Examples. (A) The relation $J \times J$ coincides with J restricted to the class of all pairs. — (B) Let σ denote the addition on N. If $\varphi \in \mathsf{N}^{\mathsf{N}}$, then $\sigma \circ (\varphi \times \varphi) = \varphi \circ \sigma$ if and only if $\varphi = \{x \to a \,.\, x\}$ for some $a \in \mathsf{N}$.

5 C.2. Definition. If $\{\varrho_a \mid a \in A\}$ is a family of (comprisable) relations, then the relation consisting of all pairs $\langle\{x_a \mid a \in A\}, \{y_a \mid a \in A\}\rangle$, where $\langle x_a, y_a\rangle \in \varrho_a$ for every $a \in A$, is called the *relational cartesian product* of $\{\varrho_a\}$ and will be denoted by $\Pi_{\text{rel}}\{\varrho_a \mid a \in A\}$ or $\prod_a{}_{\text{rel}}\varrho_a$ etc. As a rule, we write $\prod_a \varrho_a$ instead of $\prod_a{}_{\text{rel}}\varrho_a$, etc., and call $\prod_a \varrho_a$ simply the product of the family $\{\varrho_a\}$ of relations (thus, if ϱ_a are relations, $\Pi\varrho_a$ always means the relational product unless it is clear from the context or stated explicitly that the product in the sense of 5 A.4 is considered).

Remark. Clearly $\mathbf{D}\Pi\varrho_a = \Pi\mathbf{D}\varrho_a$, $\mathbf{E}\Pi\varrho_a = \Pi\mathbf{E}\varrho_a$.

Example. Let $\{X_a \mid a \in A\}$ be a family of sets. For every $a \in A$ let ϱ_a consist of all $\langle X_a, x\rangle$, $x \in X_a$. Then $\Pi\varrho_a$ is the relation consisting of all $\langle\{X_a\}, z\rangle$, $z \in \Pi X_a$.

5 C.3. Definition. If ϱ is a relation, A is a set, then the relation consisting of all $\langle\{x_a\}, \{y_a\}\rangle$ such that $x_a \varrho y_a$ is called the *relational power* of ϱ (with the exponent A) and is denoted by ϱ^A (unless it is clear from the context or stated explicitly that ϱ^A is taken in the sense of 1 E.8).

Remark. If ϱ is comprisable, then, of course, $\varrho^A = \Pi_{\text{rel}}\{a \to \varrho \mid a \in A\}$. Clearly, $\mathbf{D}(\varrho^A) = (\mathbf{D}\varrho)^A$, $\mathbf{E}(\varrho^A) = (\mathbf{E}\varrho)^A$.

Besides the products of relations described above, another kind of a product will be of use later.

5 C.4. Definition. If ϱ, σ are relations, $\mathbf{D}\varrho = \mathbf{D}\sigma$, then the relation consisting of all $\langle x, \langle y, z\rangle\rangle$ such that $\langle x, y\rangle \in \varrho$, $\langle x, z\rangle \in \sigma$ will be called the *reduced relational pair-product* (or simply the *reduced product*) of the relations ϱ, σ. (No special notation is introduced here for this product.)

5 C.5. Definition. If $\{\varrho_a \mid a \in A\}$ is a family of relations such that all $\mathbf{D}\varrho_a$ are equal to a set X, then the relation consisting of all $\langle x, \{y_a\}\rangle$ such that $\langle x, y_a\rangle \in \varrho_a$ will be called the *reduced relational product* (or simply the *reduced product*) of $\{\varrho_a\}$.

Remark. Clearly, the reduced product of $\{\varrho_a \mid a \in A\}$, if it exists, is equal to $(\Pi\varrho_a \circ \varphi)$ where φ is the relation $\{x \to A \times (x)\}$.

Finally, the relational sum of an indexed class of relations may be defined.

5 C.6. Definition. Let $\{\varrho_a \mid a \in A\}$ be an indexed class of (comprisable) relations. The relation consisting of all $\langle\langle a, x\rangle, \langle a, y\rangle\rangle$ such that $a \in A$, $\langle x, y\rangle \in \varrho_a$ will be called the *relational sum* (or simply *sum*) of $\{\varrho_a\}$ and will be denoted by $\Sigma_{\text{rel}}\varrho_a$ or, if it is clear that the relational sum and not the sum in the sense of 5 B.1 is considered, simply by $\Sigma\varrho_a$. Clearly, $\mathbf{D}\,\Sigma\varrho_a = \Sigma\mathbf{D}\varrho_a$, $\mathbf{E}\,\Sigma\varrho_a = \Sigma\mathbf{E}\varrho_a$.

The relation consisting of all $\langle\langle a, x\rangle, y\rangle$ such that $a \in A$, $\langle x, y\rangle \in \varrho_a$ will be called the *reduced relational sum* (or simply the *reduced sum*) of $\{\varrho_a\}$ and will be denoted by $\Sigma_{\mathrm{red}}\varrho_a$ or, if there is no danger of misunderstanding, simply by $\Sigma\varrho_a$. Clearly, $\mathbf{D}\,\Sigma_{\mathrm{red}}\varrho_a = \Sigma\mathbf{D}\varrho_a$, $\mathbf{E}\,\Sigma_{\mathrm{red}}\varrho_a = \bigcup\mathbf{E}\varrho_a$.

Example. If all ϱ_a are equal to ϱ, then $\Sigma\varrho_a = J_A \times_{\mathrm{rel}} \varrho$.

We conclude the exposition of these topics with the above definition, leaving to the reader, as an easy but lengthy exercise, the task of formulating and proving various propositions analogous to those which have been or will be given in this section for the product and sum in the original sense.

D. DISTRIBUTIVE LAWS

In this subsection, various propositions are established concerning what may be called "distributive laws" for set operations, in a rather general sense. As a matter of fact, there are distributive laws in the proper sense, e.g. $(\bigcup X_a) \times Y = \bigcup(X_a \times Y)$, "distributive laws" asserting the existence of a natural (canonical) bijective relation for classes resulting from certain operations, e.g. for $(\Sigma X_a) \times (\Sigma Y_b)$ and $\Sigma(X_a \times Y_b)$, and, finally, assertions of the following kind: if, for any $a \in A$, f_a is a one-to-one relation on X_a onto Y_a, then Σf_a is bijective on ΣX_a onto ΣY_a. In each case, there is an assertion implying that two certain sets (or classes) are equipollent, which will be useful in Section 9.

We begin with a proposition on union and intersection which has its proper place in Section 2 but could not be proved there, since at that stage we had not yet introduced the Axiom of Choice.

5 D.1. Theorem. *Let A, B be non-empty sets, and let $\{X_{a,b} \mid a \in A,\ b \in B\}$ be a family of sets. Then*

$$\bigcap_{a\in A} \bigcup_{b\in B} X_{a,b} = \bigcup_{f\in B^A} \bigcap_a X_{a,fa}, \quad \bigcup_{a\in A} \bigcap_{b\in B} X_{a,b} = \bigcap_{f\in B^A} \bigcup_a X_{a,fa}.$$

Proof. If $x \in \bigcap_a \bigcup_b X_{a,b}$, then, for every $a \in A$, let $B(a)$ denote the set of all $b \in B$ such that $x \in X_{a,b}$. Clearly, every $B(a)$ is non-empty and therefore there exists an $f \in B^A$ such that $fa \in B(a)$ for every $a \in A$. This implies $x \in \bigcap_a X_{a,fa}$. If, conversely, $x \in \bigcup_{f\in B^A} \bigcap_a X_{a,fa}$, choose a relation $g \in B^A$ such that $x \in \bigcap_a X_{a,ga}$. Then, for every $a \in A$, $x \in X_{a,ga} \subset \bigcup_b X_{a,b}$, hence $x \in \bigcap_a \bigcup_b X_{a,b}$. – The proof of the second equality is left to the reader.

We now proceed to assertions on "distributivity laws" involving products and sums. Only a sample of such assertions is given here.

5 D.2. Theorem. *Let A, B be non-empty sets, and let $\{X_{a,b} \mid a \in A, b \in B\}$ be a family of sets. Then*

$$\prod_a \bigcup_b X_{a,b} = \bigcup_{f\in B^A} \prod_a X_{a,fa}, \quad \prod_a \bigcap_b X_{a,b} = \bigcap_{f\in B^A} \prod_a X_{a,fa}.$$

Proof. Let $x \in \prod_a \bigcup_b X_{a,b}$. Then $x = \{x_a\}$, $x_a \in \bigcup_b X_{a,b}$; therefore, there exists an $f \in B^A$ such that $x_a \in X_{a,fa}$, for every $a \in A$. Clearly, $x \in \prod_a X_{a,fa}$. If, conversely, $x = \{x_a\} \in \bigcup_{f \in B^A} \prod_a X_{a,fa}$ then choose $g \in B^A$ such that $x \in \Pi X_{a,ga}$; then $x_a \in X_{a,ga} \subset \subset \bigcup_b X_{a,b}$, for every $a \in A$, and therefore $x \in \prod_a \bigcup_b X_{a,b}$. To prove the second equality, let $x = \{x_a\} \in \prod_a \bigcap_b X_{a,b}$; then $x_a \in X_{a,b}$ for every $a \in A$, $b \in B$ and therefore $x \in \in \prod_a X_{a,fa}$ for every $f \in B^A$. If, conversely, $x = \{x_a\} \in \bigcap_{f \in B^A} \prod_a X_{a,fa}$, suppose that $x \notin \prod_a \bigcap_b X_{a,b}$. Then, for some a' and some b', $x_{a'} \notin X_{a',b'}$; choose $f \in B^A$ such that $fa' = b'$; then $x \notin \prod_a X_{a,fa}$ and we obtain a contradiction. This proves the equality in question.

5 D.3. *Let A be a class; let B, C, be disjoint sets. Then the relation assigning to every $f \in A^{B \cup C}$ the pair $\langle f_B, f_C \rangle$ is bijective for $A^{B \cup C}$ and $A^B \times A^C$, and will be called* canonical *(for $A^{B \cup C}$ and $A^B \times A^C$).*

Proof. Clearly, if $f \in A^{B \cup C}$, then $f = f_B \cup f_C$; this implies that the relation assigning $\langle f_B, f_C \rangle$ to f is one-to-one. If $g \in A^B$, $h \in A^C$, then $f = g \cup h \in A^{B \cup C}$ (since B, C are disjoint) and $g = f_B$, $h = f_C$. This proves the proposition.

5 D.4. *Let A be a class, and let B, C be sets. Then the relation which assigns to every $g \in (A^B)^C$ the relation $\{\langle y, z \rangle \to (gz)\, y \mid y \in B, z \in C\}$ belonging to $A^{B \times C}$ is bijective for $(A^B)^C$ and $A^{B \times C}$; it will be called* canonical *(for $(A^B)^C$ and $A^{B \times C}$).*

Proof. Denote the relation in question by φ. If $h \in A^{B \times C}$, then, for any $z \in C$, let h_z denote the relation on B assigning $h\langle y, z \rangle$ to $y \in B$; put $g = \{z \to h_z\}$. Then $g \in (A^B)^C$ and clearly $(gz)\, y = h\langle y, z \rangle$; therefore, $\varphi g = h$. This proves that φ maps $(A^B)^C$ onto $A^{B \times C}$. Now, if g_1, g_2 belong to $(A^B)^C$, $g_1 \neq g_2$, then there is a $z' \in C$ such that $g_1 z' \neq g_2 z'$ and therefore there exists a $y' \in B$ such that $(g_1 z')\, y' \neq \neq (g_2 z')\, y'$; since φg_1 assigns $(g_1 z')\, y'$ to $\langle y', z' \rangle$ and φg_2 assigns $(g_2 z')\, y'$ to $\langle y', z' \rangle$, we have shown that $\varphi g_1 \neq \varphi g_2$. Thus φ is one-to-one, which completes the proof.

5 D.5. *Let A, B be classes, let C be a set. Then the relation assigning to every $\langle f, g \rangle \in A^C \times B^C$ the reduced product (see* 5 C.4*) of f and g is bijective for $A^C \times B^C$ and $(A \times B)^C$. It will be called* canonical *(for $A^C \times B^C$ and $(A \times B)^C$).*

Proof. For any $\langle f, g \rangle \in A^C \times B^C$ denote by $\varphi\langle f, g \rangle$ the reduced product of relations f, g, that is the relation which assigns to $z \in C$ the element $\langle fz, gz \rangle \in A \times B$. – If $h \in (A \times B)^C$, then, for any $z \in C$, hz is a pair which can be expressed uniquely in the form $hz = \langle fz, gz \rangle$ where $fz \in A$, $gz \in B$. Clearly, $f = \{z \to fz\}$, $g = \{z \to gz\}$ belong, respectively, to A^C and to B^C, and $\varphi\langle f, g \rangle = h$. We have shown that φ maps $A^C \times B^C$ onto $(A \times B)^C$. The rest of the proof is left to the reader.

5 D.6. The preceding propositions imply the following assertions: for any class A and any disjoint sets B, C, the classes $A^{B \cup C}$ and $A^B \times A^C$ are equipollent; for any class A and any sets B, C, the classes $(A^B)^C$ and $A^{B \times C}$ are equipollent; for any classes A, B and any set C, the classes $A^C \times B^C$ and $(A \times B)^C$ are equipollent.

5 D.7. *If $\{X_a\}$ is a non-void family of sets, Y is a set, consider the relation which assigns to every $\{f_a\} \in \prod_a(X_a^Y)$ the reduced product (see* 5 C.5*) of $\{f_a\}$. This relation is bijective for $\prod_a(X_a^Y)$ and $(\prod_a X_a)^Y$; it will be called* canonical *(for $\prod_a(X_a^Y)$ and $(\prod_a X_a)^Y$).*

The proof is left to the reader as an exercise.

5 D.8. *Let $\{Y_b \mid b \in B\}$ be a family of sets; let X be a set. For any $b \in B$ let φ_b denote the relation consisting of all $\langle y, \langle b, y\rangle\rangle$, $y \in Y_b$. Then the relation which assigns to every $f \in X^{\Sigma Y_b}$ the family $\{f \circ \varphi_b\}$ is bijective for $X^{\Sigma Y_b}$ and $\prod_b(X^{Y_b})$; it will be called* canonical *for $X^{\Sigma Y_b}$ and $\prod_b(X^{Y_b})$.*

Proof. If $\{g_b\} \in \prod_b X^{Y_b}$, put $f = \{\langle b, y\rangle \to g_b y \mid y \in Y_b\}$; it is easy to see that $\{g_b\} = \{f \circ \varphi_b\}$. Clearly, if f, f' belong to $X^{\Sigma Y_b}$, $f \neq f'$, then $f \circ \varphi_b \neq f' \circ \varphi_b$ for some b. This proves the proposition.

5 D.9. *Let $\{X_a \mid a \in A\}$, $\{Y_a \mid a \in A\}$ be families of sets. Let $\{f_a \mid a \in A\}$ be a family, f_a being a relation for X_a and Y_a. Then* (1) *Σf_a is a relation for ΣX_a and ΣY_a; $\mathbf{D}(\Sigma f_a) = \Sigma(\mathbf{D} f_a)$, $\mathbf{E}(\Sigma f_a) = \Sigma \mathbf{E} f_a$; Σf_a is single-valued (respectively, one-to-one) if and only if every f_a is single-valued (respectively, one-to-one);* (2) *Πf_a is a relation for ΠX_a and ΠY_a; $\mathbf{D}(\Pi f_a) = \Pi(\mathbf{D} f_a)$, $\mathbf{E}(\Pi f_a) = \Pi(\mathbf{E} f_a)$; Πf_a in single-valued (respectively, one-to-one) if and only if every f_a is single-valued (respectively, one-to-one).*

The proof is left to the reader as an easy exercise.

Corollary. *Let $\{X_a \mid a \in A\}$, $\{Y_a \mid a \in A\}$ be families of sets. If $\{f_a\}$ is a family of relations such that f_a is bijective for X_a and Y_a, then Σf_a (respectively, Πf_a) is bijective for ΣX_a and ΣY_a (respectively, for ΠX_a and ΠY_a).*

CHAPTER II

ALGEBRAIC STRUCTURES AND ORDER

(Sections 6 – 13)

In Section 6 some algebraic concepts are introduced: compositions ("binary operations"), semi-groups, groups, and rings. These concepts are not immediately necessary for general topology, but quite essential for any application of topology to other branches of mathematics.

In Section 7 the notion of a mapping (more generally, of a correspondence) is introduced as well as that of a "struct". The latter is a formal but useful generalization of concepts such as groups, rings, modules, topological spaces. Let us give a preliminary description of the notion of a struct; this term is due to J. W. Tukey, who used it in a rather special case, namely to denote a topological space endowed with a uniformity (see "Convergence and uniformity in topology", Princeton, 1940).

Many mathematical entities consist of a set endowed with a certain structure; thus a group may be defined as a pair $\langle X, \sigma \rangle$ where X is a set and σ is a composition satisfying certain conditions, a ring is a triple $\langle X, \sigma, \mu \rangle$ where σ, μ are compositions with certain properties. With this fact in view, we term a struct any pair $\langle X, \xi \rangle$ where X is a class; X is called the underlying class and ξ the structure of the struct $\mathscr{X} = \langle X, \xi \rangle$. Thus for a group, the underlying class is the set of elements of the group, and the structure is group multiplication; for rings the underlying class is the set of elements, and the structure is a pair of compositions. However, for the case of a ring $\langle X, \sigma, \mu \rangle$ we may also say (in a sense which will be made precise in Section 6 and 7) that the "additive" group $\langle X, \sigma \rangle$ and the "multiplicative" semi-group $\langle X, \mu \rangle$ are "underlying structs". These remarks may serve to indicate the meaning of structs; we emphasize (and will stress this point on several occasions) that this notion is purely formal, and is mainly used to simplify statements.

As concerns mappings, we abandon the attitude of conceiving mappings simply as certain sets of pairs, and include both the set mapped and the set into which the mapping goes as part of the notion of mapping itself; thus a mapping of X into Y is a triple $\langle f, X, Y \rangle$, where f is a relation (the graph of the mapping), X, Y are sets. However, even this last notion is not sufficiently rich. For example, if we say that a mapping F of a group $\langle X, \sigma \rangle$ into a group $\langle Y, \tau \rangle$ is a homomorphism, then – precisely speaking – this is not a property of the mapping $F = \langle f, X, Y \rangle$ as such, but rather a logical relation or a predicate concerning F, $\langle X, \sigma \rangle$ and $\langle Y, \tau \rangle$. This can be expressed by saying that $\langle f, X, Y \rangle$ is a homomorphism relative to σ, τ. A different approach seems more appropriate: we will term a mapping any triple $\varphi = \langle f, \mathscr{X}, \mathscr{Y} \rangle$, where $\mathscr{X} = \langle X, \sigma \rangle$ and $\mathscr{Y} = \langle Y, \tau \rangle$ are structs, and $\langle f, X, Y \rangle$ is a mapping in the previous sense.

Section 8 is, in fact, a continuation of Section 6. Some further algebraic notions are examined and some more profound properties of concepts from Section 6 are presented.

The following four sections concern further fundamental notions of set theory and of the theory of order. Cardinal numbers are considered in Section 9, order in Section 10, well-order and ordinal numbers in Section 11, and systems of sets (covers, filters and ultrafilters) in Section 12.

The concluding Section 13 is devoted to the notion of category (only definitions, examples and several elementary propositions are given); less experienced readers may well pass over this section until a second reading.

6. COMPOSITIONS

Although the examination of algebraic objects, such as groups, rings, etc., is not the purpose of this book, these objects will necessarily appear in our considerations as tools for the investigation of topological problems and, if endowed with a continuous structure, as examples on which the applications of topological concepts and theorems can be demonstrated. As a matter of fact, algebraic concepts already occur in the theory of sets. This is one reason for considering them at this stage. Another reason is that a brief treatment of various algebraic objects prepares the way for the introduction of concepts of a mapping, conceived as a triple $\langle f, A, B\rangle$, of a correspondence and especially of the important notion of a struct, i.e. of a class endowed with a structure.

First we shall recall some basic concepts and their current definitions stated in a somewhat informal manner, and give a number of examples. After this, we shall treat compositions (in general) and several types of algebraic objects such as semigroups, groups, rings in more detail.

A. PROPERTIES OF COMPOSITIONS

6 A.1. It is said that a group G is given if there is given a non-void set G and if to every pair of elements x, y from G an element $z \in G$, usually denoted by $x \,.\, y$ and called the product of x and y, is assigned in such a way that (1) $(x \,.\, y) \,.\, z = x \,.\, (y \,.\, z)$, (2) there exists an element $e \in G$, called a neutral or unit element, such that always $x \,.\, e = e \,.\, x = x$; (3) for any $x \in G$ there exists an element t, denoted usually by x^{-1}, such that $x \,.\, t = t \,.\, x = e$.

Examples. (A) If A is a set, then the set G of all permuting relations on A with the "product" of ϱ and σ defined as $\varrho \circ \sigma$ is a group (observe that J_A is a unit element). – (B) If A is a set, then $\exp A$ with the "product" of X and Y defined as $X \div Y$ is a group. – (C) The reals with the "product" $x + y$ form a group. – (D) For a natural $n \geqq 1$ consider the set of all real matrices with n rows and n columns whose determinant is distinct from 0. This set is a group under the usual matrix multiplication.

6 A.2. The requirements for a semi-group are less stringent than those for a group: thus every group is a semi-group. It is said that a semi-group G is given if there is given a non-void set G and if to every pair of elements x, y from G an element z is assigned in such a way that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Examples of semi-groups. (A) The set N with the "product" $x + y$. – (B) The set N with the usual product $x \cdot y$. – (C) For any class A, the class $\exp A$ with the "product" of X and Y equal to $X \cup Y$. – (D) For any set A, the set of all relations ϱ in A, the "product" of ϱ and σ being $\varrho \circ \sigma$. – (E) Let A be a non-void set; consider the set G of all finite sequences with values in A. For $\alpha \in G$, $\beta \in G$, let the product of α and β be the sequence $\alpha \cdot \beta$ defined in 3 F.1. Then G is a semi-group. It is, essentially, the so-called free semi-group with the set of generators A. – (F) For any natural n, $n \geqq 1$, the set of all real matrices with n rows and n columns is a semi-group under the usual matrix multiplication.

6 A.3. If a non-void set A is given and to every pair of elements x, y from A there is assigned an element from A called the sum of x and y, and denoted usually by $x + y$, as well as an element from A called the product of x and y, and denoted usually by $x \cdot y$, then it is said that a ring A is given, provided the following conditions are satisfied for every x, y, z from A: (1) $(x + y) + z = x + (y + z)$; (2) $x + y = y + x$; (3) there is an element, denoted 0 and called the zero element, in A such that always $x + 0 = x$; (4) for every $x \in A$ there exists an element in A, usually denoted by $-x$, such that $x + (-x) = 0$; (5) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$; (6) $x \cdot (y + z) = x \cdot y + x \cdot z$, $(x + y) \cdot z = x \cdot z + y \cdot z$.

Examples of rings. (A) For any set B, the set $\exp B$ with the "sum" $X \div Y$ and the "product" $X \cap Y$. – (B) The set of all reals with the usual sum and product. – (C) For any natural n, $n \geqq 1$, the set of all real matrices with n rows and n columns with the usual matrix addition and multiplication.

We are now going to give exact definitions of several general concepts relating to "algebraic operations" (compositions).

6 A.4. Definition. Let A be a class. A *binary internal composition* (or briefly a *composition*) on A is a single-valued relation σ such that $\mathbf{D}\sigma = A \times A$, $\mathbf{E}\sigma \subset A$. If σ is a composition on A and $x \in A$, $y \in A$, then the element $\sigma\langle x, y\rangle$ is sometimes called the *composite* of x and y under σ or the *σ-composite* of x and y.

We shall now describe in an exact manner the compositions occurring in examples 6 A.1, (A), (B); 6 A.2, (A)–(D); 6 A.3, (A) (the rest of the examples given above are purely illustrative and involve concepts not yet introduced in an exact manner). In 6 A.1, example (A), we have the composition $\{\langle\varrho, \sigma\rangle \to \varrho \circ \sigma \mid \varrho, \sigma$ permuting relations on $A\}$; in 6 A.1, example (B), $\{\langle X, Y\rangle \to X \div Y \mid X \subset A, Y \subset A\}$; in 6 A.2, example (A), $\{\langle x, y\rangle \to x + y \mid x \in \mathsf{N}, y \in \mathsf{N}\}$; in 6 A.2, example (B), $\{\langle x, y\rangle \to x \cdot y \mid x \in \mathsf{N}, y \in \mathsf{N}\}$; in 6 A.2, example (C), $\{\langle X, Y\rangle \to X \cup Y \mid X \subset A, Y \subset A\}$; in 6 A.2, example (D), the composition is $\{\langle\varrho, \sigma\rangle \to \varrho \circ \sigma \mid \varrho \subset A \times A, \sigma \subset A \times A\}$, thus it contains, as a subset, the composition from 6 A.1, (A); in

6 A.3, example (A), the compositions are $\{\langle X, Y\rangle \to X \div Y \mid X \subset A, Y \subset A\}$, as in 6 A.1, (B), and $\{\langle X, Y\rangle \to X \cap Y \mid X \subset A, Y \subset A\}$. – Further examples of compositions: (A) $\{X \cup Y\}$ or, more explicitly, $\{\langle X, Y\rangle \to X \cup Y \mid X, Y$ are sets$\}$; this composition on the class of all sets will be denoted by $\bigcup$. – (B) $\{X \cap Y\}$ is a composition on the class of all sets; it will be denoted by $\bigcap$. – (C) The composition $\{X \div Y\}$, that is $\{\langle X, Y\rangle \to X \div Y \mid X, Y$ are sets$\}$, will be denoted by $\div$. Observe that all these compositions are non-comprisable.

Remark. The compositions $\bigcup$, $\bigcap$, $\div$ will be occasionally called union, intersection and symmetric difference respectively (observe that we have defined in Section 2 the union of two classes, the union of a family of sets, etc., but the word "union" denoted no mathematical entity up to now).

6 A.5. Conventions. 1) If σ is a composition, then we often write $x \sigma y$ instead of $\sigma\langle x, y\rangle$. Conversely, if σ is a composition on X and if $\sigma\langle x, y\rangle$ is denoted by a symbol of the form, say, $x \mathrm{t} y$, then we often denote σ by t. – 2) If σ is a composition on X, $A \subset X$, $B \subset X$, then in accordance with the use of symbols such as $[\mathscr{A}] \cup [\mathscr{B}]$, $[\mathscr{A}] \cap [\mathscr{B}]$ etc. we shall write $[A] \sigma [B]$ to denote the set of all $x \sigma y$ where $x \in A$, $y \in B$. If there is no danger of ambiguity, $A \sigma B$ is written, for convenience, instead of $[A] \sigma [B]$.

6 A.6. Definition. Let ϱ be a composition on X, σ a composition on Y. If $\sigma \subset \varrho$, so that $Y \subset X$, we shall say that σ is the *restriction* of ϱ to a composition on Y or, briefly but not quite correctly, that σ is the restriction of ϱ to Y; the composition σ will be denoted by ϱ_Y provided there is no danger of misunderstanding.

Remark. If ϱ is a composition on X, $Y \subset X$, then clearly there exists at most one composition on Y which is a restriction of Y. However, such a restriction does not necessarily exist, for it may happen that $\{\langle x, y\rangle \to x \varrho y \mid x \in Y, y \in Y\}$ is not a composition.

6 A.7. Definition. Let σ be a composition on a class X. A class Y is called *stable under* σ if $Y \subset X$ and $y_1 \in Y$, $y_2 \in Y \Rightarrow y_1 \sigma y_2 \in Y$.

It is clear that Y is stable under σ if and only if there exists a composition on Y which is a restriction of σ.

Examples. (A) A class Y is stable under $\bigcup$ (under $\bigcap$) if and only if it is additive (multiplicative). – (B) The void class is stable under every composition. – (C) If A is a set, then the set of all $X \subset A$ such that either X or $A - X$ is finite is stable under $\bigcup$, $\bigcap$ and the symmetric difference. – (D) The class of constant relations is stable under the composition $\{\varrho \circ \sigma\}$.

6 A.8. *Let ϱ be a composition on X. Then the intersection of any indexed class of sets stable under ϱ is stable under ϱ.* – This is clear.

6 A.9. Definition. Let ϱ be a composition on X. An element $e \in X$ is called *neutral under* ϱ if $e\varrho x = x\varrho e = x$ for every $x \in X$; an element $o \in X$ is called *absorbing* *under* ϱ if $o\varrho x = x\varrho o = o$ for every $x \in X$, and a class $Y \subset X$ is called *absorbing under* ϱ if $y \in Y$, $x \in X \Rightarrow y\varrho x \in Y$, $x\varrho y \in Y$.

Remarks. 1) If ϱ is a composition, there exists at most one neutral and at most one absorbing element under ϱ. Namely, if e_1, e_2 are neutral, then $e_1\varrho e_2 = e_1$, $e_1\varrho e_2 = e_2$; if o_1, o_2 are absorbing, then $o_1\varrho o_2 = o_1$, $o_1\varrho o_2 = o_2$. — 2) If e is neutral, o is absorbing under a composition ϱ on X, and X contains two elements at least, then $e \neq o$. Indeed, choose $x \in X$, $x \neq o$; then $e\varrho x = x$ and $o\varrho x = o$ are distinct. — 3) The corresponding "one-sided" concepts may be introduced, e.g. an element e may be called left-neutral if $e\varrho x = x$ for every $x \in X$, right-neutral if $x\varrho e = x$ for every $x \in X$. For instance, if $X = (a, b)$, $a\varrho a = b\varrho a = a$, $a\varrho b = b\varrho b = b$, then a, b are left-neutral and there is no right-neutral element. Observe that if there exists a left-neutral element e and a right-neutral element e', then $e = e'$ is the only (see remark 1) neutral element.

Examples. (A) Under the composition $\{\varrho \circ \sigma \mid \varrho, \sigma$ relations in $A\}$ described in 6 A.2, example (D), J_A is neutral, $\emptyset$ is absorbing. — (B) The void set is neutral under $\bigcup$ and absorbing under $\bigcap$; there exists neither an absorbing element for $\bigcup$ nor a neutral element for $\bigcap$. However, any set A is absorbing under $\bigcup_{\exp A}$ and neutral under $\bigcap_{\exp A}$. — (C) The class of all finite sets is absorbing under $\bigcap$.

6 A.10. Definition. Let σ be a composition on X. We shall say that σ is *associative* if, for any x, y, z from X, $(x\varrho y)\,\varrho z = x\varrho(y\varrho z)$, *commutative* if $x\varrho y = y\varrho x$ for any $x \in X$, $y \in X$. An associative composition is called also a *semi-group structure.*

Examples. The compositions in all examples in 6 A.1, 6 A.2, 6 A.4 are associative, hence semi-group structures; those in 6 A.1, (B), (C); 6 A.2, (A)–(C) are also commutative, whereas the compositions in 6 A.1, example (A), 6 A.2, examples (D), (E) are, in general, not commutative. The composition $\{\langle X, Y\rangle \to X - Y \mid X, Y$ are sets$\}$ is neither commutative nor associative, the composition $\{\langle X, Y\rangle \to$ $\to X^Y \cup Y^X\}$ is commutative without being associative.

We conclude with a definition and some conventions concerning "composites of a finite number of elements".

6 A.11. Proposition and definition. *Let σ be a composition on a class X. There exists exactly one single-valued relation $\bar\sigma$ such that* (1) $\mathbf{D}\bar\sigma$ *consists of all finite non-empty sequences of elements of* X, $\mathbf{E}\bar\sigma \subset X$, (2) *if* $x_0 \in X$, *then* $\bar\sigma\{x_k \mid k \in \mathsf{N}_1\} =$ $= x_0$; (3) *if* $\xi = \{x_{k+1} \mid k \in \mathsf{N}_p\}$ *belongs to* $\mathbf{D}\bar\sigma$ *and* $x_0 \in X$, *then* $\bar\sigma\{x_k \mid k \in \mathsf{N}_{p+1}\} =$ $= x_0\sigma(\bar\sigma\xi)$. *If σ is associative, then the element* $\bar\sigma\{a_k \mid k \in \mathsf{N}_{q+1}\}$ *will be called the* σ-composite *of the sequence* $\{a_k\}$ *and will be often denoted by* $\sigma\{a_k \mid k \in \mathsf{N}_{q+1}\}$ *or* $\sigma_{k=0}^{q}\{a_k\}$ *or* $\sigma\{a_k\}$. *If, in addition, there is a neutral element e in X, then we shall say that e is the* σ-composite of the void sequence *and we shall write* $e = \sigma\emptyset$, *or e.g.* $e = \sigma_{k=0}^{-1}\{a_k\}$, *etc.; in this case,* (3) *remains valid if* $\xi = \emptyset$ *is admitted.*

If all x_k are equal to an element $x \in X$, then $\sigma\{x_k \mid k \in \mathsf{N}_n\}$ *will be called the n-th* σ-power *of x.*

Remark. Clearly, $(\sigma\{a_k \mid k \in \mathsf{N}_p\})\ \sigma\ (\sigma\{a_{k+p} \mid k \in \mathsf{N}_q\}) = \sigma\{a_k \mid k \in \mathsf{N}_{p+q}\}$.

Conventions. 1) Let $\{x_a \mid a \in A\}$ be a finite family of elements of X. In general there are many bijective relations φ on a certain N_p onto A, and elements $\sigma\{x_{\varphi k} \mid k \in \in \mathsf{N}_p\}$ may be different according to the choice of φ. However, if the choice of φ is either clear from the context or irrelevant (which is the case if σ is commutative) then we shall denote $\sigma\{x_{\varphi k} \mid k \in \mathsf{N}_p\}$ by $\sigma\{x_a \mid a \in A\}$ or $\underset{a \in A}{\sigma}\{x_a\}$ etc. — 2) If the composition σ is denoted by $+$, we shall write, as usual, $\sum_{k=0}^{n} x_k$ instead of $\sigma\{x_k \mid k \in \mathsf{N}_{n+1}\}$, etc., and call $\sum_{k=0}^{n} x_k$ the sum of $\{a_k\}$; the n-th power of x under σ will be denoted by nx provided there is no danger of misunderstanding. Similarly, if $x \,.\, y$ (or xy) is written to denote $x\sigma y$, then $\prod_{k=0}^{n} x_k$ denotes $\sigma\{x_k \mid k \in \mathsf{N}_{n+1}\}$, etc., and the n-th power of x under σ will be denoted by x^n. — 3) The symbol x^n, $n \in \mathsf{Z}$ (see Section 8) will be sometimes used in an extended sense, namely with $n \geqq 1$, to denote the n-th σ-power of x (σ being an arbitrary but fixed associative composition), with $n = 0$, to denote the neutral element under σ (if it exists), and with $n = -k$, $k \geqq 1$, to denote the k-th σ-power of the inverse of x under σ (if it exists). The symbol nx is used in a similar manner (nx is then called the n-multiple of x).

