e. $Y_2 = \overline{X_2 \cap \operatorname{int} Y_2}$.

It remains to prove statement (*). Suppose that V is open and $V \cap Y_2 \neq \emptyset$. Since V is a neighborhood of a point of Y_2, the set $V \cap X$ is non-meager, and hence the set $V \cap X_2$ is non-meager because $V \cap X_2 = (V \cap X) - X_1$ and X_1 is meager. The first statement of (*) is proved. Since Y_2 contains X_2, the first statement implies that the set $V \cap Y_2$ is non-meager. It follows that $V \cap Y_2$ is non-meager in V, in particular, $V \cap Y_2$ is not nowhere dense in Y_2. Next, Y_2 is closed, hence $V \cap Y_2$ is closed in V, and hence the interior V_1 of $V \cap Y_2$ in V is non-void. The set V_1 is open in an open set, namely in V, and hence V_1 is open in $\mathcal{P}$. Thus $\emptyset \neq V_1 \subset \operatorname{int}(V \cap Y_2) \subset V \cap \cap \operatorname{int} Y_2$. The proof is complete.

For convenience, the following corollary of the foregoing Decomposition theorem will be formulated as a theorem. Let us recall that a regular closed set in a space $\mathcal{P}$ is a closed set X of the form $X = \overline{\operatorname{int} X}$.

22 B.13. Theorem. *If $\mathcal{P}$ is a topological space and Q is the set of all points in which the space $\mathcal{P}$ is locally non-meager, then Q is a regular closed subset of $\mathcal{P}$, Q is relatively locally non-meager in $\mathcal{P}$ and the subspace Q of $\mathcal{P}$ is locally non-meager. In particular, if a topological space $\mathcal{P}$ is non-meager, then some subspace (some open subspace, some closed subspace, some regular closed subspace) of $\mathcal{P}$ is locally non-meager.*

Remark. We know that a subspace Q of a space $\mathcal{P}$ is locally connected if and only if the set Q is relatively locally connected in $\mathcal{P}$. This follows from the fact that $X \subset Q$ is connected in Q if and only if X is connected in $\mathcal{P}$. It may be in place to point out that analogous equivalences for meager and non-meager sets and spaces are not true. Of course, a subspace Q of a topological space $\mathcal{P}$ is always non-meager or locally non-meager provided that the underlying set of Q is non-meager or relatively locally non-meager in $\mathcal{P}$. But if a subspace Q of a space $\mathcal{P}$ is non-meager or locally non-meager, then Q may be nowhere dense in $\mathcal{P}$; e.g. a one-point space is non-meager and locally non-meager but a subset (x) of a space $\mathcal{P}$ is nowhere dense in $\mathcal{P}$ provided that (x) is closed and $x \in \overline{|\mathcal{P}| - (x)}$.

22 B.14. Locally non-meager spaces are sometimes called *Baire spaces* (e. g. by N. Bourbaki). Properties of the class of all locally non-meager topological spaces will be discussed in the exercises.

C. BAIRE SETS

The definition of Baire sets and various characterizations of Baire sets are followed by examination of the collection of all Baire sets of a topological space. In the second part we shall be concerned with Baire measurable mappings. The concluding part is concerned with some properties of Baire sets in a topological group. Many concepts related to our object are introduced and studied in the exercises. Recall that the *symmetric difference* of two sets X and Y, denoted by $X \div Y$, is defined to be the set $(X - Y) \cup (Y - X)$, which can also be written as $(X \cup Y) - (X \cap Y)$.

22 C.1. Definition. We shall say that a subset X of a closure space $\mathscr{P}$ has the *property of Baire*, or simply, is a *Baire set*, if the following equivalent conditions are fulfilled:

(a) X is the symmetric difference of an open set and a meager set, i.e. $X = U \div Y$ for some open set U and some meager set Y.

(b) $X = (U - Y_1) \cup Y_2$ for some open set U and some meager sets Y_1 and Y_2.

(c) The set $X \div U$ is meager for some open set U.

It is to be noted that Baire sets are often called *almost open* (and the term Baire set is often used for the elements of the smallest σ-algebra containing exact open sets, see 28 ex. 2).

For example, each meager set X is a Baire set because $\emptyset \div X = X$ is meager, and each open set U is a Baire set because $U \div U = \emptyset$ is meager.

We must show that conditions (a), (b) and (c) of 22 C.1 are actually equivalent, and this follows from the following elementary lemma.

22 C.2. Lemma. *Let $\mathscr{O}$ be a collection of sets and let $\mathscr{M}$ be an additive and hereditary collection of sets. The following conditions on a set X are equivalent*:

(a) $X = O \div Y$ *for some* O *in* $\mathscr{O}$ *and some* Y *in* $\mathscr{M}$.

(b) $X = (O - Y_1) \cup Y_2$ *for some* $O \in \mathscr{O}$ *and some* $Y_i \in \mathscr{M}$.

(c) $X \div O \in \mathscr{M}$ *for some* O *in* $\mathscr{O}$.

Proof. If $X = O \div Y$, $O \in \mathscr{O}$, $Y \in \mathscr{M}$, then $X = (O - Y) \cup (Y - O)$ is a decomposition satisfying (b), and hence (a) implies (b). If $X = (O - Y_1) \cup Y_2$ with $O \in \mathscr{O}$, $Y_i \in \mathscr{M}$, then $Y = (Y_1 - Y_2) \cup (Y_2 - O) \in \mathscr{M}$ and $X = O \div Y$, which shows that (b) implies (a); also $O \div X = (O - X) \cup (X - O) \subset Y_1 \cup Y_2$ and hence $O \div X \in \mathscr{M}$ which proves (b) implies (c). Finally, if $X \div O \in \mathscr{M}$ then $(X - O) \in \mathscr{M}$, $(O - X) \in \mathscr{M}$ and hence $X = (O - (O - X)) \cup (X - O)$ is a decomposition of X satisfying (b).

In a topological space, conditions (a)–(c) of 22 C.1 are equivalent to conditions obtained by replacing "open" by "closed".

22 C.3. *Each of the following conditions is necessary and sufficient for a subset X of a topological space $\mathscr{P}$ to be a Baire set:*

(a) *X is the symmetric difference of a closed set and a meager set.*

(b) *$X = (C - Y_1) \cup Y_2$ for some closed set C and some meager sets Y_1 and Y_2.*

(c) *$C \div X$ is meager for some closed set C.*

Proof. By Lemma 22 C.2 conditions (a)–(c) are equivalent in any space. Now let $\mathscr{P}$ be a topological space. Remember that the boundary of a closed or open set is nowhere dense. Now, if $X = (U - Y_1) \cup Y_2$ with U open and Y_i meager, then $X = (\overline{U} - (Y_1 \cup \operatorname{bd} U)) \cup Y_2$ with $\overline{U}$ closed and $Y_1 \cup \operatorname{bd} U$, Y_2 meager (remember that $\operatorname{bd} U = \overline{U} - U$). If $X = (C - Y_1) \cup Y_2$ with C closed and Y_i meager, then $X = (\operatorname{int} C - Y_1) \cup (Y_2 \cup (\operatorname{bd} C - Y_1)$ where $\operatorname{int} C$ is open and Y_1, $Y_2 \cup (\operatorname{bd} C - Y_1)$ are meager. The proof is complete.

It follows from 22 C.3 that each closed set in a topological space is a Baire set. Now we proceed to some less trivial characterizations of Baire sets. For brevity we shall introduce further notation.

22 C.4. Remark. Given a subset X of a space $\mathscr{P}$, the symbol X^* denotes the set of all points of $\mathscr{P}$ at which X is locally non-meager, and X° denotes the interior of X^*.

If $\mathscr{P}$ is a topological space then X^* is a regular closed set and $X - X^*$ is meager (by 22 B.12).

22 C.5. Theorem. *Each of the following conditions (a)–(f) is necessary and sufficient for a subset X of a topological space $\mathscr{P}$ to be a Baire set:*