B. SEMI-GROUPS

We are now going to define semi-groups. Properly speaking we shall define two concepts, namely a semi-group as a certain pair $\langle X, \sigma \rangle$ and a semi-group under a composition σ as a set satisfying certain conditions involving σ. This formal deviation from the usual approach is, nevertheless, in accordance with current use of the word "semi-group" in a formally twofold sense. We shall proceed in the same way in defining groups, rings, etc., since such a parallel use of two closely related concepts, although seemingly cumbersome, will prove itself useful.

6 B.1. Definition. A pair $\mathscr{X} = \langle X, \sigma \rangle$, where X is a non-void set, σ is a semi-group structure, i.e. an associative composition, on X, will be called a *semi-group*. We shall say that X is the *underlying set of* $\mathscr{X}$, and σ is the *structure of* $\mathscr{X}$.

If X is a non-void set and σ is a composition, then we shall call X a *semi-group under* σ if $\langle X, \sigma_X \rangle$ is a semi-group.

Conventions. 1) A semi-group $\langle X, \sigma \rangle$ will often be denoted simply by X, i.e. by the same symbol as the set X (which is a semi-group under σ), provided no misunderstanding is likely to arise. — 2) If ϱ is a composition on X, σ is a composition on a class Y, and $\varrho \subset \sigma$, then we shall occasionally write $\langle X, \sigma \rangle$ instead of $\langle X, \sigma_X \rangle$.

The fact, stated in the above definition, that X is a semi-group under σ if and only if $\langle X, \sigma_X \rangle$ is a semi-group makes possible the following mode of exposition.

Definitions and propositions concerning semi-groups will often be formulated either merely for semi-groups as pairs $\langle X, \sigma \rangle$ or merely for semi-groups under a composition, i.e. sets satisfying certain conditions; usually the task of reformulating these definitions and statements for the other concept will be left to the reader.

6 B.2. Let σ be a semi-group structure on a class X. In this case, which includes that of a semi-group $\langle X, \sigma \rangle$, we shall refer to $\langle X, \sigma \rangle$ as a "class endowed with a semi-group structure". Such entities will be considered in Section 7. Only one example is given here: the class of all finite sequences endowed with the composition assigning to $\langle \alpha, \beta \rangle$ the sequence $\alpha . \beta$ in the sense of 3 F.1.

6 B.3. Definition. Let $\langle X, \sigma \rangle$ be a semi-group. If $\langle Y, \varrho \rangle$ is a semi-group, and ϱ is a restriction of σ, we shall say that $\langle Y, \varrho \rangle$ is a *sub-semi-group* of $\langle X, \sigma \rangle$. If $Y \subset X$, we shall say that Y is a *sub-semi-group* of $\langle X, \sigma \rangle$ if Y is a semi-group under σ.

Remark. Clearly, $\langle Y, \sigma_Y \rangle$ is a sub-semi-group of $\langle X, \sigma \rangle$ if and only if Y is a semi-group under σ. It is also obvious that if $\langle X, \sigma \rangle$ is a semi-group, then $Y \subset X$ is a sub-semi-group if and only if it is non-empty and stable under σ.

Many instances of sub-semi-groups are contained implicitly in previous examples. Further examples: (A) If $\langle X, \sigma \rangle$ is a semi-group, o and e are, respectively, the absorbing and the neutral element, then (o), (e) and (o, e) are sub-semi-groups. — (B) Let A be a set; consider the semi-group from 6 A.2, example (D), i.e. the set $\exp(A \times A)$ endowed with the composition of relations. The following subsets are stable and non-empty, hence sub-semi-groups: the set of all ϱ with $\mathbf{D}\varrho = A$, of all ϱ on A onto A, of all one-to-one relations ϱ, of all bijective relations on A onto A, of all reflexive relations. — (C) In the semi-group of the preceding example, the set of all one-to-one relations ϱ with $\mathbf{D}\varrho = A$ and the set of all constant relations ϱ with $\mathbf{D}\varrho = A$ are sub-semi-groups. If A contains two elements at least, then the intersection of these two sub-semi-groups is void.

6 B.4. Convention. If $\mathscr{Y} = \langle Y, \varrho \rangle$ is a sub-semi-group of $\mathscr{X} = \langle X, \sigma \rangle$, then we shall also say that $\mathscr{Y}$ is *identically embedded* in $\mathscr{X}$.

To give a motivation for this convention we anticipate some notions which will be introduced in Section 7. If $\mathscr{X} = \langle X, \varrho \rangle$, $\mathscr{Y} = \langle Y, \sigma \rangle$, $\mathscr{Z} = \langle Z, \tau \rangle$ are semi-groups, $\mathscr{X}$ is a sub-semi-group of $\mathscr{Y}$ (thus $X \subset Y$, $\varrho = \sigma_X$) and $f = \langle \varphi, \mathscr{Z}, \mathscr{Y} \rangle$ is a mapping of $\mathscr{Z}$ into $\mathscr{Y}$ (see 7 B.1, 7 B.10) such that $f : \mathscr{Z} \to \mathscr{X}$ (see 7 B.4) is an isomorphism, we shall say (see 7 B.11) that f is an embedding of $\mathscr{Z}$ into $\mathscr{Y}$, φ is an embedding relation (under the structures τ and σ) and that $\mathscr{Z}$ is embedded in $\mathscr{Y}$ by means of f (or of φ). Now, if f is an identity mapping, then it is appropriate to speak of an identical embedding.

6 B.5. *The intersection of a non-void family of sub-semi-groups of a given semi-group is a sub-semi-group, unless it is void.*

Proof. By 6 B.3 (remark) and 6 A.8, the intersection in question is stable, hence (again by 6 B.3, remark) either a sub-semi-group or void.

Remark. If Y, Z are sub-semi-groups of a given semi-group, then $Y \cup Z$ need not be a sub-semi-group.

6 B.6. *Let $\langle G, \sigma \rangle$ be a semi-group. Let $X \subset G$, $X \neq \emptyset$. Then there exists a smallest sub-semi-group H containing X.*

This follows at once from **6 B.5** if we consider the collection, clearly non-void, of all sub-semi-groups Y containing X and take its intersection as H.

6 B.7. Definition. Let $\langle G, \sigma \rangle$ be a semi-group, $X \subset G$, $X \neq \emptyset$. Let H be the smallest sub-semi-group containing X. Then we shall say that H, as well as $\langle H, \sigma_H \rangle$, is the *semi-group generated* by X under σ (or in $\langle G, \sigma \rangle$), and that X *generates* H (or is a *generating set* for H) *as a semi-group* under σ.

Examples. (A) In the semi-group $\langle \mathsf{N}, + \rangle$, the singleton (2) generates the sub-semi-group of all positive even integers, the set $(3, 5)$ generates the sub-semi-group consisting of 3, 5, 6 and all integers $n \geqq 8$. – (B) In the semi-group $\langle \mathsf{N}, . \rangle$, there is no finite generating set; the set consisting of 0, 1 and all prime numbers is the smallest generating set for N.

6 B.8. Definition. Let σ be a composition on X. Let $x \in X$. We shall say that $y \in X$ is a *left-inverse* (respectively *right-inverse*) for x under σ if $y\sigma x$ (respectively $x\sigma y$) is neutral under σ; if $y \in X$ is both a left-inverse and a right-inverse for x, we shall call it an *inverse* for x (under σ).

Examples. (A) If e is neutral under σ, then e is an inverse for e. – (B) In semi-groups $\langle \mathsf{N}, + \rangle$, $\langle \mathsf{N}, . \rangle$ no element, except the neutral ones, possesses an inverse. – (C) Under the composition $\dotdiv$ (symmetric difference), every set X has an inverse, namely X. – (D) Let A be a set. Consider the composition ψ assigning $\varrho \circ \sigma$ to every pair $\langle \varrho, \sigma \rangle$ of single-valued relations $\varrho \in A^A$, $\sigma \in A^A$. Then $\sigma \in A^A$ has a left-inverse under ψ if and only if σ is one-to-one; if so, there are, in general, many left-inverses, namely all $\varrho \in A^A$ such that the restriction of ϱ to $\mathbf{E}\sigma$ coincides with σ^{-1}; a relation $\sigma \in A^A$ has exactly one left-inverse if and only if σ is bijective on A onto A, and similarly for the right-inverse; finally, $\varrho \in A^A$ possesses an inverse if and only if ϱ is bijective on A onto A.

6 B.9. *Let σ be an associative composition on X. If $x \in X$, y is a left-inverse for x, and y' is a right-inverse for x (under σ), then $y = y'$ is an inverse for x. In particular, every $x \in X$ has at most one inverse.*

Proof. Clearly $y = y\sigma(x\sigma y') = (y\sigma x)\,\sigma y' = y'$.

Convention. If the composition is clear from the context, then the inverse of an element x will be denoted as x^{-1} provided no misunderstanding is likely to arise.

Observe that the inverse ϱ^{-1} of a relation in the sense of **1 B.8** exists for any relation ϱ but is not necessarily the inverse for ϱ under a restriction of the composition $\{\varrho \circ \sigma\}$.

6 B.10. Definition. Let σ be a composition on X. An element $x \in X$ is called *invertible under σ*, sometimes also *σ-invertible* or simply *invertible*, if it possesses an inverse. The relation consisting of all $\langle x, y\rangle$ such that x is an inverse of y (i.e. $x\sigma y = y\sigma x$ is the neutral element) is called the *σ-inversion.*

Examples. (A) No $n \in \mathsf{N}$ except 0 is invertible under addition and no $n \in \mathsf{N}$ except 1 is invertible under multiplication. – (B) Every set X is invertible under the composition $\div$, the inverse being X.

6 B.11. Definition. Let ϱ be a composition on X. If $a \in X$, then $\{x \to a\varrho x \mid x \in X\}$ will be called the *left ϱ-translation by a*, and $\{x \to x\varrho a \mid x \in X\}$ will be called the *right ϱ-translation by a*. An element $x \in X$ is called *virtually invertible* (*under ϱ*) if the left ϱ-translation by x and the right ϱ-translation by x are both one-to-one, i.e. if for any $z \in X$ there is at most one $y \in X$ with $x\varrho y = z$ and at most one $y' \in X$ with $y'\varrho x = z$.

Clearly, if σ is an associative composition, then every invertible element is virtually invertible. The converse does not hold, however; for instance, every $n \in \mathsf{N}$ is virtually invertible under addition, and every $n \in \mathsf{N}$, $n \neq 0$, is virtually invertible under multiplication.

The concept of virtually invertible elements will be useful for the important theorem on the embedding of a semi-group into a group which we defer to Section 8.

C. GROUPS

6 C.1. Definition. An associative composition, in other words, a semi-group structure σ on X is called a *group structure* if every $x \in X$ is invertible under σ.

Example: the composition $\{\langle X, Y\rangle \to X \div Y\}$ on the class of all sets is a non-comprisable group structure.

Remark. If σ is a group structure on X, $X \neq \emptyset$, then clearly there exists a neutral element e.

6 C.2. Definition. A pair $\langle G, \sigma\rangle$ where G is a non-void set and σ is a group structure on G will be called a *group*. If G is a non-void set and σ is a composition, then G will be called a *group under σ* if $\langle G, \sigma_G\rangle$ is a group.

This definition as well as some other definitions and propositions given in the sequel are quite analogous to the corresponding statements concerning semi-groups; for this reason, proofs and examples concerning groups are sometimes omitted. It is to be pointed out, however, that, in some cases, there is an essential difference between the corresponding notions for groups and semi-groups.

The conventions in 6 B.1 apply, of course, for groups too, since every group is a semi-group. Remarks in 6 B.1 are also valid, with appropriate changes, for groups.

Examples of groups have already been given; we give only one more. Let A be a set, $P(A)$ the set of all permuting relations on A, i.e. one-to-one relations f with $\mathbf{D}f = \mathbf{E}f = A$. Put $\varrho = \{\langle f, g\rangle \to f \circ g \mid f \in P(A),\ g \in P(A)\}$; then $\langle P(A), \varrho\rangle = \langle P(A), \circ\rangle$ is a group.

6 C.3. Definition. Let $\langle G, \sigma \rangle$ be a semi-group. A group $\langle H, \varrho \rangle$ is said to be a *subgroup* of $\langle G, \sigma \rangle$ if ϱ is a restriction of σ, i.e. if $\varrho = \sigma_H$. If $H \subset G$ we shall say that the set H is a *subgroup* of $\langle G, \sigma \rangle$ if H is a group under σ.

Convention. If $\mathscr{H} = \langle H, \varrho \rangle$ is a subgroup of $\mathscr{G} = \langle G, \sigma \rangle$ we shall also say that $\mathscr{H}$ is *identically embedded* in $\mathscr{G}$ (cf. 6 B.4).

If $\langle G, \sigma \rangle$ is a group, $H \subset G$ is stable, hence a sub-semi-group, then H need not be a subgroup.

6 C.4. *Let $\langle G, \varrho \rangle$ be a group, let $\emptyset \neq H \subset G$. Then H is a subgroup if and only if it is stable under ϱ and the inverse x^{-1} of every $x \in H$ belongs to H.*

Proof. Let H be a subgroup. Then clearly H is stable. If $x \in H$, then let y be the inverse of x in H, i.e. under ϱ_H. We have $y = y\varrho(x\varrho x^{-1}) = \bar{e}\varrho x^{-1}$ where $\bar{e}$ is neutral for σ_H; since $\bar{e} = \bar{e}\varrho(\bar{e}\varrho\bar{e}^{-1}) = (\bar{e}\varrho\bar{e})\,\varrho\bar{e}^{-1} = e$, we obtain $y = x^{-1}$. The rest of the proof is left to the reader.

6 C.5. *Let $\langle G, \sigma \rangle$ be a group. If $\{H_a \mid a \in A\}$ is a non-void family of subgroups under σ, then $\bigcap\{H_a\}$ is a subgroup.*

Compare with 6 B.5 observing that $\bigcap\{H_a\} \neq \emptyset$ since the neutral element belongs to every H_a.

6 C.6. *Let $\langle G, \sigma \rangle$ be a group. Let $X \subset G$. Then there exists a smallest subgroup containing X.*

Remark. Compare with 6 B.6 and observe that the assumption $X \neq \emptyset$ is redundant here.

6 C.7. Definition. Let $\langle G, \sigma \rangle$ be a group, let $X \subset G$ and let H be the smallest subgroup containing X. Then we shall say that H, as well as $\langle H, \sigma_H \rangle$, is the *group generated* by X under σ (or in $\langle G, \sigma \rangle$) and that X *generates* H (or is a *generating set* for H) *as a group* under σ.

Example. If A is finite, then $\langle \exp A, \div \rangle$ is generated by the set of all singletons belonging to $\exp A$.

6 C.8. Before passing to semi-rings and rings we introduce the following

Definition. A semi-group or group $\langle G, \sigma \rangle$ is called *commutative* or *abelian* if the composition σ is commutative.

Example. The group $P(A)$ from 6 C.2 (example) is not commutative unless A contains two elements at most.

D. SEMI-RINGS AND RINGS

6 D.1. Definition. Let σ, μ be compositions on X. Then μ is called *distributive relative to* σ if, for any x, y, z from X, $x\mu\,(y\sigma z) = (x\mu y)\,\sigma(x\mu z)$, $(y\sigma z)\,\mu x = (y\mu x)\,\sigma(z\mu x)$.

Examples. (A) The composition $\bigcap$ is distributive relative to $\bigcup$, and conversely. – (B) The composition $\bigcap$ is distributive relative to the composition $\div$, but not conversely.

6 D.2. Definition. A pair $\langle \sigma, \mu \rangle$, where σ, μ are compositions on X, is called a *semi-ring structure on X* if σ is commutative and associative (in other words, if σ is a commutative semi-group structure), μ is associative (in other words, a semi-group structure, non-commutative in general) and μ is distributive relative to σ. If, moreover, σ is a group structure, then $\langle \sigma, \mu \rangle$ is called a *ring structure.*

Examples. (A) The pair $\langle \bigcup, \bigcap \rangle$ is a semi-ring structure on the class of all sets; it is called the *natural semi-ring structure for sets.* – (B) The pair $\langle \div, \bigcap \rangle$ is a ring structure on the same class; it is called the *Boolean ring structure for sets.* – (C) $\langle +, . \rangle$ is a semi-ring structure on N. – (D) Let A be a finite set; put $B = (\exp A)^A$. If $f \in B$, $g \in B$ (i.e. if f, g are single-valued relations on A ranging in $\exp A$), put $f\sigma g = \{a \to fa \div ga \mid a \in A\}$ and denote by $f\mu g$ the single-valued relation assigning to every $a \in A$ the set of all those $y \in A$ which satisfy the following condition: the number of elements $z \in A$ such that $z \in ga$, $y \in fz$, is an odd integer. We leave to the reader the task (not quite trivial) of proving that σ and μ are compositions on B and $\langle \sigma, \mu \rangle$ is a ring structure on B.

6 D.3. Definition. If A is a non-void set and $\langle \sigma, \mu \rangle$ is a semi-ring (respectively, ring) structure on A, then $\langle A, \sigma, \mu \rangle$ will be called a *semi-ring* (respectively, *ring*). If A is a non-void set, $\langle \sigma, \mu \rangle$ is a pair of compositions, then we shall call A a *semi-ring* (a *ring*) *under* $\langle \sigma, \mu \rangle$ if $\langle A, \sigma_A, \mu_A \rangle$ is a semi-ring (a ring).

Conventions. 1) A semi-ring $\langle A, \sigma, \mu \rangle$ will often be denoted simply by A provided its structure $\langle \sigma, \mu \rangle$ is clear from the context. – 2) If σ, μ are compositions, σ_A, μ_A are their restrictions to compositions on a class A, and $\langle A, \sigma_A, \mu_A \rangle$ is a semi-ring, we often write $\langle A, \sigma, \mu \rangle$ instead of $\langle A, \sigma_A, \mu_A \rangle$.

Examples. (A) If X is a set, then the semi-ring $\langle \exp X, \bigcup, \bigcap \rangle$ (see the above convention) is called the *natural semi-ring of parts of X.* – (B) If X is a set, then the ring $\langle \exp X, \div, \bigcap \rangle$ is called the *Boolean ring of parts of X.* – (C) Let $\mathscr{A} \neq \emptyset$ be a collection of sets. Then $\mathscr{A}$ is a semi-ring under $\langle \bigcup, \bigcap \rangle$ if and only if it is additive and multiplicative, a ring under $\langle \div, \bigcap \rangle$ if and only if it is additive and multiplicative and moreover $X \in \mathscr{X}$, $Y \in \mathscr{X}$, $X \supset Y \Rightarrow X - Y \in \mathscr{X}$.

6 D.4. Definition. If $\boldsymbol{r} = \langle \sigma, \mu \rangle$ is a semi-ring structure, then σ will be called the *underlying additive structure* of $\boldsymbol{r}$, and μ will be called the *underlying multiplicative structure* of $\boldsymbol{r}$. If $\mathscr{A} = \langle A, \sigma, \mu \rangle$ is a semi-ring, then $\langle A, \sigma \rangle$ is called the *underlying additive semi-group* of $\mathscr{A}$, and $\langle A, \mu \rangle$ is called the *underlying multiplicative semi-group* of $\mathscr{A}$, the set A is called the *underlying set* of $\mathscr{A}$ (cf. 6 B.1).

These notions may be useful if, in a certain reasoning, a semi-ring $\langle A, \sigma, \mu \rangle$ is investigated with regard to one of compositions σ, μ only. They will be considered in a wider context in Section 7.

6 D.5. Definition. Let $\langle \sigma, \mu \rangle$ be a semi-ring structure on X. A class Y is called *stable under* $\langle \sigma, \mu \rangle$ if it is stable (in the sense of 6 A.7) under both σ and μ.

Remark. Clearly, by 6 A.8, the intersection of any indexed class of stable sets is stable.

Example. A class of sets is stable under $\langle \bigcup, \bigcap \rangle$ if and only if it is additive and multiplicative.

6 D.6. Definition. Let $\mathscr{A} = \langle A, \sigma, \mu \rangle$ be a semi-ring; in particular, $\mathscr{A}$ may be a ring. We shall say that a semi-ring (respectively, ring) $\mathscr{B} = \langle B, \varrho, \nu \rangle$ is a *sub-semi-ring* (respectively, *subring*) of $\mathscr{A}$ if $\varrho = \sigma_B$, $\nu = \mu_B$. A set $B \subset A$ will be called a *sub-semi-ring* (*subring*) of $\mathscr{A}$ if it is a semi-ring (respectively, ring) under $\langle \sigma, \mu \rangle$.

Convention. If $\mathscr{B}$ is a sub-semi-ring (in particular, a subring) of $\mathscr{A}$, we shall also say that $\mathscr{B}$ is *identically embedded* in $\mathscr{A}$ (cf. 6 B.4).

6 D.7. *Let $\langle A, \sigma, \mu \rangle$ be a semi-ring. A set $B \subset A$ is a sub-semi-ring (respectively, subring) if and only if it is a sub-semi-group (respectively, subgroup) of $\langle A, \sigma \rangle$ and a sub-semi-group of $\langle A, \mu \rangle$. A non-void set $B \subset A$ is a sub-semi-ring if and only if it is stable under $\langle \sigma, \mu \rangle$, a subring if and only if, moreover, it contains with every element its inverse under σ.*

The first assertion is obvious, the second follows from the corresponding propositions concerning semi-groups and groups.

6 D.8. *Let $\mathscr{A} = \langle A, \sigma, \mu \rangle$ be a semi-ring. Then the intersection of any non-void family of sets which are sub-semi-rings (respectively, subrings) of $\mathscr{A}$ is a sub-semi-ring (subring) provided it is non-void.*

This follows at once from 6 D.7 and the corresponding propositions on semi-groups and groups.

Remark. Observe that if $\mathscr{A}$ is a semi-ring, then it may happen that there exist disjoint subrings; e.g. if $\mathscr{A} = \langle \exp A, \bigcup, \bigcap \rangle$, then any singleton (X), $X \subset A$, is a subring. On the other hand, if $\mathscr{A}$ is a ring, then every subring contains the element neutral under σ, hence the intersection of any family of subrings is non-void.

6 D.9. *Let $\mathscr{A} = \langle A, \sigma, \mu \rangle$ be a semi-ring (respectively, ring), let $X \subset A$ be non-void (if $\mathscr{A}$ is a ring, X may be void). Then there exists a smallest sub-semi-ring (respectively, subring) containing X.*

This follows at once from 6 D.8 if we consider the intersection of all sub-semi-rings (subrings) $B \subset A$ such that $B \supset X$.

6 D.10. Definition. Let $\mathscr{A} = \langle A, \sigma, \mu \rangle$ be a semi-ring (in particular, $\mathscr{A}$ may be a ring). If $\emptyset \neq X \subset A$ and B is the smallest sub-semi-ring of A containing X, then we shall say that B (as well as $\langle B, \sigma_B, \mu_B \rangle$) is the *semi-ring generated* by X under $\langle \sigma, \mu \rangle$ (or generated by X in $\mathscr{A}$) and that X *generates* B (or is a *generating set* for B) *as a semi-ring* under $\langle \sigma, \mu \rangle$ (or in $\mathscr{A}$). If $\mathscr{A}$ is a ring, $X \subset A$ and B is the smallest subring of A containing X, then we shall say that B as well as $\langle B, \sigma_B, \mu_B \rangle$

is the *ring generated* by X under $\langle \sigma, \mu \rangle$ (or generated by X in $\mathscr{A}$) and that X *generates* B (or is a *generating set* for B) *as a ring* under $\langle \sigma, \mu \rangle$ (or in $\mathscr{A}$).

Examples. (A) Consider the semi-ring $\langle \mathsf{N}, +, . \rangle$. Then every set of the form $\mathsf{E}\{x . q \mid x \in \mathsf{N}, x \geqq k\}$, where $q \geqq 1$, $k \geqq 0$ are given natural numbers, is a sub-semi-ring; its smallest generating set consists precisely of numbers kq, $(k + 1) . q$, $\ldots, (2k - 1) . q$. The only subring is the singleton (0). – (B) Consider the ring Z of integers, to be introduced in Section 8. Then the set (1) generates $\mathsf{N} - (0)$ as a sub-semi-ring, and Z as a subring. – (C) In $\langle \exp A, \div, \bigcap \rangle$ every sub-semi-ring is a subring. Clearly, if A is finite, the set of all $(a) \subset A$ is a generating set for $\exp A$.

6 D.11. Definition. Let $\langle \sigma, \mu \rangle$ be a semi-ring structure on a class A. If an element is neutral for σ, then it is called the *zero element* (or simply *zero*) for $\langle \sigma, \mu \rangle$ (or of $\langle A, \sigma, \mu \rangle$) and usually denoted by 0, i.e. by the same symbol as the number 0, provided no misunderstanding is likely to arise. The neutral element for μ, if it exists, is called the *unit element* (or simply *unit* or *unity*) for $\langle \sigma, \mu \rangle$ (or of $\langle A, \sigma, \mu \rangle$) and is often denoted by 1 provided there is no danger of misunderstanding.

Remarks. 1) It is easy to see that, for a given semi-ring structure, there exists at most one zero element and at most one unit element. – 2) If $\langle \sigma, \mu \rangle$ is a ring structure, then the zero is clearly an absorbing element for μ.

Examples. (A) In $\langle \mathsf{N}, +, . \rangle$, 0 is the zero and 1 is the unit (in the above sense). – (B) Let $\mathscr{A} = \langle \exp A, \bigcup, \bigcap \rangle$, A being a given set; if $B \subset C \subset A$, let $\mathscr{A}_{B,C}$ be the sub-semi-ring $\mathsf{E}\{X \mid B \subset X \subset C\}$. Then $\emptyset$ is the zero element and A is the unit element in $\mathscr{A}$, whereas B is the zero and C is the unit in $\mathscr{A}_{B,C}$. – (C) The structure $\langle \bigcup, \bigcap \rangle$ has no unit element, the zero element is $\emptyset$.

6 D.12. Definition. A semi-ring is called *unital* if it contains a unit element.

Example. The Boolean ring of all finite parts of a given set A is unital if and only if A is finite.

6 D.13. Definition. A semi-ring (or ring) structure $\langle \sigma, \mu \rangle$ is called *commutative* if μ is commutative. A semi-ring (a ring) is called *commutative* if its structure is commutative.

Examples. The structures in 6 D.2, examples (A), (B), (C), are commutative whereas the structure in 6 D.2, example (D), is, in general, not commutative.

E. HOMOMORPHISM-RELATIONS

Recall the following equalities proved at various places in the preceding sections, each of them under certain assumptions which need not be re-stated here: (1) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$; (2) $\varrho[X \cup Y] = \varrho[X] \cup \varrho[Y]$ where ϱ is a relation; (3) $E - (X \cup Y) = (E - X) \cap (E - Y)$; (4) $a(x + y) = a . x + a . y$

where a, x, y belong to N; (5) $a^{x+y} = a^x . a^y$ where a, x, y are elements of N. In all these cases, regardless of the differences, we have the following situation: there is given a composition ϱ on X (or on a class containing X), a composition σ on Y (or on a class containing Y) and a single-valued relation φ on X into Y such that $\varphi(x_1 \varrho x_2) = (\varphi x_1) \sigma (\varphi x_2)$ for any $x_1 \in X$, $x_2 \in X$ (the compositions in question are $\bigcup$ and $\bigcup$ in (1), $\bigcup$ and $\bigcup$ in (2), $\bigcup$ and $\bigcap$ in (3)).

Therefore it seems appropriate to examine single-valued relations satisfying, for certain compositions ϱ, σ, the condition $\varphi(x_1 \varrho x_2) = (\varphi x_1) \sigma (\varphi x_2)$. First we shall introduce a broader notion.

6 E.1. Definition. Let ϱ be a composition on a class X, and let σ be a composition on a class Y. Let φ be a relation for X and Y. If $x_1 \varphi y_1$, $x_2 \varphi y_2$ implies $(x_1 \varrho x_2) \varphi (y_1 \sigma y_2)$ for any $x_i \in X$, $y_i \in Y$, then we shall say that the relation φ is *stable under* ϱ *and* σ (or *with respect to* ϱ, σ) or else that φ is (ϱ, σ)-*stable*. If, in addition, φ is single-valued, and $\mathbf{D}\varphi = X$, then we shall say that φ is a *homomorphism-relation under* ϱ *and* σ or that φ is a (ϱ, σ)-*homomorphism-relation*. Finally, if φ is (ϱ, σ)-stable and bijective for X and Y, then φ is called an *isomorphism-relation under* ϱ *and* σ or a (ϱ, σ)-*isomorphism-relation*.

Remark. It is clear that φ is stable under ϱ and σ, in the above sense, if and only if it is stable under the composition $\varrho \times \sigma$ (see 6 E.9).

6 E.2. *If a relation* φ *is stable under compositions* ϱ *and* σ, *then* φ^{-1} *is stable under* σ *and* ϱ. *If* ϱ, σ, τ *are compositions,* φ *and* ψ *are relations,* φ *is stable (respectively, a homomorphism-relation) under* ϱ, σ, *and* ψ *is stable (respectively, a homomorphism-relation) under* σ *and* τ, *then* $\psi \circ \varphi$ *is stable (respectively, a homomorphism-relation) under* ϱ *and* τ.

The proof is easy and therefore omitted.

6 E.3. *If* ϱ, σ *are compositions and* φ *is stable under* ϱ *and* σ, *then* $\mathbf{D}\varphi$ *is stable* (see 6 A.7) *under* ϱ *and* $\mathbf{E}\varphi$ *is stable under* σ.

Proof. If $x_1 \in \mathbf{D}\varphi$, $x_2 \in \mathbf{D}\varphi$, then choose y_i with $x_i \varphi y_i$. Since φ is stable, we have, by definition, $(x_1 \varrho x_2)\ \varphi (y_1 \sigma y_2)$; hence $x_1 \varrho x_2 \in \mathbf{D}\varphi$. Similarly it can be shown that $\mathbf{E}\varphi$ is stable.

6 E.4. Definition. If σ is a composition on X and φ is a (σ, σ)-homomorphism-relation on X into X, then we shall say that φ is an *endomorphism-relation under* σ (or an *endomorphism-relation for* $\langle X, \sigma \rangle$). If, in addition, φ is one-to-one onto X, then φ is called an *automorphism-relation under* σ (or *for* $\langle X, \sigma \rangle$).

Example. Consider a finite set A and let $\bigcup$ be restricted to a composition σ on exp A. Then the endomorphism-relations under σ are exactly those relations of the form $\{X \to \varphi[X]\}$, where φ is an arbitrary relation in A, i.e. a relation such that $\mathbf{D}\varphi \subset A$, $\mathbf{E}\varphi \subset A$.

6 E.5. *If* X *is a set,* σ *is a composition on a set* $Z \supset X$, *then the set of all those* $\varphi \in X^X$ *which are endomorphism-relations under* σ *is a sub-semi-group of*

$\langle X^X, \circ\rangle$, and the set of all those $\varphi \in X^X$ which are permuting endomorphism-relations under σ is a subgroup of $\langle X^X, \circ\rangle$.

The proof, based on 6 E.2, is left to the reader.

Before proceeding further and introducing rings of homomorphism-relations, we are going to consider the product of compositions.

6 E.6. Definition. Let $\{\sigma_a \mid a \in A\}$ be a family of compositions, σ_a being a composition on a set X_a. Consider the composition on $X = \Pi X_a$ which assigns the element $\{x_a \sigma_a y_a \mid a \in A\} \in X$ to a pair $\langle x, y\rangle$, where $x = \{x_a \mid a \in A\} \in X$, $y = \{y_a \mid a \in A\} \in X$. This composition is called the *compositional product* of $\{\sigma_a\}$ or simply the *product* of the family of $\{\sigma_a\}$ of compositions and is denoted by $\Pi_{\text{comp}}\{\sigma_a \mid a \in A\}$ or simply by $\prod\limits_a \sigma_a$ etc.

Remark. It is easy to see that $\Pi_{\text{comp}}\{\sigma_a \mid a \in A\}$ is distinct, in general, from the usual cartesian product of the family $\{\sigma_a\}$ considered simply as a family of sets, as well as from the relational product (see 5 C.2) of $\{\sigma_a\}$ considered as a family of relations.

6 E.7. Definition. If σ is a composition on a class X and A is a set, then we shall call the *power* of σ with *exponent* A and denote by σ^A the composition on X^A which assigns the family $\{x_a \sigma y_a\}$ to $\langle\{x_a\}, \{y_a\}\rangle$.

Remarks. 1) Clearly, if σ is comprisable, then $\sigma^A = \Pi\{a \to \sigma \mid a \in A\}$. If $A = \emptyset$, σ is a composition, then $\sigma^A = (\langle\langle\emptyset, \emptyset\rangle, \emptyset\rangle)$ is a composition on the singleton $(\emptyset)$. — 2) The symbol σ is sometimes written instead of σ^A. This is done, as a rule, if the relation in question is denoted by + or by a dot, etc.; thus we write

$$\{x_a\} + \{y_a\} = \{x_a + y_a\}, \ \{x_a\} \cdot \{y_a\} = \{x_a \cdot y_a\} \text{ etc.}$$

Example. Let σ be a group structure on a two-element set T. Let A be a set. It is easy to see that there is a one-to-one relation on T^A onto $\exp A$ which is a homomorphism-relation under σ^A and the composition $\div$ restricted to $\exp A$.

6 E.8. *Let $\{\sigma_a \mid a \in A\}$ be a family of compositions. If every σ_a is associative (a group structure, commutative), then $\Pi\sigma_a$ is associative (a group structure, commutative). If, for any $a \in A$, μ_a is a composition which is distributive relative to σ_a, then $\Pi\mu_a$ is distributive relative to $\Pi\sigma_a$.*

Let σ, μ be compositions, and let A be a set. If σ is associative (a group structure. commutative), then σ^A is associative (a group structure, commutative). If μ is distributive relative to σ, then μ^A is distributive relative to σ^A. In particular, if $\langle\sigma, \mu\rangle$ is a semi-ring (ring) structure, then $\langle\sigma^A, \mu^A\rangle$ is also a semi-ring (ring) structure.

The reader is invited to carry out the easy proof and to formulate and prove various related propositions, e.g. to show that if e_a is neutral under σ_a, then $\{e_a\}$ is neutral under $\Pi\sigma_a$.

6 E.9. Definition. If σ_1, σ_2 are compositions on X_1 and X_2 respectively, then their *compositional product*, denoted by $\sigma_1 \times_{\text{comp}} \sigma_2$ or simply $\sigma_1 \times \sigma_2$ is the composition on $X_1 \times X_2$ which assigns to $\langle\langle x_1, x_2\rangle, \langle x_1', x_2'\rangle\rangle$ the element $\langle x_1\sigma_1 x_1', x_2\sigma_2 x_2'\rangle \in X_1 \times X_2$.

We leave to the reader the task of formulating and proving for this product propositions analogous to those valid for the compositional product of a family of compositions.

6 E.10. Proposition and definition. *If $\{\mathscr{G}_a\}$ is a family of semi-groups, $\mathscr{G}_a = \langle G_a, \sigma_a\rangle$, then $\langle \Pi G_a, \Pi\sigma_a\rangle$ is a semi-group which will be called the* product *of the family $\{\mathscr{G}_a\}$ and denoted by $\Pi\{\mathscr{G}_a\}$ or $\prod_a \mathscr{G}_a$. If $\{\mathscr{A}_b\}$ is a family of semi-rings, $\mathscr{A}_b = \langle A_b, \sigma_b, \mu_b\rangle$, then $\langle \Pi A_b, \Pi\sigma_b, \Pi\mu_b\rangle$ is a semi-ring which will be called the* product *of $\{\mathscr{A}_b\}$ and denoted by $\Pi\{\mathscr{A}_b\}$ or $\prod_b\{\mathscr{A}_b\}$, etc.*

The reader is invited to carry out the proof (based on 6 E.8) and to state similar propositions and definitions concerning groups and rings.

6 E.11. Proposition and definition. *Let B be a set. If $\mathscr{G} = \langle G, \sigma\rangle$ is a semi-group (group) and $\mathscr{A} = \langle A, \sigma, \mu\rangle$ is a semi-ring (ring), then (1) $\langle G^B, \sigma^B\rangle$ is a semi-group (group) equal to $\Pi\{b \to \mathscr{G} \mid b \in B\}$ (see 6 E.10), which will be called the* power *of $\mathscr{G}$ with* exponent *B and denoted by $\mathscr{G}^B$, and (2) $\langle A^B, \sigma^B, \mu^B\rangle$ is a semi-ring (ring) equal to $\Pi\{b \to \mathscr{A} \mid b \in B\}$ which will be called the* power *of $\mathscr{A}$ with* exponent *B and denoted by $\mathscr{A}^B$.*

6 E.12. Theorem. *Let σ be a commutative semi-group (respectively, group) structure on a class Y. Let ϱ be a composition on a set X. Consider the class H of all those $f \in Y^X$ which are homomorphism-relations under ϱ and σ. The restriction of σ^X to this class is a commutative semi-group (group) structure. If Y is a set, then $\langle H, \sigma^X\rangle$ (see 6 B.1, convention 2) is a semi-group (respectively, group).*

Proof. Since by 6 E.8, σ^X is a semi-group structure, it is sufficient to establish that H is stable under σ^X. Let $\varphi_1 \in H$, $\varphi_2 \in H$; consider $\varphi = \varphi_1\sigma^X\varphi_2 \in Y^X$. If $x \in X$, $x' \in X$, then $\varphi(x\varrho x') = (\varphi_1\sigma^X\varphi_2)(x\varrho x') = \varphi_1(x\varrho x')\,\sigma\varphi_2(x\varrho x') = ((\varphi_1 x)\,\sigma\,(\varphi_1 x'))\,\sigma$ $\sigma\,((\varphi_2 x)\,\sigma\,(\varphi_2 x'))$ which, since σ is commutative and associative, is equal to $((\varphi_1 x)\,\sigma(\varphi_2 x))\,\sigma\,((\varphi_1 x')\,\sigma\,(\varphi_2 x')) = (\varphi x)\,\sigma\,(\varphi x')$; thus, we have proved that φ is a homomorphism-relation. Therefore H is stable under σ^X. – The rest of the proof is left to the reader.

6 E.13. Theorem. *Let $\langle G, \sigma\rangle$ be a commutative semi-group (respectively, group). Then the set of all endomorphism-relations for $\langle G, \sigma\rangle$ is a semi-ring (respectively, ring) under $\langle \sigma^G, \circ\rangle$.*

Proof. By 6 E.12, σ^G is a commutative semi-group (group) structure. By 6 E.5, the composition $\circ$ is a semi-group structure which is clearly distributive with respect to σ^G.

6 E.14. Let $\langle X, \varrho\rangle$ and $\langle Y, \sigma\rangle$ be commutative semi-groups. Consider the set H of all those $f \in Y^X$ which are homomorphism-relations under ϱ and σ. By 6 E.12, H is a semi-group under σ^X, and even a group whenever $\langle Y, \sigma\rangle$ is a group. However, there is, in general, no natural composition on H making it, together with σ^X, a semi-ring (or even ring). On the other hand, there are two quite natural operations involving H: one of these assigns to every pair $\langle \varphi, \psi\rangle$, where φ is an endomorphism-relation of $\langle Y, \sigma\rangle$, ψ belongs to H, the element $\varphi \circ \psi$ from H, the second one assigns to every pair $\langle \psi, \chi\rangle$, where $\psi \in H$ and χ is an endomorphism-relation of $\langle X, \varrho\rangle$, the element $\psi \circ \chi$ from H. These "operations" satisfy, moreover, some natural conditions, namely $\varphi \circ (\psi_1 + \psi_2) = \varphi \circ \psi_1 + \varphi \circ \psi_2$, $(\varphi_1 + \varphi_2) \circ \psi = \varphi_1 \circ \psi + \varphi_2 \circ \psi$ where we have denoted, for convenience, by $+$ the "additive" composition in question.

The following apparently quite different example has much is common with the situation described above. Let $\mathscr{R}$ denote the class of all comprisable relations, let $\mathscr{S}$ denote the class of all sets and consider the relation $\{\langle \varrho, X\rangle \to \varrho[X] \mid \varrho \in \mathscr{R}, X \in \mathscr{S}\}$. This relation is single-valued with domain $\mathscr{R} \times \mathscr{S}$ and range in $\mathscr{S}$. It satisfies the equalities $\varrho[X_1 \cup X_2] = \varrho[X_1] \cup \varrho[X_2]$, $(\varrho_1 \cup \varrho_2)[X] = \varrho_1[X] \cup \varrho_2[X]$.

It is expedient to consider "operations" of such a kind in more detail. One may try to define them as single-valued relations on $A \times B$ into B satisfying certain conditions. However, a slightly different definition will prove to be more adequate and convenient.

F. MODULES

6 F.1. Definition. Let A, X be classes. If ϱ is a relation with domain A such that every $\varrho[(a)]$, $a \in A$, is a single-valued relation on X into X, we shall say that ϱ is an *external composition* (more explicitly, an *unstructured external composition*) on X over A (or with domain A). The class X will be occasionally called the *action-field* (or simply *field*) of ϱ, and we shall say that ϱ *acts* on X.

Examples. (A) If X is a set, then $\{f \to \langle x, fx\rangle \mid x \in X, f \in X^X\}$ is an external composition on X with domain X^X. — (B) $\{m \to \langle n, m \,.\, n\rangle \mid m \in \mathsf{N},\ n \in \mathsf{N}\}$ is an external composition on N with domain N. — (C) Let $\mathscr{A} = \langle A, \sigma, \mu\rangle$ be a ring. Let B be a set. Consider the relation $\{a \to \langle \{x_b\}, \{a\mu x_b\}\rangle \mid a \in A, \{x_b\} \in A^B\}$. This relation is an external composition with domain A and field A^B.

6 F.2. Convention. Let μ be an external composition over a class A on X. If $a \in A$, $x \in X$, we shall denote by $a\mu x$ the (unique) element y such that $\langle a, x, y\rangle \in \mu$.

6 F.3. Definition. Let μ be an external composition over a class A on X. Then a class $Y \subset X$ is called *absorbing under* μ if $a \in A$, $y \in Y \Rightarrow a\mu y \in Y$.

If Y is absorbing under μ, then $\mu \cap (A \times Y \times Y)$ is an external composition over A on Y; it will be called the *restriction* (more precisely *field-restriction*) of μ to an external composition on Y, and denoted by μ_Y if there is no danger of misunderstanding.

6 F.4. Before proceeding further, we shall point out the following important fact.

Very often,.simultaneously with an external composition in the above sense, there is given a structure on its domain; for instance, in example (C), there is a ring-structure on the domain A of the composition considered. As a matter of fact, such a structure is usually of a considerable importance. For this reason, we shall define a "richer" concept of an external composition endowed with a certain structure for its domain. However, at this stage, only a rather special definition is given, with merely semi-groups (or groups) and semi-rings (or rings) in view.

Definition. Let A be a class and let α be a semi-group or a semi-ring structure on A (thus α may also be a group or a ring structure). Let μ be an external composition over A (thus $\mathbf{D}\mu = A$) on a class X. Then $\langle \mu, \alpha \rangle$ will be called an *external composition* over the semi-group or semi-ring $\mathscr{A} = \langle A, \alpha \rangle$; we shall also say that $\langle \mu, \alpha \rangle$ is a *structured* (or *enriched*) *external composition*.

Example. If $\mathscr{A} = \langle A, \sigma, \mu \rangle$ is a ring, consider the relation μ^* consisting of all $\langle a, x, y \rangle$ such that $y = a\mu x$. Then μ^* is an external composition over A on A, and $\langle \mu^*, \sigma, \mu \rangle$ is a structured external composition over A on A.

We defer the examination of external compositions to Section 8. Only one special case will be considered now, namely that of an external composition over a ring $\mathscr{A}$ on a set X on which a certain group structure is given.

6 F.5. Definition. Let $\langle \alpha, \beta \rangle$ be a ring structure on a class A. Let μ be an external composition over A on a class X; let σ be a commutative group-structure on X. Then $\langle \sigma, \langle \mu, \alpha, \beta \rangle \rangle$ is called a *module structure* (more precisely, a *left module structure*) over $\mathscr{A} = \langle A, \langle \alpha, \beta \rangle \rangle$ on X, provided the following conditions are satisfied:

(1) if $a \in A$, $x \in X$, $y \in X$, then $a\mu(x\sigma y) = (a\mu x)\,\sigma(a\mu y)$,
(2) if $a \in A$, $b \in A$, $x \in X$, then $(a\alpha b)\,\mu x = (a\mu x)\,\alpha(b\mu x)$,
(3) if $a \in A$, $b \in A$, $x \in X$, then $(a\beta b)\,\mu x = a\mu(b\mu x)$.

Examples. (A) In the notation of 6 F.4, $\langle \sigma, \langle \mu^*, \sigma, \mu \rangle \rangle$ is a module structure over $\langle A, \sigma, \mu \rangle$ on A. – (B) Let $\mathscr{G} = \langle G, \varrho \rangle$, $\mathscr{H} = \langle H, \tau \rangle$ be commutative groups. Let $\langle X, \sigma \rangle$ be the group described in 6 E.12, of those single-valued relations on G into H which are homomorphism-relations under ϱ and τ, and let $\mathscr{A} = \langle A, \alpha, \beta \rangle$ be the ring of endomorphism-relations of $\mathscr{H}$ (see 6 E.13). Let μ consist of all pairs $\langle a, x, y \rangle$ where $a \in A$, $x \in X$, $y \in X$, $y = a \circ x$. Then $\langle \sigma, \langle \mu, \alpha, \beta \rangle \rangle$ is a module structure over $\mathscr{A}$ on X. – (C) In the notation of the preceding example, let $\mathscr{A}' = \langle A', \alpha', \beta' \rangle$ be the ring of endomorphism-relations of $\mathscr{G}$; let μ' consist of all pairs $\langle a', x, y \rangle$ where $a' \in A'$, $x \in X$, $y \in X$, $y = x \circ a'$. Then conditions (1), (2) in the above definition are satisfied, but condition (3) is not, except in special cases. However, we have $(a\beta' b)\,\mu' x = b\mu'(a\mu' x)$; thus, $\langle \sigma, \langle \mu', \alpha', \beta' \rangle \rangle$ is a so-called "right module structure". – (D) Let again $\mathscr{A} = \langle A, \sigma, \mu \rangle$ be a ring. Let B be a set. Consider the group $\mathscr{A}^B = \langle A^B, \sigma^B \rangle$ (see 6 E.11); denote by μ^* the relation consisting of all $\langle a, \{x_b\}, \{a\mu x_b\} \rangle$. Then $\langle \sigma^B, \langle \mu^*, \sigma, \mu \rangle \rangle$ is a module structure over $\mathscr{A}$ on A^B.

6 F.6. Convention. Let $\boldsymbol{m} = \langle\sigma, \langle\mu, \alpha, \beta\rangle\rangle$ be a module structure over $\mathscr{A}$ on X, and let $Y \subset X$. If $\langle\sigma_Y, \langle\mu_Y, \alpha, \beta\rangle\rangle$ is a module structure (over $\mathscr{A}$ on Y), then it will be denoted by $\boldsymbol{m}_Y$.

6 F.7. Definition. A pair $\langle X, \boldsymbol{m}\rangle$ where X is a non-void set, $\boldsymbol{m}$ is a module structure over $\mathscr{A}$ on X, will be called a *module over* $\mathscr{A}$ or an $\mathscr{A}$*-module*. Let $\boldsymbol{m} = \langle\sigma, \langle\mu, \alpha, \beta\rangle\rangle$ be a module structure over $\mathscr{A} = \langle \mathbf{D}\mu, \alpha, \beta\rangle$ and let X be a set; then X will be called a *module over* $\mathscr{A}$ under $\boldsymbol{m}$ if $\boldsymbol{m}_X = \langle\sigma_X, \langle\mu_X, \alpha, \beta\rangle\rangle$ is a module structure over $\mathscr{A}$ on X.

6 F.8. Definition. If $\langle X, \boldsymbol{m}\rangle$, $\langle Y, \boldsymbol{n}\rangle$ are modules over $\mathscr{A}$, $Y \subset X$, $\boldsymbol{n} = \boldsymbol{m}_Y$, then we shall say that $\langle Y, \boldsymbol{n}\rangle$ is a *submodule of* $\langle X, \boldsymbol{m}\rangle$ (or that $\langle Y, \boldsymbol{n}\rangle$ is *identically embedded in* $\langle X, \boldsymbol{m}\rangle$, see 6 B.4). A set Y will be termed a *submodule of* $\langle X, \boldsymbol{m}\rangle$ if $Y \subset X$ and Y is a module over $\mathscr{A}$ under $\boldsymbol{m}$.

6 F.9. Convention. Let $\boldsymbol{m} = \langle\sigma, \langle\mu, \alpha, \beta\rangle\rangle$ be a module structure. We shall say that σ is the *underlying additive structure* and μ is the *underlying external multiplicative structure* of $\boldsymbol{m}$. If $\mathscr{X} = \langle X, \boldsymbol{m}\rangle$ is a module, we shall say that σ is the *addition*, μ is the *external composition* of the module $\mathscr{X}$, $\langle X, \sigma\rangle$ will be called the *underlying additive group* etc. (cf. 6 D.4). The relation $\{\langle a, x\rangle \to a\mu x\}$ will be sometimes called the μ*-multiplication* (or the *external multiplication*) of the module $\mathscr{X}$.

6 F.10. Definition. If $\boldsymbol{m}$ is a module structure over $\mathscr{A}$ on X, then a class Y is called *stable under* $\boldsymbol{m}$ if $Y \subset X$ and Y is stable under the additive and absorbing under the external multiplicative structure of $\boldsymbol{m}$ (this last condition means that $a\mu y \in Y$ whenever a is an element of $\mathscr{A}$, $y \in Y$).

6 F.11. *If $\langle X, \boldsymbol{m}\rangle$ is a module, then a set $Y \subset X$ is a module under $\boldsymbol{m}$ (in other words, a submodule of $\langle X, \boldsymbol{m}\rangle$) if and only if it is stable under $\boldsymbol{m}$ and a subgroup under the additive structure of $\boldsymbol{m}$.*

6 F.12. *Let $\boldsymbol{m}$ be a given module structure. The intersection of any non-void family of modules under $\boldsymbol{m}$ (respectively, of sets stable under $\boldsymbol{m}$) is a module under $\boldsymbol{m}$ (respectively, stable).*

The proof of these both propositions is immediate.

Example. Consider the module described in 6 F.5, example (D). Let Y consist of all $\{x_b\} \in A^B$ such that $x_b = 0$ for all b except finitely many. Then Y is a submodule of the module in question.

6 F.13. *Let $\langle X, \boldsymbol{m}\rangle$ be a module. If $Y \subset X$, there exists exactly one submodule which is the smallest submodule containing Y.*

6 F.14. Definition. Let $\langle X, \boldsymbol{m}\rangle$ be a module, $Y \subset X$. Let Z be the smallest submodule containing Y. Then we shall say that Z (or also $\langle Z, \boldsymbol{m}\rangle$) is the *module generated by Y under $\boldsymbol{m}$* (or *in* $\langle X, \boldsymbol{m}\rangle$) and that Y *generates* (or is a *generating set* for) Z *as a module.*

Example. Consider the submodule described in 6 F.12 (example). For any $b \in B$ let φ_b be the single-valued relation which assigns to every $a \in A$ the element $\{x_c \mid c \in B\} \in A^B$ such that $x_b = a$, $x_c = 0$ for $c \neq b$. Then if $\{X_b\}$, $X_b \subset A$, is a family such that every X_b generates A (as a ring), then $\bigcup \varphi_b[X_b]$ generates the submodule in question.

6 F.15. Proposition and definition. *Let* $\{\mathscr{X}_b \mid b \in B\}$ *be a non-empty family of modules over a given ring* $\mathscr{A} = \langle A, \alpha, \beta \rangle$; *let* $\mathscr{X}_b = \langle X_b, \sigma_b, \mu_b, \alpha, \beta \rangle$. *Let* $X = \Pi X_b$, $\sigma = \Pi \sigma_b$ (*see* 6 E.10); *let* μ *consist of all* $\langle a, \{x_b\}, \{a\mu_b x_b\} \rangle$, *where* $a \in A$, $\{x_b\} \in X$. *Then* $\langle X, \sigma, \mu, \alpha, \beta \rangle$ *is a module over* $\mathscr{A}$. *It is called the* cartesian product of the family $\{\mathscr{X}_b\}$ and denoted by $\Pi\{\mathscr{X}_b\}$.

7. STRUCTURES AND CORRESPONDENCES

The purpose of this section has been described briefly in the introductory remarks to the present Chapter II.

It is to be pointed out that only a few non-trivial propositions are given here; the section consists, for the most part, of definitions, examples, etc. The role of the concepts introduced here will become clear gradually, e.g. in considerations involving interrelations of different kinds of spaces (topological, uniform, etc.).

Subsection 7 A requires a careful reading since the concepts introduced, although essentially well-known, are considered in an aspect and formulated in a form which is not quite usual. On the other hand, concepts such as correspondences, mappings, etc., considered in the following subsections, although defined in a somewhat unusual manner, are intuitively clear. Therefore, it may be convenient to go through these subsections quickly, noting the terminology and returning to each concept as soon as it appears in the subsequent sections.

A. STRUCTS

We begin by recalling some definitions from Section 6, stressing their "formal" aspect.

A semi-group or a group has been defined as a pair $\langle G, \sigma\rangle$ where G is a non-void set, σ is a composition satisfying certain conditions. A semi-ring (respectively, ring) is a pair $\langle A, \langle\sigma, \mu\rangle\rangle$ where A is a non-void set, $\langle\sigma, \mu\rangle$ is a semi-ring (respectively, ring) structure on A, i.e. a pair of compositions satisfying certain conditions. Pairs $\langle X, \alpha\rangle$ where X is a non-comprisable class, have also been mentioned in Section 6 (for instance, the class of all sets endowed with the semi-ring structure $\langle \bigcup, \bigcap\rangle$). In Section 10, ordered classes will be considered; these are pairs $\langle X, \varrho\rangle$, ϱ being an order on X, i.e. a relation in X satisfying certain conditions. Finally, to give a slightly more complicated example, a module over a ring $\mathscr{A}$ is a multiplet $\langle X, \sigma, \mu, \alpha, \beta\rangle = \langle X, \langle\sigma, \langle\mu, \alpha, \beta\rangle\rangle\rangle$ where X is a non-void set, $\sigma, \mu, \alpha, \beta$ are relations satisfying certain conditions.

Let us further recall that e.g. if $\mathscr{A} = \langle A, \alpha, \beta\rangle$ is a ring, then $\langle A, \alpha\rangle$ is called the underlying additive group of $\mathscr{A}$, $\langle A, \beta\rangle$ is called the underlying multiplicative semi-group of $\mathscr{A}$, A is called the underlying class of $\mathscr{A}$.

It will be of some use to introduce certain general concepts which may be applied to situations similar to those described above; in this way, the exposition can be simplified and relieved of repeated explicit statements of analogous definitions and propositions.

7 A.1. Definition. Every class X and every pair $\langle Y, \alpha\rangle$ where Y is a class will be called a *struct*. If $\mathscr{X}$ is a struct, two (mutually exclusive) cases are possible: if $\mathscr{X} = \langle X, \alpha\rangle$, where X is a class, then we shall say that X is the *underlying class of the struct* $\mathscr{X}$, α is the *structure of* $\mathscr{X}$, and $\mathscr{X}$ is *obtained by endowing* X *with* α; if $\mathscr{X} = X$ is a class, then X itself is called the *underlying class of* $\mathscr{X}$ and it is said that $\mathscr{X}$ *has no structure.* A struct will also be termed a *structured class* (thus, a structured class is either a class or a pair $\langle X, \alpha\rangle$ where X is a class). In particular, any pair $\langle \varrho, \alpha\rangle$ where ϱ is a relation, as well as any relation σ, will be called a *structured relation.*

The underlying class of a struct $\mathscr{X}$ will often be denoted by $|\mathscr{X}|$. If $\mathscr{X}$, $\mathscr{Y}$ are structs and ϱ is a relation for $|\mathscr{X}|$ and $|\mathscr{Y}|$ (respectively, on $|\mathscr{X}|$ into $|\mathscr{Y}|$ and so on; cf. 1 B.6), then we shall also say that ϱ is a relation *for* $\mathscr{X}$ and $\mathscr{Y}$ (respectively, *on* $\mathscr{X}$ *into* $\mathscr{Y}$, etc.).

In a struct $\langle X, \alpha\rangle$, the structure α may be itself an object built up in a rather complicated way. In most "practically" occurring cases α is obtained from certain relations, in particular compositions, by successive formation of pairs; e.g. the structure of a module is a regular multiplet, namely a quadruple of relations.

7 A.2. Convention. A struct and its underlying class will often be denoted by the same symbol; thus we shall often write X instead of $\langle X, \alpha\rangle$, $\mathscr{X}$ instead of $|\mathscr{X}|$.

For the sake of brevity, we shall sometimes omit the words "the underlying class of ..." or, conversely, "... endowed with ...", provided the proper meaning can be conveyed without ambiguity. Thus, we shall speak of elements and subclasses of a struct instead of elements and subclasses of the underlying class of a struct. If there is given a class X and it is clear from the context that we intend to consider X endowed with a certain structure α, then we shall speak of properties of X when those of $\langle X, \alpha\rangle$ are meant. For instance, if X is a set and it is clear that a semi-group structure σ on X is considered, we may say that X is commutative instead of saying that X endowed with σ is a commutative semi-group. Similarly, a notation such as $x \in \mathscr{X}$, $Y \subset \mathscr{X}$, etc., may be used (instead of the correct symbols $x \in |\mathscr{X}|$, $Y \subset |\mathscr{X}|$ and so on).

Observe that conventions of this kind have already been used in Section 6.

7 A.3. Let $\mathscr{X} = \langle X, \alpha\rangle$ be a struct. In many cases (for instance, if $\mathscr{X}$ is a semi-group or a semi-ring), there is a rule which determines what structs are considered as "substructs" of $\mathscr{X}$, in other words, a rule which determines, for certain subclasses $Y \subset X$, a structure β such that $\langle Y, \beta\rangle$ is considered as a "*substruct*" of $\mathscr{X}$. However, no such rule can be given *a priori*. Therefore we do not try to give a general definition of a "substruct"; nevertheless, occasionally we shall use this expression if it is obvious from the context that certain subclasses of $X = |\mathscr{X}|$ are conceived as endowed with a definite structure determined by that of $\mathscr{X}$.

7 A.4. It would be appropriate to give now a precise general meaning to clauses such as the underlying group of a module, a group endowed with an order, etc. This will be done here to a limited extent; a more complete treatment which is somewhat cumbersome (for reasons indicated in 3 F.13) will be found in the Notes at the end of the book.

We begin with an intuitive description. Suppose α, β may be expressed as multiplets (e.g. $\alpha = \langle X, \langle a, b\rangle, d, e\rangle$, $\beta = \langle X, d\rangle$) in such a way that the expression for β is obtained from that for α by cancelling symbols denoting objects $s, \ldots, t$ (in the above example, $\langle a, b\rangle$ and e are cancelled). Then we shall say that β is obtained from α by deleting $s, \ldots, t$ and that α is obtained by enriching β with $t, \ldots, s$. If α, β are structs with the same underlying class, we shall speak of endowing β with $t, \ldots, s$, as will be exactly defined below.

To give an example, $\langle X, a, b, c\rangle$ as well as $\langle X, \langle a, b\rangle, c\rangle$ is obtained by enriching $\langle X, a, b\rangle$ with c; $\langle X, a, b\rangle$ as well as $\langle X, b, a\rangle$ is obtained from $\langle X, a, b, a\rangle$ by deleting a.

The terminology just indicated agrees completely with that used (for special cases) in Section 6. It is to be noted, however, that, for instance, if $\mathscr{X} = \langle X, \sigma, \mu, \alpha, \beta\rangle$ is a module over $\mathscr{A} = \langle \mathbf{D}\mu, \alpha, \beta\rangle$, then it is not only $\langle X, \sigma\rangle$, $\langle X, \sigma, \mu\rangle$, etc., which is an underlying struct, but also the struct $\langle X, \alpha, \beta\rangle$, for which hardly any reasonable algebraic meaning can be found. This circumstance is not relevant, however, since underlying structs of such a kind, although not excluded a priori from the considerations, will never play an actual role in any reasoning.

7 A.5. If we try to give exact general definitions of notions indicated above, we encounter serious difficulties. For comprisable objects, exact definitions can be given; this has been done partly in 3 F.12 and will be completed below (7 A.7). However, the procedure indicated in Section 3 breaks down in the "non-comprisable case" because it involves, quite essentially, an infinite induction which cannot be carried out for non-comprisable objects (e.g. it is meaningless to speak of "sequences of non-comprisable objects").

On the other hand, notions such as the enriching of structures are used merely as a means for a unified description of facts which can be as well described, for each given kind of struct, without these concepts. There will occur, "in practice", a rather small number of different kinds of structs, and the structs considered will be formed from a small number of objects which cannot or need not be represented as pairs. Therefore it is sufficient, for our purpose, to define the concepts in question only for the case where the appropriate inductive procedure can be performed in at most k elementary steps, k being a fixed natural number ($k = 100$ will be more than enough). To avoid a lengthy reasoning here, this is deferred to the Notes at the end of the book, and only the "comprisable case" will be explicitly considered now.

7 A.6. The definitions given in this section will apply directly to the "comprisable case" only. The reader is invited to formulate, for the "non-comprisable case", under the limitation indicated above, the definitions corresponding to those stated explicitly for the "comprisable case".

We point out that concepts defined explicitly for the "comprisable case" will be used, whenever necessary, also in the "non-comprisable case".

7 A.7. Definition. Let α, β be elements. Let $\{c_k \mid k \in \mathsf{N}_n\}$ be a finite sequence. Let there exist a sequence $\{\alpha_k \mid k \in \mathsf{N}_{n+1}\}$ such that $\alpha_0 = \alpha$, $\alpha_n = \beta$ and, for each $k \in \mathsf{N}_n$, the element α_{k+1} is obtained by enriching α_k with c_k (see 3 F.12). Then we shall say that β is obtained by *enriching* α with $c_0, \ldots, c_{n-1}$ (or, more precisely, by enriching α with members of the sequence $\{c_k\}$), and that α is obtained from β by *deleting* $c_{n-1}, \ldots, c_0$ (or, more precisely, by deleting members of the sequence $\{c_k\}$). In particular, if $\alpha = \beta$, then β is obtained by enriching α with no elements (in other words, by enriching it with members of $\emptyset$).

If α, β are elements and there exists a finite sequence $\{c_k\}$ such that β is obtained by enriching α with members of $\{c_k\}$, we shall say that β is an *enrichment* of α or that α is *enrichable* to β.

7 A.8. Convention. Let the elements $\alpha = \langle X, a\rangle$, $\beta = \langle X, b\rangle$ be structs with the same underlying set X. Let β be obtained by enriching α with $c_0, \ldots, c_{n-1}$ (it is not difficult to prove that in this case b is obtained by enriching a with $c_0, \ldots, c_{n-1}$). Then we shall also say that β is *obtained by endowing α with* $c_0, \ldots, c_{n-1}$ (or, in a not quite correct manner, that β is equal to α endowed with $c_0, \ldots, c_{n-1}$), and that α *underlies* β or β *is underlaid by* α. The case $\alpha = \beta$ is included, as indicated above.

Remark. The expressions "α underlies β", "β is obtained by endowing α with $c_0, \ldots, c_{n-1}$" will be used solely if α, β are (possibly non-comprisable; see 7 A.6) structs with the same underlying class. Thus, if ξ, η are classes, then ξ as well as η are enrichable to $\langle \xi, \eta\rangle$; however, ξ underlies $\langle \xi, \eta\rangle$ whereas η does not.

Examples. (A) If $\mathscr{A} = \langle A, \sigma, \mu\rangle$ is a ring, there are at most four underlying structs, namely $\mathscr{A}$ itself; its underlying additive group $\langle A, \sigma\rangle$, obtained by deleting μ; the underlying multiplicative semi-group $\langle A, \mu\rangle$; the underlying class A. As a rule, all these structs are distinct; but it may happen that $\sigma = \mu$ (this is the case if and only if A is a singleton). – (B) If $\mathscr{M} = \langle M, \sigma, \langle \mu, \alpha, \beta\rangle\rangle$ is a module over a ring $\mathscr{A} = \langle \mathbf{D}\mu, \alpha, \beta\rangle$ then there are at most 16 underlying structs, e.g. $\mathscr{M}$ itself, the underlying additive group $\langle M, \sigma\rangle$, the underlying "group with operators" $\langle M, \sigma, \mu\rangle$ obtained by deleting from the structure of $\mathscr{M}$ the ring structure $\langle \alpha, \beta\rangle$, etc. – (C) A topological group (see 19 B) $\mathscr{X} = \langle X, \sigma, u\rangle$ is obtained by endowing a group $\langle X, \sigma\rangle$ with a topology u or a topological space $\langle X, u\rangle$ with a group structure σ, always under certain compatibility conditions for σ and u. Observe that, according to the definition adopted, the struct $\langle X, u, \sigma\rangle$ is also obtained by endowing $\langle X, \sigma\rangle$ with u. However, only the triple $\langle X, \sigma, u\rangle$, and not $\langle X, u, \sigma\rangle$, will be termed a topological group.

B. CORRESPONDENCES

We now proceed to the very important notions of mappings and correspondences. It is fairly clear that the concept of a single-valued relation is not suitable to express the idea of a mapping in the sense current in mathematics, for various properties ascribed to mapping cannot be properly conceived as those of relations. First, we cannot properly say that a single-valued relation is "onto" or is "into", whereas the concept of a mapping contains inherently the indication of what may be called the latent or potential range of the mapping, i.e. of the "class into which the mapping goes"; the actual range is a part of this class. Secondly, a single-valued relation as such cannot be considered e.g. as continuous; it can be thought of as continuous, or isomorphic, etc., only with regard to certain topologies, or certain group structures and so on. On the other hand, it is usual and expedient to consider such properties as intrinsic qualities of a mapping.

For these reasons, a mapping is often defined as a triple $\langle \varphi, A, B \rangle$ such that φ is a single-valued relation (usually comprisable), $A = \mathbf{D}\varphi$, $B \supset \mathbf{E}\varphi$; the explicit indication of A is superfluous in a sense, but consistent in view of the fact that a mapping may and will be conceived as a special case of a correspondence.

We shall now make a further step and also include in the concept of a mapping the structure with which A and B may be endowed. Then, a mapping may be defined as a triple $\langle \varphi, \mathscr{A}, \mathscr{B} \rangle$ such that φ is a single-valued relation, $\mathscr{A}$, $\mathscr{B}$ are structs and $\mathbf{D}\varphi = |\mathscr{A}|$, $\mathbf{E}\varphi \subset |\mathscr{B}|$ (see 7 A.1).

However, the notion of a mapping, even if conceived as above, is still not general enough in view of the important role played by "multi-valued mappings" in various considerations. For this reason, we are going to introduce a very general notion of a correspondence for structs which will include as special cases mappings of structs and correspondences for classes.