(a) *The set $X^* \cap (|\mathscr{P}| - X)^*$ is nowhere dense.*

(b) *The set $X^* - X$ is meager.*

(c) *The set $X^* \div X$ is meager.*

(d) *The set $X^\circ \cap (|\mathscr{P}| - X)^\circ$ is empty.*

(e) *The set $X^\circ - X$ is meager.*

(f) *The set $X^\circ \div X$ is meager.*

Proof. I. Since X^* is closed and X° is open, each of the conditions (b), (c), (e) and (f) is sufficient. Next, conditions (a) and (d) are equivalent because X° is open and dense in X^* (remember that the boundary of an open set is nowhere dense). Conditions (b) and (e) are equivalent because $X^* = X^\circ \cup \operatorname{bd} X^\circ$ and $\operatorname{bd} X^\circ$ is nowhere dense and so certainly meager. Similarly one finds that conditions (c) and (f) are equivalent. It remains to show that (a) is necessary, (a) implies (b) and (b) implies (c). The implication (b) $\Rightarrow$ (c) is almost self-evident because $X^* \div X = (X^* - X) \cup (X - X^*)$ and $X - X^*$ is always meager by 22 B.12. – II. Assuming that $X = (U - Y_1) \cup Y_2$ where U is open and Y_i are meager, we shall prove (a). Notice that $P - X = ((P - U) - Y_2) \cup (Y_1 - X)$ (we write $P = |\mathscr{P}|$). It follows that $X^* = U^*$ and $(P - X)^* = (P - U)^*$. For each Z we have $Z^* \subset \overline{Z}$ and therefore $U^* \subset \overline{U}$ and $(P - U)^* \subset \overline{P - U}$. Thus $X^* \cap (P - X)^* \subset \operatorname{bd} U$. Finally, U is open, hence $\operatorname{bd} U$ is nowhere dense and therefore $X^* \cap (P - X)^*$ is nowhere dense. – III. Assuming (a) we shall prove (b). We have

$$X^* - X = X^* \cap (P - X) = (X^* \cap (P - X)^* \cap (P - X)) \cup$$
$$\cup (X^* \cap ((P - X) - (P - X)^*))$$

where $X^* \cap (P - X)^*$ is nowhere dense by (a) and $(P - X) - (P - X)^*$ is meager because $Z - Z^*$ is meager for any Z. Thus $X^* - X$ is meager. The proof is complete.

We now proceed to an examination of the collection of all Baire sets of a topological space.

22 C.6. *The collection of all Baire sets of a topological space $\mathscr{P}$ is closed under complementation, countable unions and countable intersections. Each closed set, each open set and each meager set is a Baire set.*

Proof. Evidently each closed set, each open set and also each meager set is a Baire set. The complement $|\mathscr{P}| - X$ of a Baire set X is a Baire set by the preceding theorem because condition (a) is invariant under complementation, that is, if X fulfils (a) then $|\mathscr{P}| - X$ also fulfils (a). Next let X_n be a sequence of Baire sets. There exists a sequence $\{U_n\}$ of open sets such that $X_n \div U_n$ is meager for each n. Obviously

$$\bigcup\{U_n\} \div \bigcup\{X_n\} \subset \bigcup\{U_n \div X_n\}$$

and therefore $\bigcup\{X_n\}$ is a Baire set. Finally, $\bigcap\{X_n\} = |\mathscr{P}| - \bigcup\{|\mathscr{P}| - X_n\}$ and therefore the intersection of a sequence of Baire sets is a Baire set.

Let us consider the ordered set $\langle \exp P, \subset \rangle$ of subsets of a set P. Clearly $\emptyset$ is the least element, P is the greatest element and, for any family $\{X_a\}$, we have $\sup \{X_a\} = \bigcup\{X_a\}$ and $\inf \{X_a\} = \bigcap\{X_a\}$; consequently, this ordered set is complete. A subset $\mathscr{A}$ is countably meet-stable if and only if it is countably multiplicative, and $\mathscr{A}$ is countably join-stable if and only if $\mathscr{A}$ is countably additive. Thus the collection of all Baire sets of a closure space is countably lattice-stable. We know that the set-theoretical complement of a Baire set is a Baire set. Let us notice that the set-theoretical complement of a set X in P coincides with the complement of X in the ordered set $\langle \exp P, \subset \rangle$; this is an element Y such that $\sup (X, Y)$ is the greatest element and $\inf (X, Y)$ is the least element. In general lattice complementation is not unique; however in our case a complement is unique and coincides with the set-theoretical complement. A subset of an ordered set is said to be complemented if it contains all complements of each of its elements. Thus we can say that the collection of all Baire sets of a topological space $\mathscr{P}$ is complemented in $\langle \exp |\mathscr{P}|, \subset \rangle$. Proposition 22 C.6 can be restated as follows.