7 B.1. Definition. If $\mathscr{A}$, $\mathscr{B}$ are structs with underlying classes A, B, and ϱ is a relation for A and B, then the triple $\langle \varrho, \mathscr{A}, \mathscr{B} \rangle$ will be called a *correspondence for* $\mathscr{A}$ *and* $\mathscr{B}$ (or a *correspondence for* $\mathscr{A}$ *ranging in* $\mathscr{B}$). If $\boldsymbol{R} = \langle \varrho, \mathscr{A}, \mathscr{B} \rangle$ is a correspondence, then ϱ is called its *graph* and denoted by gr $\boldsymbol{R}$; $\mathscr{A}$ is called the *domain carrier* (or the *latent domain*) of $\boldsymbol{R}$ and denoted by $\mathbf{D}^*\boldsymbol{R}$; $\mathscr{B}$ is called the *range carrier* (or the *latent range*) of $\boldsymbol{R}$ and denoted by $\mathbf{E}^*\boldsymbol{R}$.

Observe that every correspondence $\boldsymbol{R} = \langle \varrho, \mathscr{A}, \mathscr{B} \rangle$ is a struct. Its underlying class $\varrho = \text{gr}\, \boldsymbol{R}$ may also be denoted by $|\boldsymbol{R}|$ (see 7 A.1).

Examples. (A) Let $\mathscr{X} = \langle X, \sigma, \mu \rangle$, $\mathscr{X}' = \langle X', \sigma', \mu' \rangle$ be rings. Let φ be a single-valued relation on X into X'. Then $\boldsymbol{F} = \langle \varphi, \mathscr{X}, \mathscr{X}' \rangle$ is a correspondence for $\mathscr{X}$ and $\mathscr{X}'$, $\boldsymbol{f} = \langle \varphi, \langle X, \sigma \rangle, \langle X', \sigma' \rangle \rangle$ is a correspondence for the groups $\langle X, \sigma \rangle$ and $\langle X', \sigma' \rangle$; $\boldsymbol{F}$ and $\boldsymbol{f}$ are different although they have the same graph φ. If φ is a homomorphism-relation under $\langle \sigma, \mu \rangle$ and $\langle \sigma', \mu' \rangle$, that is under σ and σ' as well as under μ and μ', it will be said (see Section 8) that $\boldsymbol{F} = \langle \varphi, \mathscr{X}, \mathscr{X}' \rangle$ is a homomorphism; if φ is a homomorphism-relation under σ and σ', it will be said that $f = \langle \varphi, \langle X, \sigma \rangle, \langle X', \sigma' \rangle \rangle$ is a homomorphism. – (B) Consider a composition μ on a class X, and put $\mathscr{X} = \langle X, \mu \rangle$. For $a \in X$, put $\sigma_a = \{x \to y \mid a\mu y = x\}$. Then $\boldsymbol{S}_a = \langle \sigma_a, X, X \rangle$ is a correspondence and, in general, its latent domain X is different from the domain $\mathbf{D}\sigma_a$ of its graph.

Convention. If $\boldsymbol{R}$ is a correspondence, then $\mathbf{D}$ gr $\boldsymbol{R}$, $\mathbf{E}$ gr $\boldsymbol{R}$ will sometimes be called, respectively, the *actual domain* and the *actual range* of $\boldsymbol{R}$.

7 B.2. Definition. A correspondence $\langle \varrho, \mathscr{A}, \mathscr{B} \rangle$ will be called an *abstract correspondence* if $\mathscr{A}$, $\mathscr{B}$ have no structure, i.e. if $\mathscr{A}$, $\mathscr{B}$ are classes. If $\boldsymbol{R} = \langle \varrho, \langle A, \alpha \rangle, \langle B, \beta \rangle \rangle$ is a correspondence, then $\langle \varrho, A, B \rangle$ will be called its *underlying abstract correspondence*; A will be called the *abstract domain carrier of* $\boldsymbol{R}$, and denoted, in accordance with 7 A.1, by $|\mathbf{D}^*\boldsymbol{R}|$; B will be called the *abstract range carrier of* $\boldsymbol{R}$ and denoted by $|\mathbf{E}^*\boldsymbol{R}|$.

7 B.3. Conventions. A correspondence and its graph will often be denoted by the same symbol (most often, a letter). Thus, if $\boldsymbol{R}$ is a correspondence, we shall often write $\boldsymbol{R}[X]$ instead of $(\text{gr } \boldsymbol{R})[X]$, or, provided gr $\boldsymbol{R}$ is single-valued at x, we shall write $\boldsymbol{R}x$ instead of $(\text{gr } \boldsymbol{R})\, x$; we shall speak of fibres and inverse fibres (see 1 B.7, 1 B.9) of a correspondence, etc. – Conversely, if ϱ is a given relation and the carriers $\mathscr{A}$, $\mathscr{B}$ (such that $\mathbf{D}\varrho \subset |\mathscr{A}|$, $\mathbf{E}\varrho \subset |\mathscr{B}|$) are clear from the context, then $\langle \varrho, \mathscr{A}, \mathscr{B} \rangle$ will occasionally be denoted simply by ϱ.

Moreover, let $\boldsymbol{R} = \langle \varrho, \mathscr{A}, \mathscr{B} \rangle$ be a correspondence; let $X \subset |\mathscr{A}|$. Suppose that it is clear from the context that certain subclasses of $|\mathscr{B}|$ are thought of as endowed with certain structures determined by $\mathscr{B}$, i.e. as "substructs" of $\mathscr{B}$ (see 7 A.3). Then, provided there is no danger of misunderstanding, we shall call the image of X under $\boldsymbol{R}$ not only the class $\boldsymbol{R}[X] = \varrho[X]$ but also this class $\varrho[X]$ endowed with a certain structure determined by $\mathscr{B}$. A similar convention applies, of course, for inverse images, fibres and inverse fibres.

7 B.4. Convention. Suppose that either $\boldsymbol{R}$ is a correspondence, $\varrho = \text{gr } \boldsymbol{R}$, or $\boldsymbol{R} = \varrho$ is a relation. Let $\mathscr{A}$, $\mathscr{B}$ be structs. We shall denote by $\boldsymbol{R} : \mathscr{A} \to \mathscr{B}$ the correspondence $\langle \sigma, \mathscr{A}, \mathscr{B} \rangle$ where σ is the relation $\varrho \cap (|\mathscr{A}| \times |\mathscr{B}|)$.

This notation will be used mainly in the case where ϱ is single-valued, $\mathbf{D}\varrho \supset |\mathscr{A}|$ and $\varrho[|\mathscr{A}|] \subset |\mathscr{B}|$; in such a case $\varrho : \mathscr{A} \to \mathscr{B}$ is equal to $\langle \varrho_{|\mathscr{A}|}, \mathscr{A}, \mathscr{B} \rangle$.

Example. Let R and C denote respectively the set of all real and all complex numbers. Then $\{x \to x^2 + 1\} : \mathsf{R} \to \mathsf{R}$, $\{x \to x^2 + 1\} : \mathsf{C} \to \mathsf{R}$ are different correspondences with different graphs; the actual range of the first of these is equal to the interval $[\![1, \to [\![$ (in the set R), that of the second one is equal to R.

Remark. In accordance with the above definition, an expression such as "some $\varphi : \mathscr{A} \to \mathscr{B}$" simply means "some correspondence for $\mathscr{A}$ and $\mathscr{B}$", the expression "let $\varphi = \varphi : \mathscr{A} \to \mathscr{B}$" means "let φ be such that φ is equal to $\varphi : \mathscr{A} \to \mathscr{B}$", that is "let φ be a correspondence for $\mathscr{A}$ and $\mathscr{B}$", etc.

7 B.5. Definition. If $F = \langle \varphi, A, B\rangle$ is an abstract correspondence, C is a class, then $\langle \varphi_C, A \cap C, B\rangle$, that is $\varphi: A \cap C \to B$, will be called the *domain-restriction of F to C*, and similarly for the *range-restriction* and *restriction* (cf. 1 B.10).

If F and G are abstract correspondences, G is a domain-restriction of F, then we shall say that F is a *domain-extension* of G, and similarly for the *range-extension* and *extension*. If it is clear from the context which kind of restriction or extension is considered, we shall often speak e.g. of restriction instead of domain-restriction, of extension instead of domain-extension.

Convention. Let $\mathscr{A}$ be a struct and let $\mathscr{X}$ be a "substruct" of $\mathscr{A}$ (see 7 A.3). If $F = \langle \varphi, \mathscr{A}, \mathscr{B}\rangle$ is a correspondence, then $\varphi: \mathscr{X} \to \mathscr{B}$, that is $\langle \varphi_{|\mathscr{X}|}, \mathscr{X}, \mathscr{B}\rangle = \langle \varphi \cap (|\mathscr{X}| \times |\mathscr{B}|), \mathscr{X}, \mathscr{B}\rangle$ will be called the *domain-restriction of F to $\mathscr{X}$* and will be denoted by $F_{\mathscr{X}}$. An explicit formulation of similar conventions concerning range-restriction, etc., is left to the reader.

7 B.6. Let R be a correspondence and let S be equal to R: $\mathscr{X} \to \mathscr{Y}$. In general, $\mathscr{X}$ an $\mathscr{Y}$ may be quite arbitrary and the graph of S may be any restriction of gr R (of course, gr S is determined by $|\mathscr{X}|$ and $|\mathscr{Y}|$, since it is equal to gr $R \cap (|\mathscr{X}| \times |\mathscr{Y}|)$). However, two "extreme" cases are important: (1) the case, described above, when the structure of $\mathscr{X} = \mathbf{D}^*S$, $\mathscr{Y} = \mathbf{E}^*S$ is determined by that of $\mathbf{D}^*R$, $\mathbf{E}^*R$, namely when $\mathbf{D}^*S$ is a substruct of $\mathbf{D}^*R$, $\mathbf{E}^*S$ is a substruct of $\mathbf{E}^*R$, (2) the case gr S = gr R which we are now going to consider.

Definition. Let R be a correspondence; let S be equal to $R: \mathscr{X} \to \mathscr{Y}$. If the underlying classes of $\mathbf{D}^*S$ and $\mathbf{D}^*R$ as well as those of $\mathbf{E}^*S$ and $\mathbf{E}^*R$ coincide, then we shall say that S is a *transpose* of R.

Clearly, each of the following conditions is necessary and sufficient for S to be a transpose of R: (a) abstract correspondences underlying R and S coincide; (b) $|\mathbf{D}^*R| = |\mathbf{D}^*S|$, $|\mathbf{E}^*R| = |\mathbf{E}^*S|$.

An instance of transposes has occurred in 7 B.1: the correspondences F and f from the example (A) are transposes of each other.

7 B.7. Definition. A correspondence $R = \langle \varrho, \mathscr{A}, \mathscr{B}\rangle$ is called *domain-full* if $\mathbf{D}\varrho = |\mathscr{A}|$ (i.e. if the actual range of R coincides with the underlying class of its latent range); it is called *range-full* if $\mathbf{E}\varrho = |\mathscr{B}|$.

Example. Consider 7 B.1, example (B). Clearly every S_a is range-full. If $\langle X, \mu\rangle = \langle \mathsf{N}, +\rangle$, then only S_0 is domain-full. If $\langle X, \mu\rangle = \langle \mathsf{N}, .\rangle$, then only S_1 is domain-full.

7 B.8. Conventions. Let $R = \langle \varrho, \mathscr{A}, \mathscr{B}\rangle$ be a correspondence. Parallelling 1 B.6, we are going to introduce terms indicating various combinations of the properties just introduced. According to 7 B.1, we say that R is a *correspondence for $\mathscr{A}$ and $\mathscr{B}$*. If $\mathbf{D}\varrho = |\mathscr{A}|$, i.e. if R is domain-full, we shall say that R is a *correspondence on $\mathscr{A}$ ranging in $\mathscr{B}$*; if $\mathbf{E}\varrho = |\mathscr{B}|$, we shall say that R is a *correspondence for $\mathscr{A}$ ranging on $\mathscr{B}$*; finally, if $\mathbf{D}\varrho = |\mathscr{A}|$, $\mathbf{E}\varrho = |\mathscr{B}|$, we shall say that R is *on $\mathscr{A}$ onto $\mathscr{B}$*.

If $\mathscr{A} = \mathscr{B}$, we shall say that $\boldsymbol{R}$ is a correspondence *in* $\mathscr{A}$ (or *for* $\mathscr{A}$); if, moreover, $\mathbf{D}\varrho = |\mathscr{A}|$, we shall say, of course, that $\boldsymbol{R}$ is *on* $\mathscr{A}$ *into* $\mathscr{A}$; if, in addition, $\mathbf{E}\varrho = |\mathscr{A}|$, we shall say that $\boldsymbol{R}$ is *on* $\mathscr{A}$ *onto* $\mathscr{A}$.

7 B.9. Many (but not all) properties first defined only for relations will be carried over to correspondences (usually by the following rule: $\boldsymbol{R}$ is said to possess the property $\boldsymbol{P}$ if and only if gr $\boldsymbol{R}$ possesses the property $\boldsymbol{P}$). However, this rule is to be applied with care; in most cases, its use will be explicitly indicated, unless clear from the context. We mention here explicitly two properties: a correspondence $\boldsymbol{R}$ is called *single-valued* (respectively, *one-to-one*) if its graph is single-valued (respectively, one-to-one).

7 B.10. Definition. A correspondence $\boldsymbol{F}$ is called a *mapping* if it is domain-full and single-valued. A mapping $\boldsymbol{F}$ is called *injective* if it is one-to-one, *surjective* if it is range-full, *bijective* if it is both injective and surjective. A bijective mapping $\langle \varphi, \mathscr{A}, \mathscr{B} \rangle$ is called a *permutation* if $\mathscr{A} = \mathscr{B}$.

Examples. (A) Let $\mathscr{S}$ be the class of all sets. Consider the mapping $\boldsymbol{F} = \{X \to X \cap Y\} : \mathscr{S} \to \mathscr{S}$ where Y is a fixed class. If Y is the universal class, then $\boldsymbol{F}$ is bijective; if not, then $\boldsymbol{F}$ is neither injective nor surjective. – (B) Let ϱ be a relation. Consider the single-valued correspondence $\boldsymbol{G} = \{X \to \varrho[X]\} : \mathscr{S} \to \mathscr{S}$ (see the above example). Then $\boldsymbol{G}$ is a mapping if and only if every fibre of ϱ is comprisable; $\boldsymbol{G}$ is range-full if and only if there exists a class $A \subset \mathbf{D}\varrho$ such that the domain-restriction σ of ϱ to A is one-to-one and $\mathbf{E}\sigma$ is the universal class.

7 B.11. Let $\mathscr{A}$, $\mathscr{B}$ be structs. In many cases (see 6 E.1) there is a rule determining what bijective $\varphi : \mathscr{A} \to \mathscr{B}$ are considered as "isomorphisms" and which injective $\psi : \mathscr{A} \to \mathscr{B}$ are considered as "embeddings" (cf. 6 B.4).

It is difficult to describe isomorphisms in an exact manner sufficiently general to include most of the important cases (however, this can be done for algebraic structs, see Section 8). We shall only describe embeddings in terms of isomorphisms and substructs (see 7 A.3) without giving a formal definition.

As a rule, an injective mapping $\boldsymbol{F} = \langle \varphi, \mathscr{A}, \mathscr{B} \rangle$ will be called an embedding (of $\mathscr{A}$ into $\mathscr{B}$) provided (1) the image (see 7 B.3) $\boldsymbol{F}[\mathscr{A}]$ may be considered as a substruct $\mathscr{C}$ of $\mathscr{B}$, and (2) $\boldsymbol{F} : \mathscr{A} \to \mathscr{C}$ is an isomorphism.

We point out, in addition, that an expression such as "$\mathscr{X}$ is embedded in $\mathscr{Y}$" means that there is given an embedding of $\mathscr{X}$ into $\mathscr{Y}$; if we say that $\mathscr{X}$ may be embedded in $\mathscr{Y}$, then this means that there exists an embedding of $\mathscr{X}$ into $\mathscr{Y}$.

Parallelling 1 D.3, we introduce the following definition which will be properly motivated somewhat later in this section.

7 B.12. Definition. A correspondence $\boldsymbol{R} = \langle \varrho, \mathscr{A}, \mathscr{B} \rangle$ will be called a *fibration* if it is range-full and its fibres are disjoint (in other words, ϱ^{-1} is single-valued). In particular, we shall say that $\boldsymbol{R}$ is a *fibration over* $\mathscr{A}$ or, if $\mathbf{D}\varrho = |\mathscr{A}|$, that it is a *fibration upon* $\mathscr{A}$.

C. OPERATIONS ON CORRESPONDENCES

7 C.1. Definition. Let R, S be correspondences. If the domain carrier $\mathbf{D}^*R$ of R coincides with the range carrier $\mathbf{E}^*S$ of S, then we shall denote by $R \circ S$ and call the *composite of R and S* the correspondence $\langle \operatorname{gr} R \circ \operatorname{gr} S, \mathbf{D}^*S, \mathbf{E}^*R \rangle$; in this case (i.e. if $\mathbf{D}^*R = \mathbf{E}^*S$) we shall often say that "$R \circ S$ is defined". If the condition $\mathbf{D}^*R = \mathbf{E}^*S$ is not satisfied, then the expression "$R \circ S$" has no meaning (is not defined). – If R is a correspondence, we shall call its *inverse*, and denote by R^{-1}, the correspondence $\langle (\operatorname{gr} R)^{-1}, \mathbf{E}^*R, \mathbf{D}^*R \rangle$.

Remarks. 1) Observe that $R^{-1} \circ R$ is a correspondence in $\mathbf{D}^*R$; its actual domain as well as its actual range are equal to $\mathbf{D} \operatorname{gr} R$. – 2) Clearly, R is a mapping if and only if R^{-1} is a fibration.

7 C.2. *Let R, S, T be correspondences. If $(R \circ S) \circ T$ is defined (more precisely, if $\mathbf{D}^*R = \mathbf{E}^*S$, $\mathbf{D}^*S = \mathbf{E}^*T$) then also $R \circ (S \circ T)$ is defined and is equal to $(R \circ S) \circ T$. If $R \circ S$ and $S \circ T$ are defined, then $(R \circ S) \circ T = R \circ (S \circ T)$. Finally, $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$ whenever $R \circ S$ exists.*

The proof is clear and therefore omitted.

7 C.3. *Let $\mathbf{P}$ be one of the following properties of a correspondence Φ: Φ is comprisable, domain-full, range-full, single-valued, one-to-one, a mapping, a fibration, injective, surjective, bijective. If both R and S possess property $\mathbf{P}$, and $R \circ S$ is defined, then $R \circ S$ also possesses property $\mathbf{P}$.*

We prove the first assertion leaving the rest to the reader. If $R = \langle \varrho, \mathscr{B}, \mathscr{C} \rangle$, $S = \langle \sigma, \mathscr{A}, \mathscr{B} \rangle$ are comprisable, then $\mathbf{D}\varrho$ and $\mathbf{E}\sigma$ are comprisable, and therefore $\varrho \circ \sigma$ is comprisable too, since $\mathbf{D}(\varrho \circ \sigma) \subset \mathbf{D}\sigma$, $\mathbf{E}(\varrho \circ \sigma) \subset \mathbf{E}\varrho$, $\varrho \circ \sigma \subset \mathbf{D}(\varrho \circ \sigma) \times \times \mathbf{E}(\varrho \circ \sigma) \subset \mathbf{D}\sigma \times \mathbf{E}\varrho$. Evidently, $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ are comprisable, hence $R \circ S = = \langle \varrho \circ \sigma, \mathscr{A}, \mathscr{C} \rangle$ is comprisable.

7 C.4. *If R is a (comprisable) correspondence, then there exists a (comprisable) fibration F and a (comprisable) mapping M such that $M \circ F = R$.*

Proof. If $R = \langle \varrho, \mathscr{A}, \mathscr{B} \rangle$, put $F = \langle \varphi, \mathscr{A}, \varrho \rangle$, $M = \langle \mu, \varrho, \mathscr{B} \rangle$ where $\varphi = = \{x \to \langle x, y \rangle \mid \langle x, y \rangle \in \varrho\}$, $\mu = \{\langle x, y \rangle \to y \mid \langle x, y \rangle \in \varrho\}$.

7 C.5. Definition. If $\{R_a \mid a \in A\}$ is a family of abstract correspondences (thus $\mathbf{D}^*R_a$, $\mathbf{E}^*R_a$ are sets), then their *product*, denoted by $\Pi\{R_a \mid a \in A\}$ or simply by $\Pi\{R_a\}$, is defined as $\langle \Pi\{\varrho_a\}, \Pi\{X_a\}, \Pi\{Y_a\} \rangle$ where $\varrho_a = \operatorname{gr} R_a$, $X_a = \mathbf{D}^*R_a$, $Y_a = = \mathbf{E}^*R_a$ and $\Pi\{\varrho_a\}$ is, of course, the relational product in the sense of 5 C.2.

Convention. If all R_a are equal to a fixed correspondence R, then R^A is written instead of $\Pi\{R_a\}$.

Remarks. 1) The reader is invited to formulate the definition of the product of finitely many abstract correspondences, possibly non-comprisable (cf. 7 A.6). – 2) We have limited the definition to abstract correspondences, i.e. to the case where the domain and range carriers are classes. Observe that "$\Pi\{R_a\}$" has, in general, no meaning if R_a are arbitrary comprisable correspondences; for each kind of the carrying structs $\mathbf{D}^*R_a$, $\mathbf{E}^*R_a$, a special definition is required.

7 C.6. Definition. Let $\{R_a \mid a \in A\}$ be a family of correspondences with the same domain carrier $\mathscr{X} = \mathbf{D}^*R_a$ and the same actual domain $Z = \mathbf{D}\,\mathrm{gr}\,R_a$; suppose that $\mathbf{E}^*R_a$ have no structure (hence are sets). Then $\langle \Pi_{\mathrm{red}}\varrho_a, \mathscr{X}, \Pi Y_a \rangle$ where $\varrho_a = \mathrm{gr}\,R_a$, $Y_a = \mathbf{E}^*R_a$, is called the *reduced product* of $\{R_a\}$ and denoted by $\Pi_{\mathrm{red}}\{R_a \mid a \in A\}$ or $\Pi_{\mathrm{red}}\{R_a\}$ or even simply by $\Pi\{R_a\}$, etc.

Remarks made above (7 C.5) also apply here, with appropriate modification.

7 C.7. *Let $\{R_a \mid a \in A\}$ where $A \neq \emptyset$ be a family of abstract correspondences; let $R = \Pi\{R_a\}$ (in the sense of 7 C.5). Then $\mathbf{D}^*R = \Pi\{\mathbf{D}^*R_a\}$, $\mathbf{E}^*R = \Pi\{\mathbf{E}^*R_a\}$, $\mathbf{D}\,\mathrm{gr}\,R = \Pi\{\mathbf{D}\,\mathrm{gr}\,R_a\}$, $\mathbf{E}\,\mathrm{gr}\,R = \Pi\{\mathbf{E}\,\mathrm{gr}\,R_a\}$. – Suppose, moreover, that all $\mathrm{gr}\,R_a$ are non-void. Then R is single-valued (one-to-one, domain-full, range-full) if and only if every R_a has the property in question.*

Remark. If $R = \Pi_{\mathrm{red}}R_a$ (which means, in particular, that all $\mathbf{D}^*R_a$ are mutually equal, as well as all $\mathbf{D}\,\mathrm{gr}\,R_a$), then equalities such as $\mathbf{E}\,\mathrm{gr}\,R = \Pi\{\mathbf{E}\,\mathrm{gr}\,R_a\}$, etc., do not hold in general; in particular, if R_a are surjective, $\Pi_{\mathrm{red}}R_a$ need not be surjective.

7 C.8. Definition. Let $R = \langle \varrho, \mathscr{A}, \mathscr{B} \rangle$ be a correspondence. If $\mathscr{C}$ is a struct, $|\mathscr{C}| \subset |\mathscr{A}|$ and s is a mapping on $\mathscr{C}$ into $\mathscr{B}$ such that $\mathrm{gr}\,s$ is a section (in the sense of 1 D.5) of $\mathrm{gr}\,R = \varrho$, then we shall say that s is a *section* of R. – A section of R^{-1} will be called, in accordance with 1 D.5, a *cross-section* of R.

7 C.9. Before giving an example of sections (of a fibration) we should point out that the concept of a fibration is, in a sense, redundant; we could avoid it and speak simply of inverses of mappings or even replace statements concerning fibrations by their counterparts concerning mappings.

However, the intuitive connotation of these concepts is essentially different. A mapping (or a mapping relation) is thought of as expressing the process of transforming certain objects, obtaining their "images", etc.; the inverse of such a correspondence may be conceived as expressing the inverse transformation process, the finding of "pre-images", etc. On the other hand, a fibration is thought of as expressing the following idea: every element ("point") of a given class is equipped with certain elements (in general, of a quite different kind); these elements may be conceived as possible values of a variable at a given point or possible situations in some sense, etc. If to every point there is assigned a certain unique value from among all the possible ones, then a section is obtained.

One of the most well-known examples of a fibration and its sections is the following one. Let M be a differentiable manifold. Put $x\tau\xi$ if and only if ξ is a tangent vector for the manifold M at x. Then $\langle \tau, M, B \rangle$, where B denotes $\mathbf{E}\tau$ endowed with a suitable structure, is a fibration upon M. Its sections on $M_1 \subset M$ are correspondences assigning (uniquely) to every $x \in M_1$ a certain tangent vector at x (as a rule, only differentiable sections are of interest). In this case, the idea of equipping points with vectors in various ways is clearly prevalent over that of assigning to every vector ξ in question its "origin" x.

The above example is purely illustrative since it involves concepts which will not be introduced in this book. The following one can be formulated in an exact manner.

Let σ be a semi-group structure on G; put $\mathscr{G} = \langle G, \sigma \rangle$. If $M \subset G$ is not void, denote by M^* the class of all non-empty finite sequences of elements of M; denote by $\mathscr{M}^*$ the class M^* endowed with the composition (denoted by μ here) which assigns $\{a_0, \ldots, a_n, b_0, \ldots, b_m\}$ to the pair of sequences $\{a_0, \ldots, a_n\}$, $\{b_0, \ldots, b_m\}$. Let φ be defined recursively as follows: (a) $\varphi\{x\} = x$ for any $x \in M$, (b) for any $\xi \in M^*$, $x \in M$, $\varphi(\{x\} \mu \xi) = x \mu(\varphi \xi)$. Then φ is a homomorphism-relation under μ and σ, hence φ^{-1} is composition-stable (see 6 E.1) under σ and μ; $\mathscr{R} = \langle \varphi^{-1}, \mathscr{G}, \mathscr{M}^* \rangle$ is a fibration correspondence. A section s of $\mathscr{R}$ assigns to every element $g \in \mathbf{D}s$ one of its "representations" as a „product" of elements of M. It is easy to see that a section s on a semi-group $H \subset G$ is a homomorphism, i.e. gr s is a homomorphism-relation, in exceptional cases only (e.g., if $M = (a)$, G consists of all a^n).

D. MAPPINGS

In this subsection some facts concerning classes of comprisable correspondences, mainly mappings, are considered.

7 D.1. As pointed out above (7 B.9), many properties are carried over to correspondences, in particular mappings, from their graphs. Thus, we have defined explicitly single-valued and one-to-one correspondences, including mappings, and some other properties (see 7 B.10). Now we introduce homomorphisms of semi-groups and semi-rings and related notions (for a more detailed treatment see Section 8).

Definition. If $\mathscr{G} = \langle G, \sigma \rangle$, $\mathscr{H} = \langle H, \tau \rangle$ are classes endowed with semi-group structures, $F = \langle \varphi, \mathscr{G}, \mathscr{H} \rangle$ is a mapping and φ is a homomorphism-relation under σ and τ, then we shall say that F is a *homomorphism*.

If $\mathscr{A} = \langle A, \sigma, \mu \rangle$, $\mathscr{A}' = \langle A, \sigma', \mu' \rangle$ are classes endowed with semi-ring structures, $F = \langle \varphi, \mathscr{A}, \mathscr{A}' \rangle$ is a mapping and φ is a homomorphism-relation under $\langle \sigma, \mu \rangle$ and $\langle \sigma', \mu' \rangle$, that is under σ and σ' as well as under μ and μ' (cf. 8 A.4), then F is called a *homomorphism.*

7 D.2. Definition. A homomorphism (see above) $F = \langle \varphi, \mathscr{X}, \mathscr{Y} \rangle$ will be called an *endomorphism* if $\mathscr{X} = \mathscr{Y}$. A bijective endomorphism is called an *automorphism.*

7 D.3. Definition. If $\mathscr{X}, \mathscr{Y}$ are comprisable structs, then the set of all correspondences $\varrho : \mathscr{X} \to \mathscr{Y}$ will be denoted by $\mathbf{Corr}(\mathscr{X}, \mathscr{Y})$.

We point out, in particular, that $\mathbf{Corr}(\mathscr{X}, \mathscr{Y})$ consists of all correspondences for $\mathscr{X}$ and $\mathscr{Y}$, irrespective of their properties.

7 D.4. Proposition and definition. *Let $\mathscr{X}$, $\mathscr{Y}$ be comprisable structs. The relation which assigns* gr R *to* $R \in \mathbf{Corr}(\mathscr{X}, \mathscr{Y})$ *is bijective for* $\mathbf{Corr}(\mathscr{X}, \mathscr{Y})$ *and* $\exp(X \times Y)$, *where* $X = |\mathscr{X}|$, $Y = |\mathscr{Y}|$; *it will be called* canonical *for* $\mathbf{Corr}(\mathscr{X}, \mathscr{Y})$ *and* $\exp(X \times Y)$. *Its inverse, also called canonical (for* $\exp(X \times Y)$ *and* $\mathbf{Corr}(\mathscr{X}, \mathscr{Y})$) *assigns to every* $\varrho \subset X \times Y$ *its transpose* $\varrho : \mathscr{X} \to \mathscr{Y} = \langle \varrho, \mathscr{X}, \mathscr{Y} \rangle$.

If $\mathscr{X}'$, $\mathscr{Y}'$ *are structs,* $|\mathscr{X}| = |\mathscr{X}'|$, $|\mathscr{Y}| = |\mathscr{Y}'|$, *consider the relation which assigns to a correspondence* R *for* $\mathscr{X}$ *and* $\mathscr{Y}$ *its transpose* $R : \mathscr{X}' \to \mathscr{Y}'$. *This relation is bijective for* $\mathbf{Corr}(\mathscr{X}, \mathscr{Y})$ *and* $\mathbf{Corr}(\mathscr{X}', \mathscr{Y}')$; *it will be called* canonical *for* $\mathbf{Corr}(\mathscr{X}, \mathscr{Y})$ *and* $\mathbf{Corr}(\mathscr{X}', \mathscr{Y}')$.

All above assertions are evident. They express the possibility of transition, in a specified sense, from relations to correspondences, etc.

7 D.5. Convention. We have introduced canonical relations for certain classes at various places (see e.g. subsection 5 D). If the classes in question are endowed with certain structures, then the correspondence (as a rule, a bijective correspondence) obtained will usually be called a *canonical correspondence* (or *mapping*, as the case may be).

7 D.6. *If* $\mathscr{X}$ *is a comprisable struct,* $X = |\mathscr{X}|$, *then* $\mathbf{Corr}(\mathscr{X}, \mathscr{X})$ *endowed with the composition of correspondences (see* 7 C.1) *is a semi-group. The canonical mapping (see* 7 D.5, 7 D.4) *of* $\mathbf{Corr}(\mathscr{X}, \mathscr{X})$ *onto* $\exp(X \times X)$ *endowed with the composition of relations is a bijective homomorphism (hence an isomorphism, see* 8 C.3).

This follows from the equality $\mathrm{gr}(R \circ S) = \mathrm{gr}\, R \circ \mathrm{gr}\, S$.

7 D.7. Definition. If $\mathscr{X}$, $\mathscr{Y}$ are comprisable structs, then the set of all mappings of $\mathscr{X}$ into $\mathscr{Y}$ will be denoted by $\mathbf{F}(\mathscr{X}, \mathscr{Y})$.

7 D.8. *Let* $\mathscr{X}$, $\mathscr{Y}$ *be comprisable structs,* $X = |\mathscr{X}|$, $Y = |\mathscr{Y}|$. *The relation assigning* gr f *to* $f \in \mathbf{F}(\mathscr{X}, \mathscr{Y})$ *is bijective for* $\mathbf{F}(\mathscr{X}, \mathscr{Y})$ *and* Y^X; *its inverse assigns to every* $\varphi \in Y^X$ *its transpose* $\varphi : \mathscr{X} \to \mathscr{Y}$. *If* $\mathscr{X}'$, $\mathscr{Y}'$ *are structs,* $|\mathscr{X}'| = |\mathscr{X}|$, $|\mathscr{Y}'| = |\mathscr{Y}|$, *consider the relation such that* $f \in \mathbf{F}(\mathscr{X}, \mathscr{Y})$ *is assigned to its transpose* $f : \mathscr{X}' \to \mathscr{Y}'$. *This relation is bijective for* $\mathbf{F}(\mathscr{X}, \mathscr{Y})$ *and* $\mathbf{F}(\mathscr{X}', \mathscr{Y}')$; *the relations described above will be called* canonical *for the respective sets.*

This is a complete analogue of 7 D.4, and the canonical relations in question are restrictions of those described in 7 D.4.

7 D.9. *Let* $\mathscr{X}$ *be a comprisable struct. Then* $\mathbf{F}(\mathscr{X}, \mathscr{X})$ *is a sub-semi-group of the semi-group* $\mathbf{Corr}(\mathscr{X}, \mathscr{X})$ *(see* 7 D.6*) and the set of all permuting mappings of* $\mathscr{X}$ *onto* $\mathscr{X}$ *is a subgroup of* $\mathbf{Corr}(\mathscr{X}, \mathscr{X})$.

We conclude with a somewhat less evident assertion.

7 D.10. *Let* $\mathscr{X}, \mathscr{Y}, \mathscr{X}', \mathscr{Y}'$ *be comprisable structs. Let* $u \in \mathbf{F}(\mathscr{X}', \mathscr{X})$, $v \in \mathbf{F}(\mathscr{Y}, \mathscr{Y}')$. *Then* $\{f \to v \circ f \circ u\}$ *maps* $\mathbf{F}(\mathscr{X}, \mathscr{Y})$ *into* $\mathbf{F}(\mathscr{X}', \mathscr{Y}')$. *This mapping is* (a) *injective if and only if* u *is surjective and* v *is injective,* (b) *surjective if and only if* u *is injective and* v *is surjective,* (c) *bijective if and only if both* u *and* v *are bijective.*

We shall prove assertion (a) only. Clearly, we can suppose that $\mathscr{X}, \mathscr{Y}, \mathscr{X}', \mathscr{Y}'$ are sets, since, by 7 D.4, there is a canonical bijective relation for $\mathbf{F}(\mathscr{X}, \mathscr{Y})$ and $\mathbf{F}(|\mathscr{X}|, |\mathscr{Y}|)$ as well as for $\mathbf{F}(\mathscr{X}', \mathscr{Y}')$ and $\mathbf{F}(|\mathscr{X}'|, |\mathscr{Y}'|)$, etc. Now, suppose that $\mathbf{E}\,\mathrm{gr}\,u = \mathscr{X}$, v is one-to-one. If mappings f and g from $\mathbf{F}(\mathscr{X}, \mathscr{Y})$ are different, then $fx \neq gx$ for some $x \in \mathscr{X}$. Since $\mathbf{E}\,\mathrm{gr}\,u = \mathscr{X}$, there exists an element $x' \in X'$ such that $ux' = x$. Then $(f \circ u)\,x' \neq (g \circ u)\,x'$ and $(v \circ f \circ u)\,x' \neq (v \circ g \circ u)\,x'$ because v is injective. If u is not surjective, choose f, g in $\mathbf{F}(\mathscr{X}, \mathscr{Y})$ such that $f \neq g$, but f and g coincide on $\mathbf{E}u$; then $f \circ u = g \circ u$, hence $v \circ f \circ u = v \circ g \circ u$. If v is not injective, let $va = vb$, $a \neq b$. Let f and g be constant mappings of $\mathscr{X}$ into $\mathscr{Y}$, $\mathbf{E}f = (a)$, $\mathbf{E}g = (b)$. Then $v \circ f = v \circ g$, hence $v \circ f \circ u = v \circ g \circ u$.

8. ALGEBRAIC SYSTEMS

The purpose of this section is, first, to state some basic definitions and propositions concerning "algebraic systems" in a rather general sense, and secondly, to derive some results of a more special kind needed in what follows.

As a matter of fact, except for mere examples, only few kinds of "algebraic systems" occur in this book (semi-groups including groups; semi-rings including rings; modules; algebras).

The concept of an "algebraic struct" presented here is more than sufficient to cover all such structs; on the other hand, it is very narrow in comparison with a "general algebraic system" as currently defined. The introduction of algebraic structs is not necessary, since all definitions can be given separately for each kind of struct actually considered. Nevertheless, it is often useful to give certain definitions and propositions in a general form which covers all algebraic systems under consideration.

The second aim indicated above concerns some propositions on homomorphisms, congruences and ideals, and some more or less isolated results including theorems on the embedding of commutative semi-groups into groups and rings into fields which make possible an exact introduction of integers and rational numbers.

We shall define algebraic structures gradually. First, structures formed from internal compositions on a class are defined. This makes possible a definition of external compositions over a struct (a special case has been considered in Section 6). Some of their basic properties are examined. Then we define algebraic structs in the sense loosely indicated above.

It is to be pointed out that, in accordance with the character of this book, questions concerning algebraic structs are often only sketched.

A. EXTERNAL COMPOSITIONS

8 A.1. We begin with a remark of a general character. As in Section 7, definitions are often given for the „comprisable case" only; the reader is invited to carry them over to the more general case (i.e. that of possibly non-comprisable objects under the limitation described in 7 A.5). Concepts defined for the „comprisable case" will be used, if necessary, in this generalized sense. We recall that the reasons for this mode of exposition have been explained in 7 A.5.

8 A.2. Definition. Let X be a set. Every regular multiplet, say $\alpha = \langle \alpha_1, \dots, \alpha_n \rangle$, of internal compositions on X will be called an *internal algebraic structure on* X; $\alpha_1, \dots, \alpha_n$ will be called *constituents* of α. If α is an internal algebraic structure on X, then $\langle X, \alpha \rangle$ will be called an *internal algebraic struct.*

As pointed out above, we shall also consider non-comprisable internal algebraic structures and structs. For convenience, we recall that, in this sense, internal algebraic structures on a class X are, in particular, compositions on X, pairs $\langle \alpha, \beta \rangle$ of compositions α and β on X, triples $\langle \alpha, \beta, \gamma \rangle$ where α, β, γ are compositions on X; internal algebraic structs are $\langle X, \alpha \rangle$, $\langle X, \alpha, \beta \rangle$, $\langle X, \alpha, \beta, \gamma \rangle$, etc., where α, β, γ are compositions on X.

Examples. (A) Semi-groups and semi-rings are internal algebraic structs. – (B) $\langle \bigcup, \bigcap, \div \rangle$ is a non-comprisable internal algebraic structure on the class of all sets. – (C) If A is a class, and $\mathscr{R}$ is the class of all comprisable relations in A (i.e. of all sets $\varrho \subset A \times A$), then $\langle \mathscr{R}, \bigcup_{\mathscr{R}}, \bigcap_{\mathscr{R}}, \div_{\mathscr{R}}, \circ_{\mathscr{R}} \rangle$ is an internal algebraic struct.

We are now going to carry over to internal algebraic structures several concepts and propositions introduced in Section 6 for compositions. For reasons of convenience, this will be done explicitly only for the very special case of a structure $\alpha = \langle \sigma, \mu \rangle$ where σ, μ are compositions. The reader is invited to formulate the corresponding statements for the general case. We point out that we shall occasionally use these general statements though the explicit formulations apply only to the special case mentioned above.

No examples will be given here since they may be found in Section 6, and others will be given in the subsequent exposition.

8 A.3. Definition. Let $\alpha = \langle \sigma, \mu \rangle$ be an internal algebraic structure on a class X. If $Y \subset X$ and σ', μ' are restrictions of σ, μ to compositions on Y (see 6 A.6), then $\langle \sigma', \mu' \rangle$ will be called the *restriction of* α *to an algebraic structure on* Y and will sometimes be denoted by α_Y. – A class Y will be called *stable under* α if $Y \subset X$ and Y is stable under both σ and μ (see 6 A.7).

Remark. Clearly, the intersection of any non-void indexed class of sets stable under α is stable under α (cf. 6 A.8).

8 A.4. Definition. Let $\alpha = \langle \sigma, \mu \rangle$ be an internal algebraic structure on a class X, and let $\alpha' = \langle \sigma', \mu' \rangle$ be an internal algebraic structure on a class X'. A relation φ for X and X' is called *composition-stable* (or simply *stable*) under α and α' or also (α, α')-*stable* if it is a stable relation under σ and σ' as well as under μ and μ' (i.e. if $x_1 \varphi y_1$, $x_2 \varphi y_2$ imply $(x_1 \sigma x_2)\ \varphi\ (y_1 \sigma' y_2)$, $(x_1 \mu x_2)\ \varphi\ (y_1 \mu' y_2)$.

If, in addition, φ is single-valued and $\mathbf{D}\varphi = X$, then we shall say that φ is a *homomorphism-relation under* α *and* α'.

If $\mathscr{X} = \langle X, \alpha \rangle$, $\mathscr{X}' = \langle X', \alpha' \rangle$, $F = \langle \varphi, \mathscr{X}, \mathscr{X}' \rangle$ and $\varphi = \operatorname{gr} F$ is stable under α

and α', then F will be called *stable* (more explicitly, *composition-stable*). If φ is a homomorphism-relation on $\mathscr{X}$ (i.e. if, in addition, $\mathbf{D}\varphi = |\mathscr{X}|$, φ is single-valued), then we shall say that F is a *homomorphism.*

8 A.5. Definition. Let $\{\beta_a \mid a \in A\}$ be a family of internal algebraic structures; for any $a \in A$, let $\beta_a = \langle \sigma_a, \mu_a \rangle$ where σ_a, μ_a are compositions on sets X_a. The pair $\langle \Pi\sigma_a, \Pi\mu_a \rangle$ (more explicitly, $\langle \Pi_{\text{comp}}\{\sigma_a \mid a \in A\}, \Pi_{\text{comp}}\{\mu_a \mid a \in A\} \rangle$; see 6 E.6) will be denoted by $\Pi_{\text{comp}}\{\beta_a \mid a \in A\}$ or by $\Pi\{\beta_a\}$ and will be called the *product* (more explicitly, *compositional product*) of $\{\beta_a\}$.

Clearly, $\Pi\{\beta_a\}$ is an internal algebraic structure on ΠX_a.

Convention. Instead of $\Pi_{\text{comp}}\{a \to \beta \mid a \in A\}$ (i.e. $\Pi\beta_a$ where all β_a are equal to an internal algebraic structure β) we write β^A. Thus, if $\beta = \langle \sigma, \mu \rangle$ is an internal algebraic structure on X, then $\beta^A = \langle \sigma^A, \mu^A \rangle$ is an internal algebraic structure on X^A.

8 A.6. Definition. If $\{\mathscr{X}_a \mid a \in A\} = \{\langle X_a, \beta_a \rangle \mid a \in A\}$ is a family of internal algebraic structs, then $\langle \Pi X_a, \Pi\beta_a \rangle$ is called the *product of* $\{\mathscr{X}_a\}$ and denoted by $\Pi\{\mathscr{X}_a \mid a \in A\}$ or $\Pi\mathscr{X}_a$. If all $\mathscr{X}_a$ are equal to a struct $\mathscr{X}$, then $\mathscr{X}^A$ is written instead of $\Pi\{\mathscr{X}_a \mid a \in A\}$.

Remark. This definition as well as the preceding one pertains to the "comprisable case". For non-comprisable structures and structs, we can of course define the products $\alpha \times \alpha'$, $\langle X, \alpha \rangle \times \langle X', \alpha' \rangle$, etc., in a way which may be indicated as follows: if σ, σ' are compositions on classes X, X', we define their compositional pair-product, denoted for convenience by $\sigma \times \sigma'$, as the composition assigning $\langle x\sigma y, x'\sigma' y' \rangle$ to the pair $\langle \langle x, x' \rangle, \langle y, y' \rangle \rangle$.

8 A.7. We recall that, according to 6 F.1, a relation μ is called an external composition over a class A acting on a class X, if $\mathbf{D}\mu = A$ and, for any $a \in A$ and any $x \in X$, there is exactly one $y \in X$ such that $\langle a, x, y \rangle \in \mu$. The reasons for considering pairs $\langle \mu, \alpha \rangle$, where μ is an external composition over a class, have been explained in 6 F.4.

Definition. If $\boldsymbol{m} = \langle \mu, \alpha \rangle$, and μ is an external composition over a class A acting on a class X, then we shall say that $\boldsymbol{m}$ is an *external composition over* $\mathscr{A} = \langle A, \alpha \rangle$ *acting on* X.

Let $\boldsymbol{m}$ be an external composition over a struct $\mathscr{A}$ acting on X; thus, either $\boldsymbol{m} = \mu$, $\mathscr{A} = A$ (see 6 F.1) or $\boldsymbol{m} = \langle \mu, \alpha \rangle$, $\mathscr{A} = \langle A, \alpha \rangle$, the relation μ being an external composition over the class A. Then $\mathscr{A}$ will be called the *structured domain of* $\boldsymbol{m}$, $|\mathscr{A}|$ (see 7 A.1) will be called the *abstract domain of* $\boldsymbol{m}$, μ will be called the *underlying relation of* $\boldsymbol{m}$.

To distinguish the two cases in question, we shall sometimes say that an external composition $\boldsymbol{m}$ is *structured* if $\boldsymbol{m} = \langle \mu, \alpha \rangle$, *unstructured* otherwise.

In this section we shall mainly consider external compositions of the form $\langle \mu, \alpha \rangle$, α being an internal algebraic structure. Other kinds of structured external compositions will be considered e.g. in Section 19.

8 A.8. Definition. An external composition over a class or over an internal algebraic struct will be called a *purely algebraic* (*or pure*) *external composition.*

8 A.9. Important examples of structured external compositions over a ring have been given in 6 F. We now give two examples of a different kind. – (A) Let M be a set. Let $B(M)$ be the set of all bijective mappings of M onto M and let $B(M)$ be endowed with the composition of mappings, denoted by $\circ$; let μ consist of all $\langle \varphi, x, \varphi x \rangle$, where $\varphi \in B(M)$, $x \in M$. Then $\boldsymbol{m} = \langle \mu, \circ \rangle$ is an external composition on M over the group $\langle B(M), \circ \rangle$, the structured domain of $\boldsymbol{m}$. – (B) If μ is a composition on X, let μ^* consist of all $\langle x, y, x\mu y \rangle$ where $x \in X$, $y \in X$, and let μ^{**} consist of all $\langle x, y, y\mu x \rangle$; then μ^*, μ^{**} are (unstructured) external compositions on X over X ,and $\mu^* \neq \mu^{**}$ unless μ is commutative. Now let $\mathscr{X} = \langle X, \langle \mu_1, \mu_2, \ldots, \ldots, \mu_n \rangle \rangle$, be an internal algebraic struct. Then, in the above notation, $\langle \mu_1^*, \mu_1, \ldots, \mu_n \rangle$, $\langle \mu_1^{**}, \mu_1, \ldots, \mu_n \rangle$, $\langle \mu_2^*, \mu_1, \mu_2, \ldots, \mu_n \rangle$, etc., are pure external compositions on X over $\mathscr{X}$.

8 A.10. We add a definition of a general character. In 8 A.7, we have introduced the concept of the structured domain of an external composition. It is useful to define the structured domain of a relation for a very general case.

Definition. Let ϱ be a relation. If $\boldsymbol{R} = \langle \varrho, \alpha \rangle$ or $\boldsymbol{R} = \varrho$, then $\langle \mathbf{D}\varrho, \alpha \rangle$ or $\mathbf{D}\varrho$, respectively, will be called the *structured domain of* $\boldsymbol{R}$. In other words, if $\boldsymbol{R}$ is a structured relation (see 7 A.1), then $\mathbf{D}|\boldsymbol{R}|$ endowed with the structure of $\boldsymbol{R}$ is called the structured domain of $\boldsymbol{R}$.

Convention. If it is intended to consider, along with $\boldsymbol{R} = \langle \varrho, \alpha \rangle$, its structured domain $\langle \mathbf{D}\varrho, \alpha \rangle$, we shall sometimes say that $\boldsymbol{R}$ is a *domain-structured relation.*

This notion is clearly too general to be reasonably applicable in all situations; e.g. there is little sense in considering $\langle A, A, B \rangle$ as the structured domain of a mapping $\langle f, A, B \rangle$. However, it is rather useful if, for instance, nets (see 15 B.2) are considered: a net is a pair $\langle \{x_a \mid a \in A\}, \leqq \rangle$, where A is a set, $\leqq$ is a certain order on A; the domain of this net is A, its structured domain is the ordered set $\langle A, \leqq \rangle$.

8 A.11. Convention. 1) If $\boldsymbol{m}$ is an external composition on X over $\mathscr{A}$, $\boldsymbol{m} = \mu$ or $\boldsymbol{m} = \langle \mu, \alpha \rangle$, μ being an unstructured composition, then for every $a \in |\mathscr{A}|$ and $x \in X$, there exists exactly one y such that $\langle a, x, y \rangle \in \mu = |\boldsymbol{m}|$. This element y will be denoted by $a\boldsymbol{m}x$ or by $a\mu x$. It will be sometimes convenient to call the relation assigning $a\boldsymbol{m}x$ to $\langle a, x \rangle$ the *external multiplication associated with* $\boldsymbol{m}$ (or *with* μ) or simply *external* μ*-multiplication.* Conversely, let A, X be classes; for any $a \in A$, $x \in X$, let the symbol $a\boldsymbol{m}x$ denote an element of X. Then the relation consisting of all $\langle a, x, a\boldsymbol{m}x \rangle$, $a \in A$, $x \in X$, is an external composition on X; it will often be denoted by $\boldsymbol{m}$. – 2) As usual, we shall sometimes use the same symbol for an external composition $\boldsymbol{m} = \langle \mu, \alpha \rangle$ and its underlying unstructured composition μ. – 3) If $\boldsymbol{m}$ is a composition on X over $\mathscr{A}$, and $B \subset |\mathscr{A}|$, $Y \subset X$, then $[B]\, \boldsymbol{m}\, [Y]$ will denote the class of all $b\boldsymbol{m}y$, where $b \in B$, $y \in Y$. Instead of $[B]\, \boldsymbol{m}\, [Y]$, we sometimes simply write $B\boldsymbol{m}Y$.

8 A.12. Definition. Let $\boldsymbol{m}$ be an external composition on X over $\mathscr{A}$. If $Y \subset X$, and $a \in |\mathscr{A}|$, $y \in Y \Rightarrow a\boldsymbol{m}y \in Y$, then Y is called *absorbing under* $\boldsymbol{m}$; an element $y \in X$ is called *absorbing* (*under* $\boldsymbol{m}$) if the singleton (y) is absorbing. An element $e \in |\mathscr{A}|$ is called *neutral under* $\boldsymbol{m}$ if $x \in X \Rightarrow e\boldsymbol{m}x = x$.

8 A.13. Definition. Let $\boldsymbol{m}$, $\boldsymbol{n}$ be external compositions on X over $\mathscr{A}$ and on Y over $\mathscr{A}$, respectively. Let $\mu = |\boldsymbol{m}|$, $\nu = |\boldsymbol{n}|$ denote the underlying relations of $\boldsymbol{m}$ and $\boldsymbol{n}$. If ν is a restriction of μ, we shall call $\boldsymbol{n}$ a *field-restriction* or simply a *restriction of* $\boldsymbol{m}$. We shall also say, in a more explicit manner, that $\boldsymbol{n}$ is the *restriction of* $\boldsymbol{m}$ *to an external composition on* Y; we shall sometimes denote $\boldsymbol{n}$ by $\boldsymbol{m}_Y$.

Remark. Clearly, if $\boldsymbol{m}$ is a composition on X over $\mathscr{A}$, and $Y \subset X$, then there exists at most one external composition on Y which is a field-restriction of $\boldsymbol{m}$. In order that such a field-restriction exist, it is sufficient and necessary that Y be absorbing under $\boldsymbol{m}$; in this case, its underlying relation (which is an unstructured external composition) is equal to $|\boldsymbol{m}| \cap (|\mathscr{A}| \times Y \times Y)$.

We now introduce a notion which has been considered for a special case (without introducing a special name) in 6 F.

8 A.14. Definition. Let $\boldsymbol{m}$ be an external composition on X over $\mathscr{A}$; put $A = |\mathscr{A}|$. Let σ be an internal composition on X.

If $a\boldsymbol{m}(x\sigma y) = (a\boldsymbol{m}x)\,\sigma(a\boldsymbol{m}y)$ for any $a \in A$, $x \in X$, $y \in X$, then we shall say that $\boldsymbol{m}$ is *action-distributive with respect to* σ.

Let ϱ be a composition on A which is a constituent (see 8 A.2) of the structure of $\boldsymbol{m}$. If $\boldsymbol{m}$ is action-distributive with respect to σ and, in addition, $(a\varrho b)\,\boldsymbol{m}x = (a\boldsymbol{m}x)\,\sigma(b\boldsymbol{m}x)$ holds for any $a \in A$, $b \in A$, $x \in X$, then $\boldsymbol{m}$ is said to be *ϱ-distributive with respect to* σ.

Finally, an unstructured action-distributive $\boldsymbol{m}$ will sometimes be simply called *distributive*.

For instance, if $\langle X, \sigma, \mu, \alpha, \beta\rangle$ is a module, then $\langle \mu, \alpha, \beta\rangle$ is α-distributive with respect to σ.

8 A.15. Proposition and definition. *Let $\mathscr{B}$ be a struct and let $\{\boldsymbol{m}_a \mid a \in A\}$ be a family of external compositions over $\mathscr{B}$. Put $\mu_a = |\boldsymbol{m}_a|$, $B = |\mathscr{B}|$ and if $\mathscr{B}$ is not a class, put $\mathscr{B} = \langle B, \beta\rangle$. For any $a \in A$, let $\boldsymbol{m}_a$ act on X_a. For any $b \in B$, $x = \{x_a\} \in \Pi X_a$, put $b\mu x = \{b\mu_a x_a\} \in \Pi X_a$. Then, according to* 8 A.11, *μ denotes an external composition (on ΠX_a over B) consisting of all $\langle b, \{x_a\}, \{b\mu_a x_a\}\rangle$. Let $\boldsymbol{m}$ be equal to μ if $\boldsymbol{m}_a$ are unstructured, i.e. if $\mathscr{B}$ is a class, and let $\boldsymbol{m} = \langle \mu, \beta\rangle$ if $\mathscr{B} = \langle B, \beta\rangle$.*

Then m is an external composition on X over $\mathscr{B}$; it is called the compositional product, *or simply the* product *of $\{\boldsymbol{m}_a\}$ and it is denoted by $\Pi_{\text{comp}}\{\boldsymbol{m}_a\}$ or simply $\Pi\{\boldsymbol{m}_a\}$. If all $\boldsymbol{m}_a$ are equal to a fixed $\boldsymbol{p}$, then we denote $\Pi\{\boldsymbol{m}_a\}$ by $\boldsymbol{p}^A$ and call it a* power *of $\boldsymbol{p}$ (with* exponent *A).*

8 A.16. We have defined the compositional product twice: in 6 E.6 for internal compositions, in 8 A.15 for external ones. The question arises whether these definitions overlap, i.e. whether an internal composition can coincide with an external one. It turns out that this may happen only in exceptional cases which do not occur "in practice". Namely, the following proposition holds:

The void set $\emptyset$ is an external composition on $\emptyset$ over $\emptyset$ as well as an internal composition. If $\mu \neq \emptyset$ is an internal composition on a class X as well as an external composition on a class Y over a class A, then, for some y, we have $Y = (y)$, $\langle y, y\rangle \in X$, $A = X \times X$, $\mu = (X \times X) \times (y) \times (y)$. Conversely, if X is a class, $\langle y, y\rangle \in X$, then $\mu = (X \times X) \times (y) \times (y)$ is an internal composition on X and an external composition on (y) over $X \times X$.

Indeed, if $\mu \neq \emptyset$ is as described above, then μ is a single-valued relation (since it is an internal composition). On the other hand, since μ is an external composition, $\mu[(a)] \in Y^Y$ for every $a \in A$ (observe that $A = \mathbf{D}\mu$ is non-empty). Therefore, Y is a singleton; put $Y = (y)$. Clearly, $\mu = A \times (y) \times (y)$, and $A = \mathbf{D}\mu = X \times X$ (since μ is an internal composition on X). In addition, $\mathbf{E}\mu = (\langle y, y\rangle)$, hence $\langle y, y\rangle \in X$. In particular, we obtain the following proposition.

If X is a class and μ is an internal composition on X as well as an external composition on X over a class A, then either $X = \emptyset$, $\mu = \emptyset$, or there exists an element y such that $\langle y, y\rangle = y$, $X = (y)$, $A = (y)$, $\mu = (y)$.

This is clear, for, by the above proposition, $X = Y$ is a singleton (y) and $\langle y, y\rangle \in X$; hence $\langle y, y\rangle = y$, $A = X \times X = (\langle y, y\rangle) = (y)$, $\mu = X \times X \times (y) \times (y) = (y)$. Of course, this case is clearly "pathological" from the intuitive point of view. It is not ruled out by our system of axioms; however, the axiom $a \neq \langle a, b\rangle \neq b$ could be added to this system without affecting the exposition.

Returning to the question of compositional products, let X be a set and suppose that the set A of all a a such that $\langle a, a\rangle \in X$ is non-empty. For every $a \in A$ put $\mu_a = X \times X \times (a) \times (a)$. Then $\Pi\mu_a$ in the sense of 8 A.15 is equal to $X \times X \times (\mathrm{J}_A) \times (\mathrm{J}_A)$, whereas $\Pi\mu_a$ in the sense of 6 E.6 is equal to $(X^A \times X^A) \times (\alpha)$ where α is the set of all $\langle a, a, a\rangle$, $a \in A$.

Therefore, properly speaking, it would be necessary to modify the above definition and to call the product in question e.g. the external compositional product. Nevertheless, we retain the terminology introduced above since "practically" there will occur no compositions of the kind just described.

8 A.17. It seems appropriate to insert a remark of a general character here. We have already stressed, at various places, that the axiomatic system, as well as the approach to basic mathematical concepts adopted in this book is merely one of a multitude of more or less equivalent modes of presenting certain mathematical facts in a rigorous form. In addition, within the general framework developed in this book, there are many "degrees of freedom". Thus, a certain discrepancy pointed out in 8 A.16 (namely the fact that an internal composition can simultaneously be an external one) is by no means inherent in the idea of compositions but can be ruled out by adopting a formally different definition.

Namely, we may define an external composition on X over $\mathscr{A}$ as a pair $\langle \mu, \mathscr{A} \rangle$ where (1) μ is a relation satisfying conditions indicated in 6 F.1, i.e. μ is what is called, according to 8A.6, an unstructured external composition on X over A, (2) $\mathscr{A}$ is a struct with the underlying class equal to the domain A of μ. Then an external composition is always a pair, whereas an internal one is a class; therefore, the difficulties described in 8A.16 are ruled out. We have adopted a formally simpler definition for the reason that the cases indicated in 8 A.16 are "practically" irrelevant.

8 A.18. Definition. Let $\mathscr{A}$ be a struct, $A = |\mathscr{A}|$, let $\boldsymbol{m}$ be an external composition on X over $\mathscr{A}$, $\boldsymbol{m}'$ an external composition on X' over $\mathscr{A}$. Then a relation φ for X and X' will be called *composition-stable* (or simply *stable*) *under the external compositions* $\boldsymbol{m}$ and $\boldsymbol{m}'$ or also $(\boldsymbol{m}, \boldsymbol{m}')$-*stable* if $(a\boldsymbol{m}x)\,\varphi\,(a\boldsymbol{m}x')$ whenever $a \in A$, $x\varphi x'$.

If, in addition, φ is single-valued and $\mathbf{D}\varphi = X$, then we shall say that φ is a *homomorphism-relation under* $\boldsymbol{m}$ *and* $\boldsymbol{m}'$, or an $(\boldsymbol{m}, \boldsymbol{m}')$–*homomorphism-relation.*

Example. Let S be a set, P the set of all permutations of S. Consider the composition μ on S over P consisting of all $\langle p, x, px \rangle$. Then a relation φ in S is stable under μ and μ if and only if $p \circ \varphi = \varphi \circ p$ for every $p \in P$. It is easy to prove that if S contains at least three elements, then $\varphi = \mathsf{J}_S$ is the only (μ, μ)-homomorphism-relation.

B. ALGEBRAIC STRUCTURES

8 B.1. Definition. Let X be a class. Let $\boldsymbol{r}$ be a regular multiplet, $\boldsymbol{r} = \boldsymbol{r}_1$ or $\boldsymbol{r} = \langle \boldsymbol{r}_1, \ldots, \boldsymbol{r}_n \rangle$. Suppose that there exists a natural m, $0 \leqq m \leqq n$, such that (a) if $k = 1, \ldots, m$, then $\boldsymbol{r}_k$ is an internal composition on X, (b) if $k = m + 1, \ldots, n$, then $\boldsymbol{r}_k$ is a purely algebraic (see 8 A.8) external composition of X. Then $\boldsymbol{r}$ is called a *purely algebraic* (or, for short, *algebraic*) *structure on* X and $\langle X, \boldsymbol{r} \rangle$ is called a *purely algebraic* (or, for short, *algebraic*) *struct*. Each composition $\boldsymbol{r}_k$ is called a *constituent composition* (*internal* or *external*) of $\boldsymbol{r}$. More explicitly, we shall say that $\boldsymbol{r}$ is an algebraic structure and $\langle X, \boldsymbol{r} \rangle$ is an algebraic struct of the type $\langle m, 0 \rangle$ if $m = n$, $\langle \langle m, n - m \rangle, \mathscr{A}_{m+1}, \ldots, \mathscr{A}_n \rangle$ if $n > m$ and $\mathscr{A}_k$, $k = m + 1, \ldots, n$, is the structured domain of $\boldsymbol{r}_k$.

We do not formulate a definition for the "non-comprisable" case; see 8 A.1.

Examples of algebraic structs are semi-groups (they are of type $\langle 1, 0 \rangle$), semi-rings (of type $\langle 2, 0 \rangle$), modules (which are structs of type $\langle \langle 1, 1 \rangle, \mathscr{A} \rangle$, $\mathscr{A}$ being a ring). They have been considered in Section 6, and further examples will follow in this section.

8 B.2. It is conceivable that an algebraic structure $\boldsymbol{r}$ on X admits of two expressions with the properties described above and thus has two types in the sense of 8 B.1.

It is easy to see that, in such a case, these expressions are as follows: $\boldsymbol{r} = \langle \boldsymbol{r}_1, \ldots, \boldsymbol{r}_p \rangle$ and $\boldsymbol{r} = \langle \boldsymbol{r}_1, \ldots, \boldsymbol{r}_{p-1}, \boldsymbol{r}'_p, \ldots, \boldsymbol{r}'_n \rangle$ where $n > p$, $\boldsymbol{r}_p$ is an external composition of the form $\langle \boldsymbol{r}'_p, \ldots, \boldsymbol{r}'_n \rangle$; a simple reasoning based on 8 A.16 then shows that either $X = \emptyset$ or X is a singleton. Therefore, we may assert that any algebraic structure $\boldsymbol{r}$ on a class X containing more than one element can be expressed uniquely in the form indicated in 8 B.1.

It may also happen that the number m from definition 8 B.1 is not uniquely determined. Clearly, in such a case some $\boldsymbol{r}_k$ is an external as well as an internal composition on X; this implies (see 8 A.16) that either $X = \emptyset$ or $X = (z)$, $\boldsymbol{r}_k = (z)$ where z is an element such that $\langle z, z \rangle = z$. Thus, this last "pathological" case of $\langle z, z \rangle = z$ excepted (cf. 8 A.16), the number m from 8 B.1 is uniquely determined provided the underlying set X is not empty. Summing up, we may say that, apart from the exceptional cases indicated, the expression $\boldsymbol{r} = \langle \boldsymbol{r}_1, \ldots, \boldsymbol{r}_n \rangle$ described in 8 B.1 and the type of an algebraic structure $\boldsymbol{r}$ are uniquely determined.

8 B.3. It is easy to prove the following: Let $\boldsymbol{r}$ be an algebraic structure on X with a uniquely determined type (this is true whenever X has more than one element). Then $\boldsymbol{r}$ can be expressed in exactly one manner in the form $\boldsymbol{r} = \langle \boldsymbol{r}_1, \ldots, \boldsymbol{r}_n \rangle$ described in 8 B.1.

Therefore, the following definition is correct.

Definition. Let an algebraic structure $\boldsymbol{r} = \langle \boldsymbol{r}_1, \ldots, \boldsymbol{r}_p \rangle$ on X possess a uniquely determined type, say $\langle m, n \rangle$ or $\langle \langle m, n \rangle, \ldots \rangle$. We shall call $\boldsymbol{r}' = \langle \boldsymbol{r}_1, \ldots, \boldsymbol{r}_m \rangle$ the *internal component of* $\boldsymbol{r}$ provided $m \geqq 1$; if $m = 0$, we shall say that $\boldsymbol{r}$ *has no internal component*. Similarly, $\boldsymbol{r}'' = \langle \boldsymbol{r}_{m+1}, \ldots, \boldsymbol{r}_p \rangle$ will be called the *external component of* $\boldsymbol{r}$ provided $m < p$; if $m = p$, we shall say that $\boldsymbol{r}$ *has no external component*.

We shall sometimes term $\langle X, \boldsymbol{r}' \rangle$ (respectively $\langle X, \boldsymbol{r}'' \rangle$) the *underlying internal (external) struct of* $\langle X, \boldsymbol{r} \rangle$.

It follows the from the considerations in 8 B.2, that, with the exception of cases $X = \emptyset$ and $X = (z)$, $\langle z, z \rangle = z$, an algebraic structure on X always either possesses a uniquely determined internal component or has no such component, and the same holds for the external component.

We are going to introduce, for algebraic structures, various concepts defined previously for compositions. Some of these notions have already been defined for the special case of internal algebraic structures (see 8 A.3 – 8 A.6).

8 B.4. Definition. Let $\boldsymbol{r}$ be an algebraic structure on X; let $Y \subset X$. If $\boldsymbol{r} = \langle \boldsymbol{r}_1, \ldots, \boldsymbol{r}_p \rangle$ where $\boldsymbol{r}_k$ are internal or external compositions on X, $\boldsymbol{s} = \langle \boldsymbol{s}_1, \ldots, \boldsymbol{s}_p \rangle$ and, for each $k = 1, \ldots, p$, $\boldsymbol{s}_k$ is the restriction of $\boldsymbol{r}_k$ to a composition on Y, then we shall say that $\boldsymbol{s}$ is a *restriction of* $\boldsymbol{r}$ *to a structure on* Y or also a *field-restriction of* $\boldsymbol{r}$ *to* Y, and we write $\boldsymbol{s} = \boldsymbol{r}_Y$.

Remark. In this definition, as well as in the following statements concerning arbitrary algebraic structures in the sense of 8 B.1, we have in mind the "normal" case where the type of structure considered is uniquely determined. Nevertheless, the definitions and propositions are also valid, if properly interpreted, for the rather uninteresting "exceptional" cases (see 8 A.16).

8 B.5. Definition. Let $\boldsymbol{r}$ be an algebraic structure on X; let $Y \subset X$. Then Y is called *stable under* $\boldsymbol{r}$ if it is stable under every internal constituent of $\boldsymbol{r}$, and absorbing under every external constituent; Y is called *absorbing under* $\boldsymbol{r}$ if it is absorbing under every constituent of $\boldsymbol{r}$.

For instance, in a module $\mathscr{M} = \langle M, \boldsymbol{m}\rangle$ there are only two absorbing sets: M and $\emptyset$; every submodule is stable.

It is clear that the intersection of any indexed class of sets stable (respectively, absorbing) under $\boldsymbol{r}$ is stable (absorbing) under $\boldsymbol{r}$.