22 C.7. *The collection of all Baire sets of a topological space $\mathscr{P}$ is a countably lattice-stable and complemented subset of* $\langle \exp |\mathscr{P}|, \subset \rangle$, *and it contains all open, closed and meager sets.*

If a space $\mathscr{P}$ is meager then each subset is meager and hence each subset is a Baire set. There are non-meager spaces such that each subset is a Baire set, e.g. each non-void discrete space has these properties. In 22 C.21 a subset of R which is not a Baire set will be constructed. By the preceding theorem the collection of all Baire sets of a topological space $\mathscr{P}$ is a complemented countably lattice-stable subcollection of $\langle \exp |\mathscr{P}|, \subset \rangle$ containing all open sets. We shall prove in the exercises that the smallest complemented countably lattice-stable subset of $\langle \exp \mathsf{R}, \subset \rangle$ containing all closed sets is strictly smaller than the collection of all Baire sets of R.

22 C.8. Definition. If $\mathscr{P}$ is a closure space and if $\mathscr{B}$ is the smallest complemented countably lattice-stable subcollection of $\exp |\mathscr{P}|$ containing all open sets, then the elements of $\mathscr{B}$ are called the *Borel sets* of $\mathscr{P}$.

Of course the indicated subcollection exists; in fact, if Y is any subset of a countably complete complemented ordered set $\langle X, \leqq \rangle$ then there exists the smallest complemented countably lattice-stable subset Z of X such that $Y \subset Z$; Z is the intersection of all complemented countably lattice-stable sets containing Y.

22 C.9. *In a topological space each Borel set is a Baire set.* — 22 C.8.

On the other hand a Baire set need not be a Borel set; in the exercises we shall show that there exists a Baire set in R which is not a Borel set. A countable union of closed sets is a Borel set which need not be closed, and a countable intersection of open sets is a Borel set which need not be open. These two kinds of Borel sets are very important and therefore we shall introduce the following terminology.

22 C.10. Definition. A subset X of a space $\mathscr{P}$ is said to be an $\mathbf{F}_\sigma$ (a $\mathbf{G}_\delta$) if X is the union (the intersection) of a countable number of closed (open) sets.

22 C.11. Examples. (a) In a topological space each meager set is contained in a meager $\mathbf{F}_\sigma$. In fact, if $X = \bigcup\{X_n\}$ with X_n nowhere dense, then $\bigcup\{\overline{X}_n\}$ is a meager $\mathbf{F}_\sigma$ containing X.

(b) We know that the intersection of two open dense sets in a topological space is dense. In general the intersection of two dense sets need not be dense, e.g. both Q and R $-$ Q are dense in R but $\emptyset$ is not dense. It is easy to show that the intersection of a countable number of dense $\mathbf{G}_\delta$ in a locally non-meager space is dense (see ex. 5).

22 C.12. Definition. A mapping f of a space $\mathscr{P}$ into a space $\mathscr{Q}$ is said to be *Borel measurable* (*Baire measurable*) if $f^{-1}[U]$ is a Borel set (Baire set) in $\mathscr{P}$ for each open set U in $\mathscr{Q}$.

22 C.13. Theorem. *Each continuous mapping is Borel measurable as well as Baire measurable, and each Borel measurable mapping of a topological space is Baire measurable. A mapping f of a topological space $\mathscr{P}$ into a topological space $\mathscr{Q}$ is Baire measurable provided that there exists a subspace $\mathscr{R}$ of $\mathscr{P}$ such that $|\mathscr{P}| - |\mathscr{R}|$ is meager and the domain-restriction g of f to $\mathscr{R}$ is continuous.*

Proof. The first statements are evident. Let $\mathscr{R}$ be a subspace of $\mathscr{P}$ such that the set $X = |\mathscr{P}| - |\mathscr{R}|$ is meager and the domain-restriction g of f to $\mathscr{R}$ is continuous. Consider an open subset U of $\mathscr{Q}$. The set $g^{-1}[U]$ is open in $\mathscr{R}$ because g is continuous and there exists an open subset V of $\mathscr{P}$ such that $g^{-1}[U] = |\mathscr{R}| \cap V$ (because $\mathscr{P}$ is topological). Now clearly $f^{-1}[U] = g^{-1}[U] \cup (X \cap f^{-1}[U]) = (V - (X \cap V)) \cup \cup (X \cap f^{-1}[V])$ which shows that $f^{-1}[U]$ is a Baire set in $\mathscr{P}$.

The converse of the last statement of 22 C.13 is not true in general. However, the following important result holds.

22 C.14. Theorem. *Let f be a Baire measurable mapping of a topological space $\mathcal{P}$ into a topological space $\mathcal{Q}$ with a countable total character. Then there exists a subspace $\mathcal{R}$ of $\mathcal{P}$ such that the set $|\mathcal{P}| - |\mathcal{R}|$ is meager in $\mathcal{P}$ and the domain-restriction of f to $\mathcal{R}$ is continuous.*

Proof. Let $\mathcal{B}$ be a countable open base for $\mathcal{Q}$. For each B in $\mathcal{B}$ let U_B be an open subset of $\mathcal{P}$ such that the set $X_B = U_B \div f^{-1}[B]$ is meager in $\mathcal{P}$. Consider the subspace $\mathcal{R}$ of $\mathcal{P}$ such that $|\mathcal{P}| - |\mathcal{R}|$ is the union of $\{X_B \mid B \in \mathcal{B}\}$. We shall prove that the domain-restriction g of f to $\mathcal{R}$ is continuous. It is sufficient to show that $g^{-1}[B]$ is open in $\mathcal{R}$ for each B in $\mathcal{B}$, and this follows from the following equality: $g^{-1}[B] = U_B \cap |\mathcal{R}|$.

22 C.15. Remark. Each lower or upper semi-continuous function is a Borel measurable mapping. Therefore 22 C.14 applies to semi-continuous functions on a topological space $\mathcal{R}$.

Notice that we have proved in 22 B.5 somewhat more for semi-continuous functions, namely that $\mathcal{R}$ can be so chosen that f is continuous at each point of $\mathcal{R}$ relative to $\mathcal{P}$; this is essentially more than the continuity of g. A similar result for Borel measurable mappings is not true: e.g. consider the characteristic function f of Q in R; clearly the function f is continuous at no point but f is Baire measurable and R is non-meager.

We proceed to the examination of some properties of Baire sets in topological groups and modules. We shall need an important property of Baire sets in any topological space. If X is a Baire set in a topological space P, then $X^\circ \cap (P - X)^\circ = \emptyset = (X \cap (P - X))^\circ$ and hence $X^\circ \cap Y^\circ = (X \cap Y)^\circ$ for $Y = P - X$. Now we shall prove that the last equality holds for any Baire sets X and Y.

22 C.16. *If X and Y are Baire sets in a topological space, then $(X \cap Y)^\circ = X^\circ \cap Y^\circ$.*

Proof. Evidently $(X \cap Y)^\circ \subset X^\circ \cap Y^\circ$ without any assumption on X, Y. To prove the converse inclusion it will suffice to show that the set $X \cap Y$ is locally non-meager at each point of $X^\circ \cap Y^\circ$. Indeed, this implies that $X^\circ \cap Y^\circ \subset (X \cap Y)^*$, and $X^\circ \cap Y^\circ$ being open, we obtain $X^\circ \cap Y^\circ \subset \operatorname{int}(X \cap Y)^* = (X \cap Y)^\circ$. Let U be any open neighborhood of any point x of $X^\circ \cap Y^\circ$; the set $V = U \cap X^\circ \cap Y^\circ$ is a neighborhood of x, and we shall prove that $V \cap X \cap Y$ is non-meager. We have $V \cap X \cap Y = V - ((V - X) \cup (V - Y))$, where the set V is non-meager because V is a non-void open subset of a relatively locally non-meager set, namely $X^\circ \cap Y^\circ$, and the sets $V - X$ and $V - Y$ are meager because $V - X \subset X^* - X$, $V - Y \subset Y^* - Y$, and the sets $X^* - X$, $Y^* - Y$ are meager since the sets X and Y are Baire sets (see 22 C.5). The proof is complete.

Remark. It follows from 22 C.16 that the intersection of two locally non-meager Baire sets is a locally non-meager set, in particular, a dense set. If fact, if $X^* = P$, $Y^* = P$, then $X^\circ = P$, $Y^\circ = P$, and hence (by 22 C.16) $(X \cap Y)^\circ = P$. It is easily seen that the intersection of any countable collection of locally non-meager Baire sets is a locally non-meager set. It is sufficient to notice that any locally non-meager Baire set of a space P is of the form $P - X$ where X is meager, and in a locally non-meager space any set of this form is a locally non-meager Baire set. Thus the conclusion of 22 C.16 is rather weak for locally non-meager sets. On the other hand, 22 C.16 is not true for countable intersections; e.g. if $X_n = \rrbracket -1/n, 1/n \llbracket$, then $X_n^\circ = X_n$ and hence $\bigcap X_n^\circ = (0)$; however $\bigcap X_n = (0)$ and $(0)^\circ = \emptyset$.