8 B.6. *Let $\langle X, \boldsymbol{r}\rangle$ be an algebraic struct, let $Y \subset X$. Then Y is stable under $\boldsymbol{r}$ if and only if there exists a field-restriction of $\boldsymbol{r}$ to a structure on Y.*

This follows at once from the assertions in 6 A.7 and 8 A.13 (remark).

8 B.7. At first sight it seems reasonable to define a substruct of an algebraic struct $\langle X, \boldsymbol{r}\rangle$ as an algebraic struct of the form $\langle Y, \boldsymbol{r}_Y\rangle$. As a rule, however, we only consider structs satisfying certain conditions (such as stated in the definition of a semi-group, a group, a semi-ring, etc.). If $\langle X, \boldsymbol{r}\rangle$ is such a struct and $Y \subset X$ is stable (i.e. $\langle Y, \boldsymbol{r}_Y\rangle$ is meaningful), then it can still happen that $\langle Y, \boldsymbol{r}_Y\rangle$ does not satisfy the conditions set up for the kind of structs considered; thus if $\langle X, \boldsymbol{r}\rangle$ is a group, $\emptyset \neq Y \subset X$, and Y is stable, then $\langle Y, \boldsymbol{r}_Y\rangle$ is a sub-semi-group but it need not be a subgroup.

For these reasons, we do not give a general definition of substructs of an algebraic struct. The above remarks serve only to indicate a mode of defining substructs.

8 B.8. Definition. Let $\{\mathscr{X}_a \mid a \in A\}$, where $\mathscr{X}_a = \langle X_a, \boldsymbol{r}_a\rangle$, be a family of algebraic structs. Let all $\boldsymbol{r}_a$ be of the same type. Suppose that every $\boldsymbol{r}_a$ has an internal or an external component (this supposition excludes, by 8 B.3, only the cases $X = \emptyset$, and $X = (z)$, $\langle z, z\rangle = z$).

If $\boldsymbol{r}_a = \langle r_a^{(1)}, \ldots, r_a^{(n)}\rangle$, then we put $\Pi\{\boldsymbol{r}_a\} = \langle \Pi_a r_a^{(1)}, \ldots, \Pi_a r_a^{(n)}\rangle$ (for the definition of $\Pi_a r_a^{(k)}$ see 6 E.6 if $r_a^{(k)}$ are internal, 8 A.15 if $r_a^{(k)}$ are external), $\Pi\{\mathscr{X}_a\} = \langle \Pi X_a, \Pi \boldsymbol{r}_a\rangle$. The structure $\Pi\{\boldsymbol{r}_a\}$ denoted also more explicitly by $\Pi_{\text{comp}}\{\boldsymbol{r}_a\}$ is called the *cartesian product of* $\{\boldsymbol{r}_a\}$, and $\Pi\{\mathscr{X}_a\}$ is called the *cartesian product of* $\{\mathscr{X}_a\}$. As usual, we write $\boldsymbol{r}^A$ instead of $\Pi\{\boldsymbol{r}_a\}$ if all $\boldsymbol{r}_a$ are equal to $\boldsymbol{r}$, and similarly for $\mathscr{X}^A$.

Clearly, this definition includes definitions 8 A.5 and 8 A.6 and also, as a very special case, definition 6 F.15.

We conclude this subsection with the definition of a special kind of algebraic struct which includes all algebraic structs actually considered here and admits a simple description of homomorphisms.

8 B.9. Definition. An algebraic struct $\mathscr{X} = \langle X, \boldsymbol{r}\rangle$, as well as its structure $\boldsymbol{r}$, is called *module-like* provided the following holds: if $\boldsymbol{r} = \boldsymbol{r}_1$ or $\boldsymbol{r} = \langle \boldsymbol{r}_1, \ldots, \boldsymbol{r}_n\rangle$, $\boldsymbol{r}_k$

being an internal (for $1 < k \leqq m$) or external (for $m < k \leqq n$) composition on X, then (1) $\boldsymbol{r}_1$ is an associative internal composition, (2) every internal $\boldsymbol{r}_k$, $k > 1$, is distributive with respect to $\boldsymbol{r}_1$, and every external $\boldsymbol{r}_k$, $k > 1$, is action-distributive (see 8 A.14) with respect to $\boldsymbol{r}_1$. The composition $\boldsymbol{r}_1$ will be sometimes called the *basic constituent of* $\boldsymbol{r}$. For convenience we shall sometimes refer to the compositions $\boldsymbol{r}_k$, $k > 1$, as the *multiplicative constituents of* $\boldsymbol{r}$.

Stated in a more intuitive way, a module-like algebraic struct is obtained by endowing a semi-group $\langle X, \sigma\rangle$ with arbitrary internal and external compositions, distributive or action-distributive with respect to σ.

Observe that a module-like struct may be of the form say $\mathscr{X} = \langle X, \sigma, \sigma\rangle$. In such a case, σ is a basic as well as a multiplicative constituent of the structure of $\mathscr{X}$. An example: $\mathscr{X} = \langle S, \bigcup, \bigcup\rangle$, S being the class of all sets.

Examples. A) Clearly, semi-groups, semi-rings and modules (as well as "right modules", see 6 F.5, example (C)) are module-like structs. – B) Let v consist of all $\langle \varrho, X, \varrho[X]\rangle$ such that X is a set, ϱ is a comprisable relation. Then the structure $\langle \bigcup, \bigcap, v\rangle$ is module-like, whereas $\langle \bigcap, \bigcup, v\rangle$ is not.

8 B.10. *If $\langle X, \boldsymbol{r}\rangle$ is module-like and $Y \subset X$ is stable under $\boldsymbol{r}$, then $\langle Y, \boldsymbol{r}_Y\rangle$ is module-like.* – This is clear.

As an example of module-like structs, we shall now introduce algebras over rings.

8 B.11. Definition. Let $\langle \alpha, \beta\rangle$ be a ring structure on a class A; put $\mathscr{A} = \langle A, \alpha, \beta\rangle$; let $\langle \sigma, \mu\rangle$ be a ring structure on a class X. Let $\boldsymbol{r} = \langle \varrho, \alpha, \beta\rangle$ be an external composition on X over $\mathscr{A}$ such that $\langle \sigma, \boldsymbol{r}\rangle$ is a module structure on X over $\mathscr{A}$. If, in addition, $a\boldsymbol{r}(x\mu y) = (a\boldsymbol{r}x)\mu y = x\mu(a\boldsymbol{r}y)$ for any $a \in A$, $x \in X$, $y \in X$, then we shall say that $\langle \sigma, \mu, \boldsymbol{r}\rangle = \langle \sigma, \mu, \varrho, \alpha, \beta\rangle$ is an *algebra structure on X over $\mathscr{A}$*; the struct $\langle X, \sigma, \mu, \boldsymbol{r}\rangle$ will be called an *algebra over $\mathscr{A}$* provided X is comprisable non-void and $\mathscr{A}$ is a ring.

In a more intuitive way, we may say that an algebra is a ring endowed with an external composition satisfying certain conditions.

Examples. (A) Let $\mathscr{A} = \langle A, \alpha, \beta\rangle$ be a ring. Let β^* consist of all $\langle a, x, a\beta x\rangle$ where $a \in A$, $x \in A$. Then $\langle A, \alpha, \beta, \beta^*, \alpha, \beta\rangle$ is an algebra over $\mathscr{A}$. More intuitively, this means that with every ring $\mathscr{A}$ there is associated an algebra obtained by endowing $\mathscr{A}$ with a natural external composition over A itself. – (B) Let $\mathscr{A} = \langle A, +, .\rangle$ denote an arbitrary ring. Let P be a set. Denote by τ the external composition on A^P over $\mathscr{A}$ such that $a\,\tau\,\{b_p\} = \{a\,.\,b_p\}$. Then $\langle A^P, \alpha^P, \beta^P, \tau, \alpha, \beta\rangle$ is an algebra over $\mathscr{A}$. The algebra obtained by transferring (see 8 C.4) the structure just described from A^P to $\mathbf{F}(P, \mathscr{A})$ by means of the canonical relation is usually called the algebra of functions on P with values in $\mathscr{A}$.

C. HOMOMORPHISMS. CONGRUENCES

In this subsection we consider composition-stable correspondences, in particular homomorphisms and congruences, i.e. composition-stable equivalences.

8 C.1. Definition. Let $\mathscr{X} = \langle X, \boldsymbol{r} \rangle$, $\mathscr{Y} = \langle Y, \boldsymbol{s} \rangle$ be algebraic structs. Let φ be a relation for X and Y. If $\boldsymbol{r} = \langle \boldsymbol{r}_1, \dots, \boldsymbol{r}_n \rangle$ and $\boldsymbol{s} = \langle \boldsymbol{s}_1, \dots, \boldsymbol{s}_n \rangle$ are of the same type t and φ is stable under $\boldsymbol{r}_k$ and $\boldsymbol{s}_k$ for each $k = 1, \dots, n$, then we shall say that φ is *composition-stable* (or simply *stable*) *of type* t under $\boldsymbol{r}$ and $\boldsymbol{s}$ or that φ is $(\boldsymbol{r}, \boldsymbol{s})$-*stable* (the words "of type t" are usually omitted in this expression and in other similar expressions introduced here); the correspondence $\langle \varphi, \mathscr{X}, \mathscr{Y} \rangle$ will be called *composition-stable.*

If, in addition, φ is single-valued and $\mathbf{D}\varphi = X$, then φ will be called a *homomorphism-relation.*

A mapping $\langle \varphi, \mathscr{X}, \mathscr{Y} \rangle$ which is composition-stable will be called a *homomorphism* (of $\mathscr{X}$ into $\mathscr{Y}$); a homomorphism of the form $\langle \varphi, \mathscr{X}, \mathscr{X} \rangle$ will be called *an endomorphism* (of $\mathscr{X}$). A homomorphism $\langle \varphi, \mathscr{X}, \mathscr{Y} \rangle$ the inverse of which is also a homomorphism will be called an *isomorphism*; an isomorphism of the form $\langle \varphi, \mathscr{X}, \mathscr{X} \rangle$ will be called an *automorphism.*

Let $\mathscr{X}, \mathscr{Y}$ be algebraic structs of the same type. If there exists an isomorphism of $\mathscr{X}$ onto $\mathscr{Y}$, then we shall say that $\mathscr{X}$ and $\mathscr{Y}$ are *isomorphic.* If there exists a homomorphism of $\mathscr{X}$ onto $\mathscr{Y}$, we shall say that $\mathscr{Y}$ is a *homomorphic image of* $\mathscr{X}$.

Observe that an injective homomorphism is not an isomorphism if it is not surjective. It is true (see 8 C.3) but not quite trivial, that a bijective homomorphism is an isomorphism.

Examples. (A) If A is a finite set, $\mathscr{S} = \exp A$, then every homomorphism-relation φ under $\langle \bigcup, \bigcap \rangle_{\mathscr{S}}$ and $\langle \bigcap, \bigcup \rangle_{\mathscr{S}}$ is of the form $\{X \to B - \bigcup_{x \in X} B_x\}$ where $B \subset A$ and $\{B_x \mid x \in A\}$ is a disjoint family of subsets of B. – (B) Consider composition-stable correspondences $\langle \varphi, \langle \mathsf{N}, + \rangle, \langle \mathsf{N}, + \rangle \rangle$. Clearly, every homomorphism-relation φ is of the form $\{x \to k \,.\, x\}$, k fixed. On the other hand, it is difficult to describe all composition-stable relations φ. Certain relations φ of this kind can be obtained as follows: let $M \subset \mathsf{N}$ be an arbitrary non-void set; let $\langle 0, 0 \rangle \in \varphi$ and, for $m \in \mathsf{N}$, let $\langle m, x \rangle \in \varphi$ if and only if x is expressible as a sum of m numbers from M.

8 C.2. *Let* $\mathbf{K}_t$ *denote the class of all comprisable composition-stable correspondences of type* t. *If* $F \in \mathbf{K}_t$, *then* $F^{-1} \in \mathbf{K}_t$; *if* $F \in \mathbf{K}_t$, $G \in \mathbf{K}_t$ *and* $\mathbf{D}^*F = \mathbf{E}^*G$, *then* $F \circ G \in \mathbf{K}_t$.

8 C.3. *Let* $\mathscr{X} = \langle X, \boldsymbol{r} \rangle$, $\mathscr{Y} = \langle Y, \boldsymbol{s} \rangle$ *be algebraic structs. If* $F = \varphi : \mathscr{X} \to \mathscr{Y}$ *is a bijective homomorphism, then it is an isomorphism; if, in addition,* $\varphi = \mathsf{J}$, *then* $\mathscr{X}$ *and* $\mathscr{Y}$ *coincide.*

Proof. By 8 C.2, F^{-1} is composition-stable; since F is bijective, F^{-1} is a mapping; thus F^{-1} is a homomorphism, which proves the first assertion. If, in addition, $\varphi = \mathsf{J}$, then $X = Y$. If e.g. $\boldsymbol{r}, \boldsymbol{s}$ are of type $\langle 1, 0 \rangle$, i.e. if $\boldsymbol{r}, \boldsymbol{s}$ are internal compositions, then $\varphi(x\boldsymbol{r}x') = (\varphi x)\, \boldsymbol{s}\, (\varphi x')$, that is $x\boldsymbol{r}x' = x\boldsymbol{s}x'$ for any $x \in X$, $x' \in X$ and therefore $\boldsymbol{r} = \boldsymbol{s}$. The assertion for the general case follows immediately.

8 C.4. *Let $\mathscr{X}$ be an algebraic struct and let φ be a bijective relation for $|\mathscr{X}|$ and a class Y. Then there exists exactly one algebraic structure s on Y such that $\langle \varphi, \mathscr{X}, \langle Y, s \rangle \rangle$ is an isomorphism.*

Proof. If $\mathscr{X} = \langle X, r \rangle$, $r = \langle r_1, \ldots, r_n \rangle$, put (1) $ys_k y' = \varphi\left((\varphi^{-1} y)\, r_k\, (\varphi^{-1} y')\right)$ if r_k is internal, (2) $as_k y = \varphi\left(a\, r_k (\varphi^{-1} y)\right)$ if r_k is external. It is easy to show that $s = \langle s_1, \ldots, s_n \rangle$ is an algebraic structure for which $\varphi : \mathscr{X} \to \langle Y, s \rangle$ is an isomorphism. The uniqueness of s follows from 8 C.3 since if s' is another structure with the property in question, then $\mathsf{J} : \langle Y, s \rangle \to \langle Y, s' \rangle$ is an isomorphism.

Convention. We shall say that s is obtained by *transferring r to Y by means of φ.*

8 C.5. Convention. Let $\mathscr{X}$ be a comprisable struct and let $\mathscr{Y}$ be a comprisable algebraic struct; let $X = |\mathscr{X}|$, $\mathscr{Y} = \langle Y, s \rangle$. The algebraic structure on $\mathbf{F}(\mathscr{X}, \mathscr{Y})$ obtained by transferring s by means of the canonical relation for Y^X and $\mathbf{F}(\mathscr{X}, \mathscr{Y})$ (see 7 D.8) will be denoted by s^X or $s^{\mathscr{X}}$, occasionally simply by s (cf. 6 E.6, remark 1). The struct $\langle \mathbf{F}(\mathscr{X}, \mathscr{Y}), s^X \rangle$ will often be denoted simply by $\mathbf{F}(\mathscr{X}, \mathscr{Y})$.

8 C.6. Definition. If $\mathscr{X}$ and $\mathscr{Y}$ are algebraic structs of the same type t, then the set of all homomorphisms $h = h : \mathscr{X} \to \mathscr{Y}$ will be denoted by $\mathrm{Hom}(\mathscr{X}, \mathscr{Y})$.

In general, there is no reasonable internal composition on $\mathrm{Hom}(\mathscr{X}, \mathscr{Y})$. On the other hand, there are two natural external compositions, and if $\mathscr{X} = \mathscr{Y}$, there is a natural internal composition.

If $\mathscr{X}$ is an algebraic struct (of type t*), then* $\mathrm{Hom}(\mathscr{X}, \mathscr{X})$ *is a semi-group under the composition of mappings.*

This follows from 6 E.5 and 8 C.2.

8 C.7. If $\mathscr{X}, \mathscr{Y}$ are algebraic structs of the same type, we shall often consider two external compositions on $\mathrm{Hom}(\mathscr{X}, \mathscr{Y})$. One of these is over $\mathrm{Hom}(\mathscr{X}, \mathscr{X})$; it (respectively, its underlying unstructured composition) consists of all $\langle h, g, g \circ h \rangle$, $h \in \mathrm{Hom}(\mathscr{X}, \mathscr{X})$, $g \in \mathrm{Hom}(\mathscr{X}, \mathscr{Y})$. The other is over $\mathrm{Hom}(\mathscr{Y}, \mathscr{Y})$; it (respectively its underlying composition) consists of all $\langle f, g, f \circ g \rangle$ where $f \in \mathrm{Hom}(\mathscr{Y}, \mathscr{Y})$, $g \in \mathrm{Hom}(\mathscr{X}, \mathscr{Y})$.

8 C.8. In special cases, $\mathrm{Hom}(\mathscr{X}, \mathscr{Y})$ is stable under the structure of $\mathbf{F}(\mathscr{X}, \mathscr{Y})$ (see 8 C.5) and therefore may be considered as an algebraic struct. One such case is described below.

Let $\mathscr{X} = \langle X, r \rangle$, $\mathscr{Y} = \langle Y, s \rangle$ be structs of the same type t, t $= \langle 1, 0 \rangle$ *or* t $=$ $= \langle \langle 1, q \rangle, \ldots \rangle$. *Suppose that the internal constituent σ of s is associative and commutative and every external constituent of s is action-distributive with respect to σ. Then* $\mathrm{Hom}(\mathscr{X}, \mathscr{Y})$ *is a stable subset of* $\mathbf{F}(\mathscr{X}, \mathscr{Y})$ *and the field-restriction of $s^{\mathscr{X}}$ to* $\mathrm{Hom}(\mathscr{X}, \mathscr{Y})$ *is an algebraic structure on* $\mathrm{Hom}(\mathscr{X}, \mathscr{Y})$.

Proof. It is sufficient to show that, for any homomorphism-relations φ and ψ under r and s, $\gamma = \{x \to (\varphi x)\, \sigma\, (\psi x)\}$ is also a homomorphism-relation under r and s. For convenience, we indicate the proof only for the special case $r = \langle \varrho, \mu \rangle$,

$s = \langle \sigma, \nu \rangle$ (ϱ, σ are internal compositions, μ and ν are over $\mathscr{A}$); instead of $y\sigma y'$, $y + y'$ is written.

Now, $\gamma(x\varrho x') = (\varphi x + \varphi x') + (\psi x + \psi x')$ and therefore, by the associativity and the commutativity of σ, $\gamma(x\varrho x') = (\varphi x + \psi x) + (\varphi x' + \psi x') = \gamma x + \gamma x'$. If $a \in |\mathscr{A}|$, then $\gamma(a\mu x) = a\nu(\varphi x) + a\nu(\psi x)$ and therefore, by the action-distributivity of ν, $\gamma(a\mu x) = a\nu(\varphi x + \psi x) = a\nu(\gamma x)$.

Remarks. 1) The suppositions from the above proposition are satisfied e.g. if $\mathscr{Y}$ is a commutative semi-group or a semi-ring or a module. – 2) Under the suppositions indicated above, the external compositions described in **8 C.7** are action-distributive with respect to σ^X. – 3) If, in addition, $\mathscr{X} = \mathscr{Y}$, then Hom $(\mathscr{X}, \mathscr{X})$ endowed with $\langle \sigma^X, \circ \rangle$ is a semi-ring (a ring, if σ is a group structure). It will be denoted simply by Hom $(\mathscr{X}, \mathscr{X})$. In this connection, see **6 E.13**.

8 C.9. *Let $\mathscr{X} = \langle X, \boldsymbol{r} \rangle$, $\mathscr{Y} = \langle Y, \boldsymbol{s} \rangle$ be algebraic structs; let F be a homomorphism of $\mathscr{X}$ into $\mathscr{Y}$. Then $F[X]$ is stable under $\boldsymbol{s}$; $\langle F[X], \boldsymbol{s}' \rangle$, where $\boldsymbol{s}'$ is the restriction of $\boldsymbol{s}$ to a structure on $F[X]$, is an algebraic struct, and $F : \mathscr{X} \to \langle F[X], \boldsymbol{s}' \rangle$ is a surjective homomorphism.*

Proof. If σ is an internal constituent of $\boldsymbol{s}$, $y \in F[X]$, $y' \in F[X]$, then let ϱ be the corresponding constituent of $\boldsymbol{r}$. Choose $x \in X$, $x' \in X$ such that $y = Fx$, $y' = Fx'$. Clearly $y\sigma y' = \varphi(x\varrho x')$. Thus it is proved that $F[X]$ is stable under σ. In a similar way it can be proved that $F[X]$ is stable under external constituent compositions of $\boldsymbol{s}$. The rest of the proof is left to the reader.

Convention. In accordance with **7 B.3**, $\langle F[X], \boldsymbol{s}' \rangle$ will be called the *image of $\mathscr{X}$ under F*; it will often be denoted by $F[\mathscr{X}]$.

Remark. There are many useful properties such that if $\boldsymbol{r}$ has the property in question, then $\boldsymbol{s}'$, as described above, has this property as well. For example if $\boldsymbol{r}$ is an associative (or commutative) composition, then $\boldsymbol{s}'$ is associative (or commutative).

We are going to state a proposition concerning a connection between homomorphisms and certain equivalences.

8 C.10. Definition. Let $\mathscr{X} = \langle X, \boldsymbol{r} \rangle$ be an algebraic struct. Let λ be an equivalence on $\mathscr{X}$ (that is, on X). If λ is $(\boldsymbol{r}, \boldsymbol{r})$-stable, then it will be called a *congruence on $\mathscr{X}$* or an *$\boldsymbol{r}$-congruence.*

Examples. (A) If k is a natural number and λ is the least equivalence containing all $\langle x, x + k \rangle$, then λ is a congruence on N. – (B) Let $k \in \mathsf{N}$; let λ consist of all $\langle x, y \rangle \in \mathsf{N} \times \mathsf{N}$ such that $x \geqq k$, $y \geqq k$ or $x = y$; then λ is a congruence on N. – (C) Let $\langle G, \sigma \rangle$ be a commutative semi-group; denote $x\sigma y$ by $x \,.\, y$. Put $x\lambda y$ if (and only if) there are natural m, n and u, v from G such that $x \,.\, u = y^m$, $y \,.\, v = x^n$. Then λ is a congruence on G. – (D) If X, Y are sets, put $X\lambda Y$ if (and only if) $X \div Y$ is finite. Then λ is a $\bigcup$-congruence as well as a $\bigcap$-congruence and a $\langle \bigcup, \bigcap \rangle$-congruence.

8 C.11. Theorem. *Let $\mathscr{X} = \langle X, r \rangle$ be an algebraic struct (of type $\mathfrak{t}$). Let F be a mapping of $\mathscr{X}$ into a class Y. Then there is at most one algebraic structure s (of type $\mathfrak{t}$) on $F[X]$ such that $F : \mathscr{X} \to \langle F[X], s \rangle$ is a homomorphism. Such a structure exists if and only if the graph of $F^{-1} \circ F$, that is the equivalence $\{x \to x' \mid Fx = Fx'\}$, is a congruence on X.*

Proof. Suppose that s_i, $i = 1, 2$, are structures such that the mapping F_i equal to $F_i : \mathscr{X} \to \langle F_i[X], s_i \rangle$ is a homomorphism, $i = 1, 2$. Then $F_1 \circ F_2^{-1}$ is equal to $\mathrm{J} : \langle F[X], s_2 \rangle \to \langle F[X], s_1 \rangle$ and, by 8 C.2, composition-stable. Therefore, by 8 C.3, $s_1 = s_2$.

Now suppose that such a structure s exists. Then by 8 C.2, $F^{-1} \circ F$ is composition-stable, hence its graph, being an equivalence, is a congruence. On the other hand, let $\{x \to x' \mid Fx = Fx'\}$ be a congruence. Let ϱ be an arbitrary internal constituent composition of r. If, for $k = 1, 2$, $x_k \in X$, $x'_k \in X$, $Fx_k = Fx'_k$, then, by definition, $F(x_1 \varrho x_2) = F(x'_1 \varrho x'_2)$. Therefore if we put, for every $y \in F[X]$, $y' \in F[X]$, $y \bar{\varrho} y' = F(x \varrho x')$ where $y = Fx$, $y' = Fx'$, we obtain an internal composition $\bar{\varrho}$ on $F[X]$ such that $\{x \to Fx\}$ is a homomorphism-relation under ϱ and $\bar{\varrho}$. The proof is completed by considering an arbitrary external constituent of r.

8 C.12. Definition. Let $\mathscr{X} = \langle X, r \rangle$ be an algebraic struct, let Y be a class and let f be a relation (or a correspondence) such that $f : \mathscr{X} \to Y$ is a mapping which satisfies conditions indicated in 8 C.11 (i.e. the graph of $f^{-1} \circ f$ is a congruence on $\mathscr{X}$). Then the uniquely determined structure s such that $f : \mathscr{X} \to \langle Y, s \rangle$ is a homomorphism will be denoted by r/f and $\langle f[X], s \rangle$ will be denoted by $\mathscr{X}/f$. The struct $\mathscr{X}/f$ will occasionally be called the *quotient-struct* (e.g. the quotient-semi-group, the quotient-group etc.) of $\mathscr{X}$ under f, and similarly for the *quotient-structure* r/f of r under f.

Convention. If λ is a congruence on an algebraic struct $\mathscr{X} = \langle X, r \rangle$, then the symbol $\mathscr{X}/\lambda$ will be used, in general, to denote any $\mathscr{X}/f$ where f is such that $x \lambda x' \Leftrightarrow fx = fx'$. In particular, if the fibres of λ are comprisable, then $\mathscr{X}/\lambda$ denotes, as a rule, the struct $\mathscr{X}/g$ where $g = \{x \to \lambda[(x)]\}$.

Examples. (A) If $k \in \mathsf{N}$ and $fx = k \cdot x$, then $\langle \mathsf{N}, + \rangle/f$ is isomorphic with $\langle \mathsf{N}, + \rangle$ provided $k > 0$. – (B) Let $G = \mathbf{E}\{n \mid n \in \mathsf{N}, n \geqq 1\}$; let σ be a composition on G, $x \sigma y = x \cdot y$. Consider the congruence λ described in 8 C.10, example (C). Then, by the above convention, we may write $\langle G, \sigma \rangle/\lambda = \langle K, \bigcup \rangle$ where K is the collection of all finite sets of prime numbers.

8 C.13. Theorem. *Let $\mathscr{X} = \langle X, r \rangle$, $\mathscr{Y} = \langle Y, s \rangle$ be algebraic structs of the same type. Let F be a homomorphism of $\mathscr{X}$ into $\mathscr{Y}$.*

Then $\mathscr{X}/F$ and $F[\mathscr{X}]$ coincide, F is the composite of $F : \mathscr{X} \to \mathscr{X}/F$ and $\mathrm{J} : \mathscr{X}/F \to \mathscr{Y}$ (in symbols, $F = (\mathrm{J} : \mathscr{X}/F \to \mathscr{Y}) \circ (F : \mathscr{X} \to \mathscr{X}/F)$); the mapping $F : \mathscr{X} \to \mathscr{X}/F$ is a surjective homomorphism, $\mathrm{J} : \mathscr{X}/F \to \mathscr{Y}$ is an injective homomorphism.

If $\mathscr{X}$ is an algebraic struct, $H \in \text{Hom}(\mathscr{X}, \mathscr{Z})$, $G \in \text{Hom}(\mathscr{Z}, \mathscr{Y})$, $F = G \circ H$, H is surjective and G is injective, then there exists exactly one isomorphism Φ of $\mathscr{X}/F$ onto $\mathscr{Z}$ such that $H = \Phi \circ (F : \mathscr{X} \to \mathscr{X}/F)$.

Proof. The first assertion follows at once from the definition of $\mathscr{X}/F$ and $F[\mathscr{X}]$ and from 8 C.9, 8 C.11. To prove the second, it is sufficient to put $\Phi = H \circ F^{-1}$.

Remark. The theorem may be stated in a more intuitive way as follows: *every homomorphism can be expressed, in an essentially unique manner, as the composite of a surjective homomorphism and an injective one.*

8 C.14. *Let $\mathscr{X}, \mathscr{Y}$ be algebraic structs, $F \in \text{Hom}(\mathscr{X}, \mathscr{Y})$. If λ is a congruence on $\mathscr{Y}$, then $F^{-1} \circ \lambda \circ F$ is a congruence on $\mathscr{X}$.*

The proof is left to the reader.

We conclude this subsection with a proposition on the congruence generated by a given relation.

8 C.15. *Let $\mathscr{X}$ be an algebraic struct. If $\{\lambda_a \mid a \in A\}$ is an indexed class of congruences on $\mathscr{X}$, then $\bigcap \lambda_a$ is a congruence on $\mathscr{X}$.*

The straightforward proof may be omitted.

8 C.16. *Let $\mathscr{X}$ be an algebraic struct. Let κ be a relation in $\mathscr{X}$ (i.e. in $|\mathscr{X}|$). Then there exists a smallest congruence on $\mathscr{X}$ containing κ; more explicitly, there exists a uniquely determined congruence λ on $\mathscr{X}$ such that (1) $\kappa \subset \lambda$, (2) if λ' is a congruence on $\mathscr{X}$, $\kappa \subset \lambda'$, then $\lambda \subset \lambda'$.*

Proof. Clearly, $X \times X$, where $X = |\mathscr{X}|$, is a congruence on $\mathscr{X}$ containing κ. Now let λ denote the class of all $\langle x, y \rangle \in X \times X$ such that $\langle x, y \rangle \in \varrho$ whenever ϱ is a congruence on $\mathscr{X}$, $\kappa \subset \varrho$. It is easy to see that λ is a congruence possessing the required properties.

D. IDEALS

In many cases, it is important to investigate a special type of homomorphism and congruence, namely those "generated", in a specific sense, by a subset of the struct in question.

We shall limit ourselves to module-like structs (see 8 B.9). First we consider inverse images of zero.

8 D.1. Definition. Let $\mathscr{X} = \langle X, \boldsymbol{r} \rangle$ be a module-like struct (this means that either $\boldsymbol{r} = \varrho$ is an associative internal composition or $\boldsymbol{r} = \langle \varrho, \ldots \rangle$ where ϱ is an associative internal composition and the other constituents of $\boldsymbol{r}$ are distributive, respectively action-distributive with respect to ϱ). If an element $o \in X$ is neutral under ϱ and absorbing under each of the remaining constituents of r, then it is termed a *zero* of the module-like struct $\mathscr{X}$.

Clearly, the zero of a module is a zero in the above sense.

8 D.2. *Let $\mathscr{X}, \mathscr{Y}$ be module-like structs of the same type; let $\mathscr{X} = \langle X, \boldsymbol{r} \rangle$, and denote by ϱ the basic constituent of $\boldsymbol{r}$. Let $\mathscr{Y}$ possess a zero, denoted by* 0. *Let $F \in$ $\in \operatorname{Hom}(\mathscr{X}, \mathscr{Y})$ and denote by T the inverse image of* 0 *under F.*

Then the following conditions are satisfied: (1) *for any $t \in T$,* (a) $x\varrho t \in T \Leftrightarrow x \in T$, (b) $t\varrho y \in T \Leftrightarrow y \in T$, (c) $x\varrho t\varrho y \in T \Leftrightarrow x\varrho y \in T$; (2) *$T$ is absorbing under every multiplicative constituent of $\boldsymbol{r}$.*

Proof. Let $\mathscr{Y} = \langle Y, \boldsymbol{s} \rangle$ and denote by σ the basic constituent of $\boldsymbol{s}$. For any $t \in T$, we have $Ft = 0$; therefore, $F(x\varrho t) = Fx\sigma Ft = Fx$ from which assertion (a) follows, and similarly for (b) and (c).

Now let ϱ' be an internal constituent of $\boldsymbol{r}$ distinct from ϱ, and let σ' be the corresponding constituent of $\boldsymbol{s}$. Then, for any $x \in X$, $t \in T$, we have $F(x\varrho' t) = (Fx)\,\sigma'(Ft) =$ $= (Fx)\,\sigma' 0 = 0$, $F(t\varrho' x) = 0\,\sigma' F(x) = 0$; thus, T is absorbing under ϱ'. Similarly, it may be shown that T is absorbing under every external constituent of $\boldsymbol{r}$.

8 D.3. It turns out (see 8 D.15) that, for module-like structs, the above conditions (1), (2) are also sufficient for a non-empty subset T of a module-like struct $\mathscr{X}$ to be the inverse image of zero under an appropriate homomorphism. Therefore it is useful to consider sets satisfying these conditions; they will be called ideals here, although in various cases they bear a different name (e.g. normal or invariant subgroups if $\mathscr{X}$ is a group). We now state the definition explicitly.

8 D.4. Definition. Let $\mathscr{X} = \langle X, \boldsymbol{r} \rangle$ be a module-like struct; let ϱ be the basic constituent of $\boldsymbol{r}$. A non-void class $T \subset X$ is called an *ideal of $\mathscr{X}$* (or an *ideal under $\boldsymbol{r}$*, or simply an *$\boldsymbol{r}$-ideal*) if

(1) for any $t \in T$ the following holds: (a) $x\varrho t \in T \Leftrightarrow x \in T$, (b) $t\varrho y \in T \Leftrightarrow y \in T$, (c) $x\varrho t\varrho y \in T \Leftrightarrow x\varrho y \in T$;

(2) T is absorbing under every multiplicative constituent of $\boldsymbol{r}$.

Convention. Let τ denote the smallest congruence on $\mathscr{X}$ containing $T \times T$. Then $\mathscr{X}/T$ is written instead of $\mathscr{X}/\tau$.

Before proceeding to general propositions on ideals, we shall consider ideals in some special cases.

8 D.5. *Let $\mathscr{X}$ be an additive class of sets. A non-void class $\mathscr{T} \subset \mathscr{X}$ is an $\bigcup_{\mathscr{X}}$-ideal if and only if it is additive and hereditary (i.e. if it contains, with any T, each set $X \in \mathscr{X}$ such that $X \subset T$).*

Proof. There is only one composition, namely $\bigcup$, and this composition is commutative. Hence, a class $\mathscr{T}$ of sets is an ideal if and only if, for any $T \in \mathscr{T}$, and any set $X \in \mathscr{X}$, $X \in \mathscr{T}$ implies $X \cup T \in \mathscr{T}$ and $X \cup T \in \mathscr{T}$ implies $X \in \mathscr{T}$. From this the assertion follows at once.

8 D.6. *Let $\mathscr{X}$ be a multiplicative class of sets. A non-void class $\mathscr{T} \subset \mathscr{X}$ is an $\bigcap_{\mathscr{X}}$-ideal (in other words an ideal of $\langle \mathscr{X}, \bigcap \rangle$) if and only if it is multiplicative and $T \in \mathscr{T}$, $X \in \mathscr{X}$, $T \subset X$ imply $X \in \mathscr{T}$.*

Remark. If $\mathscr{X} = \exp A$, A being a set, then an $\bigcap_{\mathscr{X}}$-ideal will be called a filter on A; see Section 12.

8 D.7. *Let $\mathscr{X} = \langle X, \mathbf{r} \rangle$ be a module-like struct; let ϱ be the basic constituent of $\mathbf{r}$. Let $T \subset X$ be non-empty. If $\mathscr{X}$ has a zero, then T is an ideal of $\mathscr{X}$ if and only if it is absorbing under every multiplicative constituent of $\mathbf{r}$, and, for any $t \in T$, $x\varrho t \varrho y \in$ $\in T \Leftrightarrow x \varrho y \in T$. If ϱ is commutative, then T is an ideal if and only if it is stable under ϱ, absorbing under every multiplicative constituent of $\mathbf{r}$, and $t \in T$, $x \varrho t \in T$ implies $x \in T$. If ϱ is a group structure, then T is an ideal if and only if T is absorbing under every multiplicative constituent and stable under ϱ, $t \in T \Rightarrow \bar{t} \in T$, and $x \in X$, $t \in T \Rightarrow x \varrho t \varrho \bar{x} \in T$, where, for any $z \in X$, $\bar{z}$ denotes the inverse of z under ϱ.*

Proof. The first two assertions being almost evident, we shall prove the third. Clearly, the condition is satisfied if T is an ideal. Let the condition be fulfilled. If $x \varrho y \in T$, and $t \in T$, we have $x \varrho t \varrho \bar{x} \in T$, $(x \varrho t \varrho \bar{x}) \varrho (x \varrho y) \in T$, hence $x \varrho t \varrho y \in T$; therefore, if $x \varrho t \varrho y \in T$, $t \in T$, then $x \varrho \bar{t} \varrho (t \varrho y) \in T$, $x \varrho y \in T$. This proves the assertion.

8 D.8. It follows at once from 8 D.7 that (1) ideals (in the sense of 8 D.4) of a group coincide with its *invariant (normal) subgroups* in the sense of the current definition; (2) ideals (in the sense of 8 D.4) of a ring coincide with its *two-sided ideals* in the current sense. – Observe, however, that ideals of a semi-group $\langle G, \sigma \rangle$ are, in general, different from its ideals in the sense frequently met in the literature; the latter are termed absorbing sets here.

8 D.9. *If $\mathscr{X}$ is a module-like struct with zero* 0, *then* (0) *is an ideal and* 0 *belongs to every ideal.*

Convention. Observe that, for any module-like struct $\mathscr{X}$, $|\mathscr{X}|$ is an ideal. Ideals distinct from $|\mathscr{X}|$ and (0) will occasionally be called *proper ideals* of $\mathscr{X}$.

8 D.10. *The intersection of a family of ideals is either an ideal or void. More precisely, if $\mathscr{X}$ is a module-like struct, $\{T_a \mid a \in A\}$ is a family of ideals of $\mathscr{X}$, then either $\bigcap T_a = \emptyset$, or $\bigcap T_a$ is an ideal of $\mathscr{X}$.*

It follows from 8 B.5 that $\bigcap T_a$ satisfies (2) from 8 D.2; condition (1) from 8 D.2 is verified in a straightforward manner: e.g. if $t \in \bigcap T_a$, $x \varrho t \in \bigcap T_a$, then, for any a, $t \in T_a$, and $x \varrho t \in T_a$, hence $x \in T_a$; thus, $x \in \bigcap T_a$.

8 D.11. Theorem. *Let $\mathscr{X} = \langle X, \mathbf{r} \rangle$ be a module-like struct. If $Y \subset X$, $Y \neq \emptyset$, then there exists a smallest ideal of $\mathscr{X}$ containing Y (more precisely: there exists an ideal T of $\mathscr{X}$ such that $Y \subset T$ and $T \subset V$ whenever V is an ideal of $\mathscr{X}$, $Y \subset V$).*

Proof. Let T denote the class of all $t \in X$ such that $t \in V$ for any ideal V of $\mathscr{X}$ such that $V \supset Y$. It is easy to verify that T is an ideal (if X is comprisable, we may use 8 D.10). Clearly, T is the smallest ideal containing Y.

8 D.12. Definition. Let $\mathscr{X} = \langle X, \mathbf{r} \rangle$ be a module-like struct. If $Y \subset X$ and T is the smallest ideal of $\mathscr{X}$ containing Y, then we shall say that *Y generates T as an ideal of $\mathscr{X}$* or that Y is a *generating class for T as an ideal of $\mathscr{X}$*.

Remark. The void set generates the intersection of all ideals, provided this latter is not empty (hence, it generates (0) if 0 is the zero of $\mathscr{X}$); otherwise, it generates no ideals.

8 D.13. Before proving the main theorem announced in 8 D.3, some lemmas are given concerning congruences.

Let $\mathscr{X} = \langle X, \boldsymbol{r} \rangle$ be an algebraic struct; let λ be a symmetric relation on X; let the following condition be satisfied:

(S) *if ϱ is an internal constituent of $\boldsymbol{r}$, then, for any elements a, b, c of X, $b\lambda c$ implies $(a\varrho b)\,\lambda\,(a\varrho c), (b\varrho a)\,\lambda\,(c\varrho a)$; if an external composition ϱ over $\mathscr{A}$ is a constituent of $\boldsymbol{r}$, then, for any $a \in |\mathscr{A}|$, $b \in X$, $c \in X$, $b\lambda c$ implies $(a\varrho b)\,\lambda\,(a\varrho c)$.*

Then the smallest equivalence containing λ is a congruence on $\mathscr{X}$.

Proof. Put $\bar{\lambda} = \lambda \cup \mathsf{J}_x$. Denote the equivalence in question by v. We are to prove that v is an $\boldsymbol{r}$-congruence, i.e. a ϱ-congruence for each constituent ϱ of $\boldsymbol{r}$. We shall give the proof for an internal ϱ only; the "external" case is quite analogous. — Let xvy, $x'vy'$. Then there exist (see 3 F.4) sequences $\{x_k\}_{k=0}^{p}$, $\{x'_k\}_{k=0}^{q}$, such that $x_0 = x$, $x_p = y$, $x'_0 = x'$, $x'_q = y'$, $x_k \lambda x_{k+1}$, $x'_k \lambda x'_{k+1}$; clearly, we may suppose that $p = q$. Then $(x_0 \varrho x'_0)\,\bar{\lambda}(x_0 \varrho' x'_1)$, $(x_0 \varrho x'_1)\,\bar{\lambda}(x_1 \varrho x'_1)$, ..., $(x_{p-1} \varrho x'_p)\,\bar{\lambda}(x_p \varrho x'_p)$, from which $(x_0 \varrho x'_0)\; v\; (x_p \varrho x'_p)$ follows directly.

8 D.14. *Let $\mathscr{X} = \langle X, \boldsymbol{r} \rangle$ be a module-like struct, let ϱ_0 be the basic constituent of $\boldsymbol{r}$. Let λ be a symmetric relation on X satisfying condition* (S) *from* 8 D.13 *for each multiplicative constituent ϱ of $\boldsymbol{r}$. Let λ^* consist of all $\langle a\varrho_0 x \varrho_0 b, a\varrho_0 y \varrho_0 b \rangle$ where a, b, x, y are elements of X and $x\lambda y$. Then λ^* is a symmetric relation satisfying condition* (S) *for each constituent ϱ (including ϱ_0) of $\boldsymbol{r}$.*

This follows at once from the distributivity of the multiplicative constituents with respect to ϱ_0.

Observe that $\lambda \subset \lambda^*$ does not necessarily hold.

We are now ready to prove the main theorem on ideals. The decisive step is contained in the following proposition.

8 D.15. *Let $\mathscr{X} = \langle X, \boldsymbol{r} \rangle$ be a module-like struct with zero and let $T \subset X$ be an ideal; let v be the smallest congruence containing $T \times T$. Then T is a fibre of v and there exists a homomorphism F of $\mathscr{X}$ onto a module-like struct $\mathscr{Y}$ such that T is the inverse image of zero.*

Proof. Denote the basic constituent of $\boldsymbol{r}$ by ϱ_0. Put $\lambda = (T \times T) \cup \mathsf{J}_X$. From the properties of ideals it readily follows that λ satisfies condition (S) from 8 D.13 for each multiplicative constituent ϱ of $\boldsymbol{r}$. Now let λ^* be the relation described in 8 D.14; since X contains a zero, clearly $\lambda \subset \lambda^*$. Denote by v the smallest equivalence containing λ^*. By 8 D.13, v is a congruence; it is easy to see that it is the smallest one containing λ. Evidently, $t_1 v t_2$ for any $t_1 \in T$, $t_2 \in T$. Now let $t \in T$, $u \in X$, $t \neq u$, $u\lambda^* t$; then there exist a, b, x, y such that $u = a\varrho_0 x \varrho_0 b$, $t = a\varrho_0 y \varrho_0 b$, $x\lambda y$, that is $x \in T$, $y \in T$. From the properties of T, we have $a\varrho_0 b \in T$, $c\varrho_0 x \varrho_0 b \in T$. Thus $t \in T$, $u \in X$, $u\lambda^* t$ implies $u \in T$. Now let $t \in T$, $u \in X$, tvu. Since v is the smallest equivalence containing λ^*

there exist, by 3 F.4, $u_k \in X$ such that $u_0 = t$, $u_p = u$, $u_k\lambda^* u_{k+1}$. From this it follows at once that $u \in T$. This proves the assertion on v. The existence of an (essentially unique) homomorphism with the required properties follows from 8 C.11.

8 D.16. Let $\mathscr{X} = \langle X, \boldsymbol{r}\rangle$ be a module-like struct. If o is an element, $o \notin X$, then clearly we get a module-like struct $\mathscr{X}' = \langle X', \boldsymbol{r}'\rangle$ possessing a zero, if every constituent ϱ_k of $\boldsymbol{r}$ is extended to a composition ϱ'_k on $X \cup (o)$ in such a way that, for the basic constituent ϱ_0, $o\varrho'_0 x = x\varrho'_0 o = x$ for every $x \in X \cup (o)$, for every multiplicative internal constituent ϱ_k, $o\varrho'_k x = x\varrho'_k o = o$ for every $x \in X \cup (o)$, and similarly for an external ϱ_k. If T is an ideal of $\mathscr{X}$, then evidently $T \cup (o)$ is an ideal of $\mathscr{X}'$.

8 D.17. Theorem. *Let $\mathscr{X} = \langle X, \boldsymbol{r}\rangle$ be a module-like struct. In order that there exist a homomorphism F of $\mathscr{X}$ onto a (module-like) struct $\mathscr{Y}$ with zero 0 such that T is the inverse image of 0 it is necessary and sufficient that T be an ideal of $\mathscr{X}$.*

Proof. Necessity: 8 D.2. Sufficiency: 8 D.15, 8 D.16; in the "exceptional case" that X is the universal class, we may go over to an appropriate isomorphic struct.

8 D.18. *Let $\mathscr{X}$ be a module-like struct. Let $\boldsymbol{T}$ be a monotone collection of ideals of $\mathscr{X}$. Then $\bigcup \boldsymbol{T}$ is an ideal. (In other words, the class of all comprisable ideals of $\mathscr{X}$ is monotonically additive.)*

Proof. The conditions stated in 8 D.3 are to be verified. Put $T^* = \bigcup \boldsymbol{T}$. Write ab instead of $a\varrho b$ where ϱ is the basic constituent of $\boldsymbol{r}$. Let $t \in T^*$; let $a \in X$, $b \in X$, $a\,t\,b \in T^*$. Then there are ideals T_1, T_2 belonging to $\boldsymbol{T}$ such that $t \in T_1$, $a\,t\,b \in T_2$. Since $\boldsymbol{T}$ is monotone, there is an ideal $T' \in \boldsymbol{T}$ such that $T_1 \subset T'$, $T_2 \subset T'$. Then, T' being an ideal, we get $ab \in T'$. Therefore $ab \in T^*$, and thus condition (1 c) from 8 D.4 is fulfilled. The remaining parts of the proof are quite similar and therefore left to the reader.

8 D.19. Definition. An ideal M of a module-like struct $\mathscr{X}$ is called *maximal* if it is distinct from $|\mathscr{X}|$ and every ideal $M_1 \supset M$ is equal either to $|\mathscr{X}|$ or to M.

Examples. (A) In $\langle \mathsf{N}, +\rangle$, the maximal ideals are precisely those consisting of all $n \,.\, p$, $n \in \mathsf{N}$, where p is a fixed prime number. — (B) In $\langle \mathsf{N}, .\rangle$ e.g. the following ideal M is maximal: M consists of all $4^k \,.\, n$ where $k \in \mathsf{N}$, n is odd.

8 D.20. Theorem. *Let X be a set and let $\mathscr{X} = \langle X, \boldsymbol{r}\rangle$ be a module-like struct. Let X be generated as an ideal by a finite set. Then, for any ideal T of $\mathscr{X}$ distinct from X, there exists a maximal ideal $M \supset T$.*

Proof. Denote by $\boldsymbol{T}$ the collection of all ideals of $\mathscr{X}$ distinct from X. Let $\boldsymbol{S} \subset \boldsymbol{T}$ be a monotone collection. By 8 D.16, $S_0 = \bigcup \boldsymbol{S}$ is an ideal of $\mathscr{X}$. Suppose that $S_0 = X$. Then, denoting by K a finite set which generates X as an ideal (see 8 D.12), we have $K \subset \bigcup \boldsymbol{S}$ and therefore $K \subset S$ for some ideal $S \in \boldsymbol{S}$. This implies $S = X$ which is a contradiction. We have proved that $\boldsymbol{T}$ is monotonically additive. The assertion of the theorem follows by 4 C.3.

Remark. Clearly, the suppositions of the theorem are fulfilled e.g. if $\mathscr{X}$ is a unital ring or if $\mathscr{X} = \langle \exp A, \cap\rangle$, A being a set.

E. EMBEDDING INTO GROUPS

In this and the following subsection we shall consider two related questions: under what conditions is it possible to embed a semi-group into a group or to embed a ring into a field, i.e. a ring every non-zero element of which is invertible under multiplication. We shall give an answer for the commutative case. Based on the results obtained, integers and rational numbers are introduced. This could be done, if necessary, immediately after the introduction of natural numbers in Section 3, but it seems natural to consider these topics in a wider context.

8 E.1. Theorem. *Let $\mathscr{H} = \langle H, \sigma \rangle$ be a commutative semi-group. Then there exists a commutative group $\mathscr{G}$ and a homomorphism φ of $\mathscr{H}$ into $\mathscr{G}$ with the following property: if $\mathscr{K}$ is a commutative group and ψ is a homomorphism of $\mathscr{H}$ into $\mathscr{K}$, then there exists exactly one $\chi \in \operatorname{Hom}(\mathscr{G}, \mathscr{K})$ such that $\psi = \chi \circ \varphi$.*

The group $\mathscr{G}$ and the homomorphism φ are essentially unique: if $\mathscr{G}'$ and φ' possess the property in question, then there is a isomomorphism f of $\mathscr{G}$ onto $\mathscr{G}'$ such that $\varphi' = f \circ \varphi$.

The graph of $\varphi^{-1} \circ \varphi$ is uniquely determined and consists of all $\langle x, y \rangle$ such that $x\sigma u = y\sigma u$ for some u. The set $\varphi[H]$ is a generating set for $\mathscr{G}$ as a group.

Proof. We shall write $x + y$ instead of $x\sigma y$. Consider the semi-group $\mathscr{H} \times \mathscr{H}$ (see 8 A.6). If $\langle x, y \rangle$, $\langle x', y' \rangle$ belong to $H \times H$ put $\langle x, y \rangle \lambda \langle x', y' \rangle$ whenever $x + y' + u = y + x' + u$ for some $u \in H$. It is easy to show that λ is a congruence on $\mathscr{H} \times \mathscr{H}$. For any $\langle x, y \rangle \in H \times H$, let $\nu\langle x, y \rangle$ denote the fibre $\lambda[\langle x, y \rangle]$ of λ at $\langle x, y \rangle$; clearly, $\nu^{-1} \circ \nu = \lambda$. Put $\mathscr{G} = (\mathscr{H} \times \mathscr{H})/\nu$; let the composition on $\mathscr{G}$ still be denoted by $+$. Evidently, $\mathscr{G}$ is a commutative semi-group, $\nu\langle x, x \rangle$ with arbitrary $x \in H$ is neutral for $\mathscr{G}$. If $\xi \in \mathscr{G}$, choose $\langle x, y \rangle \in \xi$; clearly $\nu\langle y, x \rangle$ is the inverse of ξ in $\mathscr{G}$.

If $x \in H$, then $\langle x + u, u \rangle \lambda \langle x + v, v \rangle$ for any u, v from H. Therefore we may put, for any $x \in H$, $\varphi^* x = \nu\langle x + u, u \rangle$ where $u \in H$ is arbitrary. It is easy to prove that $\varphi = \langle \varphi^*, \mathscr{H}, \mathscr{G} \rangle$ is a homomorphism.

Let ψ be a homomorphism of $\mathscr{H}$ into a commutative group $\mathscr{K} = \langle K, \tau \rangle$; for convenience we write $a + b$ instead of $a\tau b$, $a - b$ instead of $a\tau\bar{b}$ where $\bar{b}$ is the inverse of b under τ. Clearly, if $\langle x, y \rangle \lambda \langle x', y' \rangle$, then $\psi x - \psi y = \psi x' - \psi y'$; now let χ be the mapping of $\mathscr{G}$ into $\mathscr{K}$ which assigns to an element $\xi \in \mathscr{G}$ the element $\psi x - \psi y$, where x, y are such that $\nu\langle x, y \rangle = \xi$. It is easy to prove that χ is a homomorphism; clearly, $\varphi \circ \chi = \psi$. It is easy to see that χ is uniquely determined.

Assertions concerning the existence and the uniqueness of λ easily imply the assertion on the essential uniqueness of $\mathscr{G}$ and φ. In its turn, this assertion implies, since $(f \circ \varphi)^{-1} \circ f \circ \varphi = \varphi^{-1} \circ \varphi$, that the graph of $\varphi^{-1} \circ \varphi$ is uniquely determined. With φ as described above, we clearly have $\varphi^{-1} \circ \varphi = \lambda$.

Finally, if $\xi \in \mathscr{G}$, $\xi = \nu\langle x, y \rangle$, then $\varphi y + \xi = \nu\langle y + u, u \rangle + \nu\langle x, y \rangle = \nu\langle x + y + u, y + u \rangle = \varphi x$. This proves that $\mathscr{G}$ is generated by $\varphi[H]$.

8 E.2. Theorem. *Let $\mathscr{H} = \langle H, \sigma \rangle$ be a commutative semi-group. Then $\mathscr{H}$ admits of an injective homomorphism in a group if and only if every $x \in H$ is virtually invertible* (*see* 6 B.11).

Proof. Let φ be an injective homomorphism of $\mathscr{H}$ into a group $\langle G, \sigma' \rangle$. Let x, u, v be elements of H, $u \neq v$. Then, denoting by $\bar{x}$ the inverse of x under σ', we have $\bar{x}\sigma'(x\sigma u) = u \neq v = \bar{x}\sigma'(x\sigma v)$ which proves $x\sigma u \neq x\sigma v$.

If the condition is fulfilled, apply 8 E.1. By 8 E.1, the graph of $\varphi^{-1} \circ \varphi$ consists of $\langle x, y \rangle$ such that $x\sigma u = y\sigma u$ for some u. Since every $u \in H$ is virtually invertible, $x\sigma u = y\sigma u \Rightarrow x = y$. Thus $\mathrm{gr}\,(\varphi^{-1} \circ \varphi)$ consists of all $\langle x, x \rangle$, $x \in H$, and therefore φ is injective. This proves the theorem.

Remarks. 1) It is easy to see that 8 E.1 and 8 E.2 remain true, with obvious changes, if we suppose that σ is a commutative semi-group structure on a class (possibly non-comprisable). — 2) Theorem 8 E.1 and its proof describe, in a formal manner and for a rather general situation, the well known device of considering "formal differences" $x - y$ of elements of a commutative semi-group.

8 E.3. *Every commutative semi-group $\mathscr{H} = \langle H, \sigma \rangle$ such that every $x \in H$ is virtually invertible, is a sub-semi-group of a commutative group $\mathscr{G} = \langle G, \sigma' \rangle$ such that for every $\xi \in G$ there exist x, y from H with $x\sigma'\xi = y$.*

This follows at once from 8 E.2.

8 E.4. We are now ready to introduce integers as elements of a group into which $\langle \mathsf{N}, + \rangle$ is embedded. This can be done in various ways. Here we choose an axiomatic definition; the set of integers will appear as a new undefined object which necessitates the introduction either of addition (for integers) as a further basic object, or the introduction of the group of integers as a fundamental concept described by axioms.

It is irrelevant, however, which method of introduction of integers is used, since, as a consequence of the above theorems, any two such sets are isomorphic, in a sense specified in the proposition below.

There exists a group $\mathscr{G} = \langle G, \sigma \rangle$ such that $\langle \mathsf{N}, + \rangle$ is a sub-semi-group of $\mathscr{G}$ and, for any $\xi \in G$, there are $m \in \mathsf{N}$, $n \in \mathsf{N}$ such that $\xi\sigma n = m$. Let $\mathscr{G}_i = \langle G_i, \sigma_i \rangle$, $i = 1, 2$, be a group, and let $\langle \mathsf{N}, + \rangle$ be a sub-semi-group of $\langle G_i, \sigma_i \rangle$; let there exist, for any $\xi \in G_i$, elements $m \in \mathsf{N}$, $n \in \mathsf{N}$ such that $\xi\sigma_i n = m$. Then there exists an isomorphism f of $\mathscr{G}_1$ onto $\mathscr{G}_2$ such that $fn = n$ for every $n \in \mathsf{N}$.

We omit the proof since the proposition follows easily from 8 E.1, 8 E.2, 8 E.3 (besisides, it is easily proved directly).

8 E.5. Defining axioms for integers.

(a) Z_{gr} *is a commutative group*;
(b) *the semi-group of natural numbers* $\langle \mathsf{N}, + \rangle$ *is a sub-semi-group of* Z_{gr};
(c) N *generates* Z_{gr} *as a group.*

We shall call Z_{gr} the *group of integers*. Its underlying set will be denoted by Z and called the *set of all integers*; its elements will be called *integers*. The structure of Z_{gr} will be denoted by $+_{\mathsf{Z}}$ or simply by $+$ and will be called the *addition for integers.*

8 E.6. We do not give an exposition of the arithmetic of integers and assume that it is known. Only two propositions will be stated, and two basic concepts will be given an explicit definition here.

Proposition and definition. *There exists exactly one composition μ on Z such that $\langle \mathsf{Z}, +_{\mathsf{Z}}, \mu \rangle$ is a ring with a unit. The struct $\langle \mathsf{N}, +, . \rangle$ is a sub-semi-ring of $\langle \mathsf{Z}, +_{\mathsf{Z}}, \mu \rangle$. The composition μ is called* multiplication for integers. *Instead of $x\mu y$, we write $x . y$ or xy (accordingly, μ will be denoted simply by a dot.).*

We omit the proof.

Proposition and definition. *There exists exactly one order τ on Z such that (1) for any $m \in \mathsf{N}$, $n \in \mathsf{N}$, $m \leqq n \Leftrightarrow m\tau n$, (2) for any x, y, z from Z, $x\tau y \Rightarrow (x + z)\,\tau(y + z)$. – The order τ will be called the* natural order for integers *and will be denoted by $\leqq_{\mathsf{Z}}$ or simply by $\leqq$.*

F. EMBEDDING INTO FIELDS

We are going to consider commutative rings such that every element, except 0, possesses an inverse. Such rings, called fields, are quite important; well known examples are: the field of rational numbers, the field of real (respectively, complex) numbers, the field of rational functions and so on. However, none of these rings could be properly considered till now, since we had not even introduced rationals. The only field which has been exactly defined is the struct $\langle A, \div_A, \bigcap_A \rangle$ where $A = (\emptyset, (a))$.

We now give a definition of fields and consider some related questions. One might expect that a proposition similar to 8 E.1 could be stated; but no theorem analogous to 8 E.1 is true, in general, for the embedding of rings into fields. Nevertheless, there is a theorem (8 F.6) similar to 8 E.3. Using this theorem, we shall introduce rational numbers.

8 F.1. Definition. Let $\mathscr{A} = \langle A, \sigma, \mu \rangle$ be a commutative ring. An element $x \in A$, $x \neq 0$, is called a *divisor of zero in $\mathscr{A}$* if there exists an $y \in A$, $y \neq 0$, such that $x\mu y = 0$.

Examples. (A) Let P be a non-empty set. Consider the ring Z^P. Then an element $\{x_p\} \in \mathsf{Z}^P$ is a divisor of zero if and only if some $x_p = 0$ and some $x_q \neq 0$. – (B) Denote by T_n, $n \in \mathsf{N}$, the set of all $k . n$, $k \in \mathsf{Z}$. Let φ be a natural homomorphism of the ring Z onto Z/T_n. If $x \in \mathsf{Z}$, then φx is a divisor of zero in Z/T_n if and only if the greatest common divisor of x and n is different from n and 1.

8 F.2. *Let $\mathscr{A} = \langle A, \sigma, \mu \rangle$ be a commutative ring. An element of $\mathscr{A}$ distinct from 0 is a divisor of zero if and only if it is not virtually invertible under μ.*

Proof. Let $\mathscr{A} = \langle A, +, . \rangle$. If $x \in A$ is a divisor of zero, let $xy = 0$, $y \neq 0$. Then $xy = x . 0 = 0$; therefore $\{y \to x . y\}$ is not one-to-one and thus x is not virtually invertible (see 6 B.11). If x is not virtually invertible, then choosing $y, z, y \neq z$ such that $x \,.\, y = x \,.\, z$ we have $x \,.\, (y - z) = 0$.

8 F.3. Definition. Let $\mathscr{A} = \langle A, \sigma, \mu \rangle$ be a commutative ring. If no $x \in A$ is a divisor of zero in $\mathscr{A}$, then we shall say that $\mathscr{A}$ is a *ring without divisors of zero* or a *ring with cancellation* (this name is motivated by the fact that, in such a ring, $a\mu x = a\mu y$ implies $x = y$ provided $a \neq 0$). The traditional term "domain of integrity" is often used to denote such rings. If A contains two elements at least and every $x \in A$, $x \neq 0$, has an inverse, then we shall say that $\mathscr{A}$ is a *field*.

Examples. (A) $\langle \mathsf{Z}, +, . \rangle$ is a ring with cancellation but not a field. – (B) The ring Z/T_n described in Example (B) from 8 F.1 is a field if and only if n is prime. – (C) Let $\mathscr{A} = \langle A, +, . \rangle$ be a given ring. Let $S(\mathscr{A})$ consist of all families $\{x_k\} \in A^{\mathsf{Z}}$ such that, for some $m \in \mathsf{Z}$, $k \leqq m \Rightarrow x_k = 0$. Define compositions σ and μ on $S(\mathscr{A})$ as follows: $\sigma = +^{\mathsf{Z}}$, i.e. $\{x_k\} \,\sigma \{y_k\} = \{x_k + y_k\}$. If $\{x_k\} \in S(\mathscr{A})$, $\{y_k\} \in S(\mathscr{A})$, put, for every $k \in \mathsf{Z}$, $z_k = x_p y_{k-p} + \ldots + x_{k-q} y_q$ where p, q are such that $x_i = 0$ for $i < p$, $y_i = 0$ for $i < q$. It is easy to see that z_k is uniquely determined; we put $\{x_k\} \,\mu\, \{y_k\} = \{z_k\}$. We leave to the reader the verification of the fact that $S(\mathscr{A}) = \langle S(\mathscr{A}), \sigma, \mu \rangle$ is a commutative ring. If $\{x_k\} \,\mu\, \{y_k\} = \{z_k\}$, $\{x_k\} \neq 0$, $\{y_k\} \neq 0$, let m, n be the least integers such that $x_m \neq 0$, $y_n \neq 0$. Clearly, $z_{m+n} = x_m \,.\, y_n$; thus, if $\mathscr{A}$ is a ring with cancellation, then $S(\mathscr{A})$ is also a ring with cancellation. – We outline a proof that if $\mathscr{A}$ is a field, then $S(\mathscr{A})$ is also a field. Clearly, the unit of $S(\mathscr{A})$ is the family $\{\alpha_k\}$ where $\alpha_k = 0$ except for $\alpha_0 = 1$. If $\{\xi_k\} \in S(\mathscr{A})$, $\xi_0 = 1$, $\xi_k = 0$ for $k < 0$, then $\{\eta_k\}$ is an inverse of $\{\xi_k\}$ under μ if $\eta_k = 0$ for $k < 0$, and $\xi_0\eta_0 = 1$, $\xi_1\eta_0 + \xi_0\eta_1 = 0$, $\xi_2\eta_0 + \xi_1\eta_1 + \xi_0\eta_2 = 0, \ldots$. This system of infinitely many equations clearly has a solution obtained recursively: $\eta_0 = 1$, $\eta_1 = \xi_1\eta_0$, $\eta_2 = -\xi_1\eta_0 + \xi_0\eta_1, \ldots$. For the general case observe that every $x \in S(\mathscr{A})$, $x \neq 0$, is of the form $x = \{z_k\} \,\mu\, \{x_k\}$ where $x_0 = 1$, $x_k = 0$ for $k < 0$, and $z_k = 0$ for every k with exactly one exception.

Remark. If $\mathscr{A} = \langle A, \sigma, \mu \rangle$ is a field, let B be the set of all $x \in A$ distinct from 0. Then $\langle B, \mu_B \rangle$ is a commutative group. It will be called the *multiplicative group* of A.

8 F.4. *Every field is a ring with cancellation. Every subring of a ring with cancellation is a ring with cancellation.*

This is evident.

8 F.5. Theorem. *Let $\mathscr{A}$ be a commutative ring with a unit element, T an ideal of $\mathscr{A}$. Then $\mathscr{A}/T$ is a field if and only if T is a maximal ideal of $\mathscr{A}$* (see 8 D.17).

Proof. Suppose $\mathscr{A}/T$ is a field; let $\mathscr{A} = \langle A, +, . \rangle$, $\mathscr{A}/T = \langle A/T, +, . \rangle$. Denote by 1 the unit element of $\mathscr{A}$, and by f the canonical mapping of $\mathscr{A}$ onto $\mathscr{A}/T$.

Now suppose T_1 is an ideal of $\mathscr{A}$, $T_1 \supset T$, $T_1 \neq T$. Choose $t \in T_1$, $t \notin T$. Then $ft \neq 0$, hence for any $x \in A$ there is $u \in A/T$ with $u \,.\, ft = fx$. If $u = fv$, then $f(vt - x) = 0$, $vt \in T_1$, $vt - x \in T_1$, hence $x \in T_1$. We have shown that $T_1 = A$. This proves that T is maximal.

Suppose T is maximal. If $x \in A$, $x \notin T$, consider the set T_x of all $zx + t$, where $z \in A$, $t \in T$. This set is clearly an ideal, $T_x \supset T$, $T_x \neq T$ (for $x \in T_x$, $x \notin T$); therefore $T_x = A$. Now let $y \in A$ be such that $yx + t = 1$. Then $(fy).(fx) = 1$. This proves that $\mathscr{A}/T$ is a field.

Corollary. *Let $\mathscr{A}$ be a commutative ring with a unit element. Then $\mathscr{A}$ is a field if and only if there are exactly two ideals of $\mathscr{A}$, namely (0) and $\mathscr{A}$.*

Remark. Let $\mathscr{A} = \langle A, +, . \rangle$ be a commutative ring. Let T be an ideal of $\mathscr{A}$. It is easy to prove that $\mathscr{A}/T$ is a ring with cancellation if and only if T has the following property: if $x . y \in T$, then either $x \in T$ or $y \in T$. Ideals with this property are called *prime ideals* of $\mathscr{A}$. Occasionally, this term will also be used for ideals of semi-rings: if $\mathscr{A} = \langle A, \varrho, \sigma \rangle$ is a commutative semi-ring, then an ideal $T \subset A$ will be called *prime* if, for any $x \in A$, $y \in A$, $x\sigma y \in T$ implies that either $x \in T$ or $y \in T$.

8 F.6. Theorem. *Let $\mathscr{A} = \langle A, \sigma, \mu \rangle$ be a commutative ring with cancellation containing more than one element. Then there exists a field $\mathscr{K} = \langle K, \sigma^*, \mu^* \rangle$ such that $\mathscr{A}$ is a sub-ring of $\mathscr{K}$ and for every $\xi \in K$ there are $u \in A$, $v \in A$ such that $\xi\mu^* v = u$. If $\mathscr{K}'$ is a field with the properties just described, then there exists exactly one isomorphism f of $\mathscr{K}$ onto $\mathscr{K}'$ such that $fx = x$ for $x \in A$.*

Proof. We shall write $x + y$ instead of $x\sigma y$, xy instead of $x\mu y$. Denote by B the set of all $\langle x, y \rangle \in A \times A$ such that $y \neq 0$. We shall introduce two compositions σ' and μ' on B putting $\langle x_1, y_1 \rangle \sigma' \langle x_2, y_2 \rangle = \langle x_1 y_2 + x_2 y_1, y_1 y_2 \rangle$, $\langle x_1, y_1 \rangle \mu'$ $\mu' \langle x_2, y_2 \rangle = \langle x_1 x_2, y_1 y_2 \rangle$ (this is possible, for $y_1 \neq 0$, $y_2 \neq 0 \Rightarrow y_1 y_2 \neq 0$). It is easy to prove that $\mathscr{B} = \langle B, \sigma', \mu' \rangle$ is a commutative semi-ring. Now, for $\langle x_1, y_1 \rangle \in$ $\in B$, $\langle x_2, y_2 \rangle \in B$ put $\langle x_1, y_1 \rangle \lambda \langle x_2, y_2 \rangle$ if and only if $x_1 y_2 = x_2 y_1$. It can be easily shown that λ is a congruence on $\mathscr{B}$.