22 C.17. Theorem. *If X is a non-meager Baire set of a topological group $\langle G, ., u\rangle$ then the set $X . X^{-1}$ $(= \mathsf{E}\{x . y^{-1} \mid x \in X, y \in X\})$ is a neighborhood of the unit element.*

The main result needed is 22 C.16. For convenience the proof will be preceded by the following proposition with a self-evident proof.

22 C.18. *If f is a homeomorphism of a space $\mathscr{P}$ onto itself, then for each subset X of $\mathscr{P}$ we have $f[X^*] = (f[X])^*$, $f[X^\circ] = (f[X])^\circ$; in particular, X is meager or relatively locally non-meager if and only if $f[X]$ has the corresponding property.*

Proof of 22 C.17. It is sufficient to show that $X^\circ . (X^\circ)^{-1} \subset X . X^{-1}$. In fact, X° is non-void because X is non-meager and hence $X^\circ . (X^\circ)^{-1}$ contains the unit (if $x \in X^\circ$, then $x . x^{-1} \in X^\circ . (X^\circ)^{-1}$); on the other hand X° is open and so certainly $X^\circ . (X^\circ)^{-1}$ is open. The inclusion $X^\circ . (X^\circ)^{-1} \subset X . X^{-1}$ follows from 22 C.16 and 22 C.18 by a simple calculation. First notice that $x \in Y . Y^{-1}$ if and only if $x . Y \cap Y \neq \emptyset$ ($x \in Y . Y^{-1}$ is equivalent to $x = y_1 . y_2^{-1}$ for some y_1 and y_2 in Y, and $x . Y \cap Y \neq \emptyset$ is equivalent to $x . y_2 = y_1$ for some y_1 and y_2 in Y). Now suppose $x \in X^\circ . (X^\circ)^{-1}$, hence $x . X^\circ \cap X^\circ \neq \emptyset$. By 22 C.18 $x . X^\circ = (x . X)^\circ$ because $\{y \to x . y\}$ is a homeomorphism. Thus $\emptyset \neq x . X^\circ \cap X^\circ = (x . X)^\circ \cap X^\circ$; by 22 C.16 the latter set is $(x . X \cap X)^\circ$, and therefore $x . X \cap X \neq \emptyset$ which means that $x \in X . X^{-1}$.

22 C.19. Corollary. *If X is a subgroup of a topological group $\mathscr{G}$ and if X is a non-meager Baire set of $\mathscr{G}$, then X is open in $\mathscr{G}$, and by* 19 B.11, *X being an open subgroup, X is also closed. It follows that a connected topological group contains no proper non-meager subgroup which is a Baire set.* — Notice that X is an open subgroup if any only if X is a neighborhood of the neutral element and apply 22 C.17.

It is to be noted that theorem 22 C.17 applies to topological linear spaces. Since every topological linear space over R or C is connected, we obtain from 22 C.19:

22 C.20. Corollary. *If X is a subspace of a topological linear space $\mathscr{L}$ over* R *or* C, *and if X is a non-meager Baire set in $\mathscr{L}$, then $X = \mathscr{L}$.*

The following example shows that there exists a subgroup of the additive group R of reals which is not a Baire subset of R.

22 C.21. Example. Let us consider the space R of reals as a linear space over Q and let B be a base of R over Q, that is, each real number is a linear combination $\Sigma r_i b_i$ of elements b_i of B with rational coefficients r_i, and B is a linearly independent collection over Q, that is $\Sigma r_i b_i = 0$ with r_i in Q and different b_i in B implies $r_i = 0$ for each i. Let us choose a sequence $\{b_n\}$ in B such that $b_n \neq b_m$ if $n \neq m$, and for each $n \in$ N let us consider the linear space $\mathscr{L}_n$ which spans $B_1 \cup (b_0) \cup \ldots \cup (b_n)$, where B_1 is the set of all $b \in B$ with $b \neq b_n$ for $n \in$ N. Clearly $\{\mathscr{L}_n\}$ is an increasing sequence of linear subspaces of R (over Q) and $\bigcup\{\mathscr{L}_n\} =$ R. Since R is non-meager (by 22 B.2), some $\mathscr{L}_n$ is non-meager. If this non-meager $\mathscr{L}_n$ were a Baire set, then by 22 C.19 necessarily $\mathscr{L}_n =$ R because R is connected by 20 B.2. But $\mathscr{L}_n \neq$ R by construction.