Consider the semi-ring $\mathscr{B}/\lambda$ and let f be the canonical mapping of $\mathscr{B}$ onto $\mathscr{B}/\lambda$; we denote σ'/λ by σ^* and μ'/λ by μ^*. If $y \in A$, $y \neq 0$, then $f\langle 0, y \rangle$ is a zero element of $\mathscr{B}$, and for any $f\langle x, y \rangle$ we have $(f\langle x, y \rangle)\, \sigma^*(f\langle -x, y \rangle) = 0$; thus σ^* is a group structure. Hence $\mathscr{B}/\lambda$ is a ring. Clearly, $f\langle x, x \rangle$ (where $x \in A$ is distinct from 0, but otherwise arbitrary) is a unit element for $\mathscr{B}/\lambda$, and, for any $f\langle x, y \rangle \in \mathscr{B}/\lambda$ distinct from 0, $f\langle y, x \rangle$ is its inverse under μ^*. We have proved that $\mathscr{B}/\lambda$ is a field.

Obviously, $f\langle x, u \rangle = f\langle xv, uv \rangle$ for any u, v from A distinct from 0. Therefore we may put, for any $x \in A$, $gx = f\langle xu, u \rangle$, where $u \in A$, $u \neq 0$. It is easy to show that g is an injective homomorphism of $\mathscr{A}$ into $\mathscr{B}/\lambda$.

The rest of the proof may be left to the reader.

8 F.7. Definition. Let $\mathscr{K} = \langle K, \sigma, \mu \rangle$ be a field. We shall say that a set $X \subset K$ *generates $\mathscr{K}$ as a field* if K is the smallest field (under $\langle \sigma, \mu \rangle$) containing X.

For instance, in 8 F.6 we have shown that A generates $\mathscr{K}$ as a field.

8 F.8. We are now able to introduce rational numbers. The situation is analogous, in a sense, to that considered e.g. at the beginning of subsection 3 D as well as in 8 E. We already know, as a quite special case of 8 F.6, that there exists a field Q

such that $\langle Z, +, .\rangle$ is a subring of Q and, for every $\xi \in Q$, we have $\xi v = u$ for some $u \in Z$, $v \in Z$, $v \neq 0$; such a field Q is essentially unique (in the sense indicated in 8 F.6). We could choose (even exhibit effectively) a certain field Q with these properties to serve as the field of rational numbers. For reasons explained in connection with the introduction of natural numbers in Section 3, however, we adopt as in 3 D, a different procedure stating properties of Q mentioned above as axioms for a certain fixed object which will be called the field of rational numbers.

8 F.9. Defining axioms for rational numbers.

(a) Q_{fd} *is a field;*

(b) *the ring of integers* $\langle Z, +, .\rangle$ *is a subring of* Q_{fd};

(c) Z *generates* Q_{fd} *as a field.*

Definition. We shall denote the underlying set of Q_{fd} by Q. Every element of Q will be called a *rational number* (or, briefly, a *rational*); Q will be called the *set of (all) rational numbers*, and Q_{fd} will be called the *field of rational numbers.*

Clearly, by the defining axioms, every integer is a rational number. We state explicitly the following proposition:

If ξ is a rational number, then there are integers u, v, $v \neq 0$, such that $v\xi = u$.

It is to be pointed out that Q_{fd} is a fixed field; this field, its underlying set and every rational number are determined uniquely; however, we can assert no more of their properties than those which follow from the axioms.

8 F.10. Convention. The compositions of Q_{fd} will be denoted by $+_Q$ and $\cdot_Q$ or simply by $+$ and . and will be called the *addition (multiplication) for rationals*; thus $Q_{fd} = \langle Q, +_Q, \cdot_Q\rangle$. Usually, we shall write $x + y$ instead of $x +_Q y$, xy or $x \,.\, y$ instead of $x \,._Q y$. The symbol Q_{fd} will be used only rarely in the sequel.

We do not examine the field $\langle Q, +, .\rangle$; basic properties of this field and the current arithmetic of rationals will be assumed to be known. However, the order on Q will be defined explicitly.

8 F.11. Proposition and definition. *Let ϱ consist of all $\langle x, y\rangle \in Q \times Q$ such that there exists a number $n \in N$, $n > 0$, with $nx \in N$, $ny \in N$, $nx \leq ny$. Then ϱ is an order, called the* natural order on Q.

9. CARDINALS

In this section, questions are investigated concerning, roughly speaking, the "size" or the "number of elements" of sets. It is natural to consider two sets as possessing "the same number of elements" if and only if they are equipollent (see 3 A.1). As we know (see 4 C.9) the "size" of finite sets can be characterized by means of natural numbers; a similar characterization of the "size" of infinite sets will be achieved by means of cardinal numbers.

We intend to assign to every set X a certain object, a cardinal number, which will be called the cardinality or the power of X, in such a way that two sets are assigned the same object if and only if they are equipollent. This can certainly be done. Indeed, consider the relation $\varrho = \{X \to Y \mid X \in \mathscr{S},\ Y \in \mathscr{S},\ X \text{ is equipollent to } Y\}$ where $\mathscr{S}$ is the class of all sets. By the Axiom of Choice, there exists a class $\mathscr{B} \subset \mathscr{S}$ such that, for every $X \in \mathscr{S}$, there exists precisely one $Y \in \mathscr{B}$ with $X\varrho Y$; put $\langle X, Y\rangle \in \varphi$ if and only if $X \in \mathscr{S}$, $Y \in \mathscr{B}$, $X\varrho Y$. Now we could declare the set φX to be the cardinality of the set X; in other words, we could "choose" one "representative" from each class of mutually equipollent sets and declare it to be the cardinality of every set equipollent with this selected set.

Of course, the choice of such "privileged" sets to serve as cardinals necessarily contains certain artificial characteristics. We shall therefore choose a procedure similar to the one used in Section 3 for natural numbers. Cardinal numbers will be introduced by means of axioms containing a new basic concept, the relation of being the cardinality of a set; the fact, indicated above, that there exists a single-valued relation φ on the class of all sets such that $\varphi Y = \varphi X$ if and only if X and Y are equipollent will serve to ascertain that the conditions imposed by axioms can be satisfied.

In addition, as mentioned above, it is desirable that the cardinality of any finite set X coincide with the "number of elements" of X in the sense of 3 E.7. This is, of course, not necessary, and an approach is quite acceptable under which finite cardinals would be different from natural numbers. However, it seems quite natural to 'identify" these cardinals with natural numbers; the symbol card X has already been introduced (in 3 E.7) with a view to such an approach. Therefore, we shall introduce explicitly an axiom stating that the cardinals of finite sets coincide with the natural numbers.

A. CARDINAL NUMBERS

9 A.1. Axioms for cardinal numbers.

(a) card *is a single-valued relation whose domain is the class of all sets;*
(b) card X = card Y *if and only if* X *and* Y *are equipollent;*
(c) card $\mathsf{N}_k = k$ *for each* $k \in \mathsf{N}$.

9 A.2. Definition. If X is a set, card X will be called the *cardinality* or the *power* of the set X. If x is the power of some set, i.e. if $x \in \mathbf{E}$ card, we shall say that x is a *cardinal number* or simply a *cardinal.* A cardinal x will be called *finite* (*infinite*) if $x = \text{card } X$, where X is finite (infinite); this definition is correct, since the particular choice of the set X is clearly irrelevant.

9 A.3. We shall now order cardinals "by their size", putting card $X <$ card Y if, roughly speaking, there are "less" elements in X than in Y. The meaning of X having less elements than Y is clearly that X is equipollent to a part of Y without being equipollent to Y; it is not sufficient to require that X be equipollent with a proper subset of Y, since every infinite Y is equipollent to a proper subset of itself.

Before stating the definition we shall prove several propositions (some of which are not quite trivial) on equipollent sets, the main purpose being to establish which cases can occur from among the following ones which appear possible a priori: (a) X is equipollent to a part of Y and Y is equipollent to a part of X; (b) X is equipollent to a part of Y but Y is equipollent to no part of X; (c) Y is equipollent to a part of X but X is equipollent to no part of Y; (d) neither is X equipollent to any part of Y nor is Y equipollent to any part of X. We shall prove that in case (a), the sets X and Y are equipollent, card X = card Y; in case (b), we shall put card $X <$ card Y, and similarly, in case (c), card $Y <$ card X; as for case (d), which would mean that the sets X and Y are "incomparable", it will be proved that this cannot happen.

9 A.4. Theorem. *Let A, B and C be classes, $A \subset B \subset C$. Let A be equipollent to C. Then B is equipollent to C.*

Proof. Since A, C are equipollent, there exists a bijective relation f on C onto A. Denote by D the class of all x such that, for some $k \in \mathsf{N}$, $k \geqq 1$, $x \in f^k[C - B]$ (see 3 F.2); put $E = (C - B) \cup D$. Then, clearly, $f[E] = D$, $E \cap (B - D) = \emptyset$, $E \cup (B - D) = C$. Now put $g = f_E \cup \mathsf{J}_{B-D}$ (that is, $gx = fx$ if $x \in E$, $gx = x$ if $x \in B - D$). Then $\mathbf{D}g = E \cup (B - D) = C$, $\mathbf{E}g = f[E] \cup (B - D) = B$, and it is easy to see that g is one-to-one.

From Theorem 9 A.4 we immediately have the following assertion.

9 A.5. Theorem. *Let A and B be classes. Let there exist $C \subset A$, $D \subset B$ such that A is equipollent to D and B is equipollent to C. Then A and B are equipollent.*

Proof. As A is equipollent to D there exists a bijective relation f on A onto D. Then $f[C] \subset D \subset B$; but $f[C]$ is equipollent to C, hence to B so that, by 9 A.4, D is equipollent to B. Therefore A, being equipollent to D, is equipollent to B.

9 A.6. Theorem. *Let A and B be sets. Let A be equipollent to no part of B. Then B is equipollent to some subset of A.*

Proof. Suppose that the contrary holds. Let $\mathscr{A}$ be the set of all one-to-one relations R such that $\mathbf{D}R \subset A$, $\mathbf{E}R \subset B$. Obviously $\emptyset \in \mathscr{A}$. It is easy to see that $\mathscr{A}$ is monotonically additive, for if $\mathscr{X} \subset \mathscr{A}$ is monotone, then clearly $\xi = \bigcup \mathscr{X}$ is a relation for A and B which is one-to-one by 4 A.2. Consequently according to theorem 4 C.3 there is a maximal set μ in $\mathscr{A}$. According to the stated assumption we have $\mathbf{D}\mu \neq A$, $\mathbf{E}\mu \neq B$. Let us choose $x \in A - \mathbf{D}\mu$, $y \in B - \mathbf{E}\mu$. Then clearly $\mu \cup (\langle x, y \rangle) \in \in \mathscr{A}$, which contradicts the maximality of μ.

Remark. The theorem can also be proved by use of theorems 4 A.7, 4 A.8 (and the Axiom of Choice). The reader is invited to carry this out as an exercise.

Before introducing the definition of the relation $<$ for cardinals we shall state the following proposition which will guarantee that this definition will not depend on the special choice of the representing sets.

9 A.7. *Let A_1, A_2, B_1, B_2 be sets. Let A_1 be equipollent to A_2 and let B_1 be equipollent to B_2. Let there exist a $C_1 \subset B_1$ such that A_1 is equipollent to C_1. Then there exists a $C_2 \subset B_2$ such that A_2 is equipollent to C_2 (at the same time, of course, C_2 is equipollent to C_1).*

The simple proof is left to the reader.

9 A.8. Definition. A quasi-order ϱ is called *monotone* if, for any $x \in \mathbf{D}\varrho \cup \mathbf{E}\varrho$, $y \in \mathbf{D}\varrho \cup \mathbf{E}\varrho$, $x \neq y$, either $x\varrho y$ or $y\varrho x$.

9 A.9. *The relation $\{X$ is equipollent to a subset of $Y\}$ is a monotone quasi-order on the class of all sets.*

This follows at once from 9 A.6.

B. ORDER ON CARDINALS

9 B.1. Theorem. *There exists exactly one relation on the class of all cardinals which contains a pair $\langle \text{card } X, \text{card } Y \rangle$ if and only if X is equipollent to a subset of Y. This relation is a monotone order.*

This follows at once from the preceding propositions, in particular from 9 A.9.

Convention. The relation indicated above will be called the *natural order for cardinals* and denoted by $\leqq$. If x, y are cardinals, $x \leqq y$, we shall say that x is less than or equal to y. As usual, instead of $x \leqq y$ we also write $y \geqq x$; if $x \leqq y$, $x \neq y$, then $x < y$ or $y > x$ is written.

Remark. The restriction of the order $\leqq$ to the set N coincides with the order on N described in 3 D.2.

Now we shall give some easily proved propositions on the natural order for cardinals.

9 B.2. Theorem. *If A is a set, then* card $A <$ card (exp A).
This follows at once from 3 A.6.

9 B.3. Theorem. *Let A be a set of cardinal numbers. Then there exists a cardinal number b such that $b > a$ for each $a \in A$.*

Proof. By the Axiom of Choice, there exists a family $\{X_a \mid a \in A\}$ such that card $X_a = a$. Then $\bigcup X_a$ is a set and evidently $a \leqq$ card $(\bigcup X_a)$ for each $a \in A$. In view of 9 B.2 it is now sufficient to put $b =$ card (exp $(\bigcup X_a)$).

It can be shown that the cardinals form a non-comprisable class; we shall in fact prove a stronger assertion: a class of cardinals is a set if and only if it is "bounded from above".

9 B.4. Theorem. *Let A be a class of cardinal numbers. Then A is a set if and only if there exists a cardinal number b such that $a < b$ for each $a \in A$.*

Proof. Let A be a set. Then by 9 B.3 there is a cardinal b so that $b > a$ for each $a \in A$. Conversely let there exist such a cardinal b. Let B be a set, card $B = b$. For each $a \in A$, $a <$ card B, hence there exists a set $X \subset B$ such that card $X = a$. Consequently $A \subset$ card [exp B] from which it follows (since the relation card is single-valued) that A is a set.

9 B.5. *If f is a comprisable single-valued relation, then* card $\mathbf{E}f \leqq$ card $\mathbf{D}f$.
This follows at once from 4 B.2 (corollary).

9 B.6. Convention. The power of N (hence, of any countably infinite set) will be denoted by $\aleph_0$.

The reasons for this notation will be clear later (see 11 B.9) in a wider context.

9 B.7. *The cardinal number $\aleph_0$ is infinite. If a is an infinite cardinal, then $\aleph_0 \leqq a$.*
This follows at once from 4 C.9.

9 B.8. *If A is an infinite set, then there exists a disjoint collection $\mathscr{A}$ of countably infinite sets such that $\bigcup \mathscr{A} = A$.*

Proof. Let $\boldsymbol{M}$ be the set of all non-void disjoint collections $\mathscr{X} \subset$ exp A consisting of countably infinite sets. It is easy to see that $\boldsymbol{M}$ is monotonically additive. Hence there exists a maximal $\mathscr{X}_0 \in \boldsymbol{M}$. We are going to prove that $\bigcup \mathscr{X}_0 = A$. Suppose $S = \bigcup \mathscr{X}_0 \neq$ $\neq A$. If $A - S$ is finite, choose a set $X_0 \in \mathscr{X}_0$, put $X'_0 = X_0 \cup (A - S)$, and $\mathscr{X}'_0 =$ $= (\mathscr{X}_0 - (X_0)) \cup (X'_0)$. Then $\mathscr{X}'_0$ possesses the properties required. If $A - S$ is an infinite set, there is, by 4 C.9, a countably infinite set $B \subset A - S$. Clearly $\mathscr{X}_0 \cup (B)$ is disjoint and consists of countably infinite sets. This is a contradiction, for $\mathscr{X}_0$ is maximal in $\boldsymbol{M}$.

9 B.9. *If A is an infinite set, then there exists a set B such that A is equipollent to $B \times$ N.*

Proof. Let $\mathscr{A}$ be a disjoint system of countably infinite sets, $\bigcup \mathscr{A} = A$. It is easy to deduce from 4 B.2 that there exists a family $\{\varphi_X \mid X \in \mathscr{A}\}$ such that every φ_X

is a bijective relation on N onto X. Now, if $\langle X, n\rangle \in \mathscr{A} \times \mathsf{N}$, put $\varphi\langle X, n\rangle = \varphi_X n$. It is easy to see that the relation φ is one-to-one on $\mathscr{A} \times \mathsf{N}$ onto A.

9 B.10. *Let A be an infinite set. Then, for any countable set X, A is equipollent with $A \cup X$; if X is countable non-empty, then A is also equipollent with $A \times X$.*

Proof. We shall only show that $A \times \mathsf{N}$ is equipollent to A, leaving the rest of the proof to the reader. – By 9 B.9, there exists a set B such that A is equipollent to $B \times \mathsf{N}$; now it is clear that $A \times \mathsf{N}$ is equipollent to $(B \times \mathsf{N}) \times \mathsf{N}$ and therefore equipollent to A, because $(B \times \mathsf{N}) \times \mathsf{N}$ is evidently equipollent to $B \times (\mathsf{N} \times \mathsf{N})$ and $\mathsf{N} \times \mathsf{N}$ is, by 3 G.9, equipollent to N.

9 B.11. *If X, Y are sets,* $\operatorname{card} Y \leqq \operatorname{card} X$, *$X$ is infinite, then $X \cup Y$ is equipollent to X.*

This follows at once from the fact that $X \times (0, 1)$ is equipollent to X, and theorem 9 A.4.

9 B.12. *If X is an infinite set, $Y \subset X$,* $\operatorname{card} Y < \operatorname{card} X$, *then $X - Y$ is equipollent to X.*

Proof. Suppose $\operatorname{card}(X - Y) < \operatorname{card} X$. We have either $\operatorname{card}(X - Y) \leqq \operatorname{card} Y$ or $\operatorname{card} Y \leqq \operatorname{card}(X - Y)$. By 9 B.11, we have in the first case $\operatorname{card} X \leqq \operatorname{card} Y$, in the second case $\operatorname{card} X \leqq \operatorname{card}(X - Y)$, hence, in each case, there is a contradiction. This proves that $\operatorname{card}(X - Y) \geqq \operatorname{card} X$, hence $X - Y$ is equipollent to X.

9 B.13. Theorem. *Let $\mathscr{A}$ be a monotone collection of sets. Let m be a cardinal number and let* $\operatorname{card} X < m$ *for each $X \in \mathscr{A}$. Then* $\operatorname{card} \bigcup \mathscr{A} \leqq m$.

Proof. I. Let M be a set such that $\operatorname{card} M = m$. It is obviously sufficient to consider $\mathscr{A} \neq \emptyset$. Put $A = \bigcup \mathscr{A}$. Let $\mathscr{A}'$ be the system of all $\bigcup \mathscr{X}$, where $\mathscr{X} \subset \mathscr{A}$; we have, of course, $\mathscr{A} \subset \mathscr{A}'$. By 4 A.4, $\mathscr{A}'$ is monotone and completely additive. Let $\mathscr{F}$ be the system of all one-to-one relations φ such that $\mathbf{D}\varphi \in \mathscr{A}'$, $\mathbf{E}\varphi \subset M$. As $\emptyset \neq \mathscr{A} \subset \subset \mathscr{A}'$ and $\operatorname{card} X < m$ for $X \in A$, we obtain $\mathscr{F} \neq \emptyset$.

We shall show that $\mathscr{F}$ is monotonically additive. Let $\mathscr{X} \subset \mathscr{F}$ be monotone. By 4 A.2, $\bigcup \mathscr{X}$ is one-to-one. We have $\mathbf{D}(\bigcup \mathscr{X}) = \bigcup_{\varphi \in \mathscr{X}} \mathbf{D}\varphi$, and since $\mathscr{A}'$ is completely additive, we have $\mathbf{D}(\bigcup \mathscr{X}) \in \mathscr{A}'$. Further we have $\mathbf{E}(\bigcup \mathscr{X}) = \bigcup_{\varphi \in \mathscr{X}} \mathbf{E}\varphi \subset M$. Hence $\bigcup \mathscr{X} \in \mathscr{F}$.

II. Let $\varphi \in \mathscr{F}$, $\mathbf{D}\varphi \neq A$. Then $\mathbf{D}\varphi = \bigcup \mathscr{X}$ for some monotone collection $\mathscr{X} \subset \mathscr{A}$. Since $\bigcup \mathscr{X} \neq A = \bigcup \mathscr{A}$, there exists a $Y \in \mathscr{A}$ such that $Y \subset X$ for no $X \in \mathscr{X}$, hence $Y \supset X$ for all $X \in \mathscr{X}$ and therefore $Y \supset \bigcup \mathscr{X}$. If $Y \neq \bigcup \mathscr{X}$, put $Z = Y$; if $Y = \bigcup \mathscr{X}$, then $Y \neq A$, hence there exists a $Z \in \mathscr{A}$, $Z \supset Y$, $Z \neq Y$. Clearly, $\operatorname{card} \mathbf{E}\varphi = \operatorname{card} \mathbf{D}\varphi \leqq \operatorname{card} Y < m$. Hence, by 9 B.12, $\operatorname{card}(M - \mathbf{E}\varphi) = m$. Thus there exists a bijective relation ψ on $Z - \mathbf{D}\varphi$ into $M - \mathbf{E}\varphi$. Then $\mathbf{D}(\varphi \cup \psi) = Z$, $\varphi \cup \psi \in \mathscr{F}$.

III. By the above argument, the suppositions of theorem 4 C.6 are fulfilled. Hence there exists a $\psi \in \mathscr{F}$ with $\mathbf{D}\psi = A$. This proves that $\operatorname{card} \bigcup \mathscr{A} \leqq m$.

9 B.14. Theorem. *In every non-void class of cardinal numbers there exists a least element.*

Proof. Suppose that, on the contrary, $\mathscr{X} \neq \emptyset$ is a class of cardinals and that in $\mathscr{X}$ there does not exist a least element, and let us derive a contradiction. Evidently $\mathscr{X}$ contains only infinite cardinals and for each $a \in \mathscr{X}$ there exists a $b \in \mathscr{X}$ such that $b < a$. Let us choose $c \in \mathscr{X}$; there exists a set C of power c. Let $\mathscr{A}$ be the system of all $X \subset C$ such that card $X < m$ for each $m \in \mathscr{X}$. $\mathscr{A} \neq \emptyset$ since a finite $X \subset C$ belongs to $\mathscr{A}$. We shall show that $\mathscr{A}$ is monotonically additive. Let $\mathscr{B} \subset \mathscr{A}$ be monotone. Let $a \in \mathscr{X}$; then there is a $b \in \mathscr{X}$, $b < a$. For each $X \in \mathscr{A}$ and hence for each $X \in \mathscr{B}$ we have card $X < b$. Hence by 9 B.13 card $\bigcup \mathscr{B} \leqq b < a$. Obviously $\bigcup \mathscr{B} \subset C$; hence $\bigcup \mathscr{B} \in \mathscr{A}$.

Because $\mathscr{A}$ is monotonically additive, there exists by 4 C.3 a maximal $A \in \mathscr{A}$. Because card $A < c$, there exists an $x \in C - A$; then clearly $A \cup (x) \in \mathscr{A}$, which contradicts the maximality of A.

C. ARITHMETIC OF CARDINALS

We shall now consider the "arithmetic" of cardinals, defining for them addition, multiplication and exponentiation as generalizations of the corresponding compositions for natural numbers. This is motivated by theorem 3 E.8 in which addition, multiplication and exponentiation introduced for natural numbers by recursive definitions have been related to set-operations such as the union and the cartesian product.

9 C.1. Proposition and definition. *Consider the relation consisting of all* $\langle\langle a, b\rangle, c\rangle$ *where* a, b, c *are such that there exist sets* A, B, C, *with* card $A = a$, card $B = b$, card $C = c$, $A \cap B = \emptyset$, $C = A \cup B$. *This relation is a composition on the class of all cardinals. It will be called the* addition (for cardinals) *and denoted by* $+$; *the cardinal* $a + b$ *will be called the* sum of a and b.

Remark. In other words, if a, b are cardinals, then their sum $a + b$ is defined as the cardinality of some $A \cup B$ where A, B are arbitrary disjoint sets with card $A = a$, card $B = b$. It is asserted that this definition is "correct", i.e. that $a + b$ does not depend on the particular choice of A, B. This remark also applies, with appropriate changes, to the multiplication (9 C.3) and exponentiation (9 C.5) of cardinals.

Proof. Suppose that A', B' are disjoint, card $A' = a$, card $B' = b$. Clearly, if φ (respectively ψ) is bijective on A onto A' (respectively, on B onto B') then $\varphi \cup \psi$ is bijective on $A \cup B$ onto $A' \cup B'$. Thus, $A \cup B$ and $A' \cup B'$ are equipollent, which proves the proposition.

9 C.2. *Addition for cardinals restricted to the set of natural numbers coincides with addition on* N (*as defined in* 3 E.1).

This follows at once from 3 E.8.

9 C.3. Proposition and definition. *Consider the relation consisting of all* $\langle\langle a, b\rangle, c\rangle$ *such that for some sets* A, B, $a =$ card A, $b =$ card B, $c =$ card $(A \times B)$. *This relation is a composition on the class of all cardinals. It will be called* multiplication (for cardinals); *the element assigned by this composition to* $\langle a, b\rangle$ *will be denoted by* $a \, . \, b$ *or* ab *and will be called the* product *of* a *and* b.

This follows at once from the fact that $A \times B$ and $A' \times B'$ are equipollent provided A and A', B and B' are equipollent.

9 C.4. *Multiplication for cardinals restricted to the set of natural numbers coincides with multiplication on* N *(as defined in* 3 E.2*)*.

9 C.5. Proposition and definition. *Consider the relation consisting of all* $\langle\langle a, b\rangle, c\rangle$ *where* a, b, c *are cardinals and, for some* A, B, C, *we have* $\operatorname{card} A = a$, $\operatorname{card} B = b$, $\operatorname{card} C = c$, $A^B = C$. *This relation is a composition on the class of all cardinals. It will be called* exponentiation (for cardinals) *and the element assigned thereby to* $\langle a, b\rangle$ *will be denoted* a^b.

This follows at once from the fact that A^B, P^Q are equipollent provided A and P, B and Q are equipollent.

9 C.6. *Exponentiation for cardinals restricted to* N *coincides with exponentiation* $\{\langle a, b\rangle \to a^b\}$ *on natural numbers (as defined in* 3 E.3*)*.

9 C.7. *The pair* $\langle +, .\rangle$ *formed by the addition and the multiplication on cardinals is a commutative semi-ring structure on the class* **E** card *of all cardinals; for this structure* 0 *is the zero element,* 1 *is the unit element.*

All the assertions in question, such as $x + (y + z) = (x + y) + z$, etc., follow at once from the corresponding facts concerning sets and their equipollence. Thus, $x(y + z) = xy + xz$ follows from $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$, $x \, . \, (y \, . \, z) = (x \, . \, y) \, . \, z$ follows from the fact that $X \times (Y \times Z)$ and $(X \times Y) \times Z$ are equipollent. Therefore the detailed proof may be left to the reader.

9 C.8. *If* a, b, c *are cardinals, then* (1) $a^{b+c} = a^b \, . \, a^c$; (2) $a^{bc} = (a^b)^c$; (3) $(ab)^c = a^c \, . \, b^c$.

The proof consists in applying propositions on equipollence contained in 5 D.6.

R e m a r k s. 1) Let a be a cardinal. Clearly, $0^a = 0$ provided $a \neq 0$; $a^0 = 1$ (in particular, $0^0 = 1$); $a^1 = a$, $1^a = 1$. – 2) If A is a set, $\operatorname{card} A = a$, then $2^a = \operatorname{card}(\exp A)$, hence, $a < 2^a$ (this follows from 9 B.2). In particular, $\aleph_0 < 2^{\aleph_0}$. – 3) The assertion "if $\aleph_0 \leqq a \leqq 2^{\aleph_0}$, then either $a = \aleph_0$ or $a = 2^{\aleph_0}$" is known as the "continuum hypothesis" (observe that $2^{\aleph_0}$ is the power of the set R of all real numbers, see 10 H). With a view to some recent results, it seems that the continuum hypothesis is an undecidable sentence in the axiomatic system presented here (after a suitable formalisation). We do not consider these questions here since they lie outside the scope of this book.

9 C.9. *If* a *is an infinite cardinal, then* $\aleph_0 \leqq a$, $a + \aleph_0 = a$, $a \, . \, \aleph_0 = a$. *In particular,* $\aleph_0^n = \aleph_0$ *for every finite* $n \geqq 1$. *If* n *is a finite cardinal (i.e. a natural number), then* $n < \aleph_0$, $n + \aleph_0 = \aleph_0$, $n \, . \, \aleph_0 = \aleph_0$.

These assertions follow at once from the preceding propositions, particularly 9 B.10, 9 B.11.

9 C.10. Theorem. *If a is an infinite cardinal, then $a + a = a$, $a \, . \, a = a$.*

Proof. The first assertion follows at once from 9 B.11. To prove the second one, let A be a set, card $A = a$. Consider the class $\mathcal{F}$ of all one-to-one relations φ such that $\mathbf{D}\varphi = X \subset A$, $\mathbf{E}\varphi = X \times X$. If $\mathcal{M} \subset \mathcal{F}$ is monotone non-void, then clearly $\bigcup \mathcal{M}$ is a one-to-one relation; let it be denoted by g. Clearly $\mathbf{D}g = \bigcup\{\mathbf{D}\varphi \mid \varphi \in \mathcal{M}\}$, $\mathbf{E}g = \bigcup\{\mathbf{E}\varphi \mid \varphi \in \mathcal{M}\}$; the monotonicity of $\mathcal{M}$ and the fact that $\mathbf{E}\varphi = \mathbf{D}\varphi \times \mathbf{D}\varphi$, for every $\varphi \in \mathcal{F}$, imply at once that $\mathbf{E}g = \mathbf{D}g \times \mathbf{D}g$. Therefore, $\mathcal{F}$ is monotonically additive. There exists, by 4 C.9, a countably infinite $X \subset A$ and, by 3 G.9, X is equipollent to $X \times X$; hence $\mathcal{F}$ is not void. By 4 C.3 there exists a maximal $f \in \mathcal{F}$.

Put $B = \mathbf{D}f$, $b = \text{card } B$; clearly, $b^2 = b$ since $\mathbf{E}f = \mathbf{D}f \times \mathbf{D}f$. Suppose $b < a$. Then, by 9 B.12, card $(A - B) = a$. Hence, there exists a set $C \subset A - B$ with card $C = b$. We already know that $b^2 = b$; since b is infinite, we have (by the first assertion of the theorem we are now proving) $b = b^2 + b^2 + b^2$. Therefore, there exists a one-to-one relation h on C onto $(B \times C) \cup (C \times B) \cup (C \times C)$.

Then $f' = f \cup h$ is a one-to-one relation, $\mathbf{D}f' = B \cup C$, $\mathbf{E}f' = (B \times B) \cup \mathbf{E}h = (B \cup C) \times (B \cup C)$. Therefore $f' \in \mathcal{F}$, which is a contradiction since f is maximal on $\mathcal{F}$, $f' \supset f$, $f' \neq f$.

We have proved that $b = a$, hence $a^2 = a$.

9 C.11. *Let a, b, p, q be cardinals. If $a \leqq p$, $b \leqq q$, then $a + b \leqq p + q$, $ab \leqq pq$, $a^b \leqq p^q$. If $a < p$, $b < q$, then $a + b < p + q$, $ab < pq$.*

Proof. The case $a \leqq p$, $b \leqq q$ is quite easy. Consider the case $a < p$, $b < q$. If a, b are finite, the assertion is evident, Therefore, suppose $a \leqq b$, b infinite. Then, by 9 C.10, $a + b = b$, $ab = b$, hence $a + b < q \leqq p + q$, $ab < q \leqq pq$.

D. FAMILIES OF CARDINALS

We conclude this section with some propositions concerning sums and products of families of cardinal numbers.

9 D.1. *Let $\{x_a \mid a \in A\}$ be a family of cardinal numbers. Then (1) there is exactly one cardinal number s such that there exists a family of sets $\{X_a \mid a \in A\}$ for which* card $X_a = x_a$, *for any $a \in A$, and* card $\sum_{a \in A} X_a = s$; *(2) there is exactly one cardinal number p such that there exists a family of sets $\{X_a \mid a \in A\}$ for which* card $X_a = x_a$, *for any $a \in A$, and* card $\prod_{a \in A} X_a = p$.

Proof. It follows from 4 B.2 that there exists a family of sets $\{X_a \mid a \in A\}$ such that card $X_a = x_a$. By 5 D.9, if $\{X'_a \mid a \in A\}$ is a family of sets such that card $X'_a = x_a$, then ΣX_a and $\Sigma X'_a$, as well as ΠX_a and $\Pi X'_a$ are equipollent. This proves the proposition.

Definition. The cardinal s described above will be denoted by $\Sigma\{x_a \mid a \in A\}$ or by Σx_a and called the *cardinal sum* (usually simply *sum*) *of the family* $\{x_a \mid a \in A\}$ or, for convenience, the sum of cardinal numbers x_a, $a \in A$. The cardinal p described above will be denoted by $\Pi\{x_a \mid a \in A\}$ or by Πx_a and called the *cardinal product* (usually simply *product*) *of the family* $\{x_a \mid a \in A\}$ or, for convenience, the product of cardinal numbers x_a, $a \in A$.

Remarks. 1) Observe that the cardinal sum of the void family is the number 0, and the cardinal product of the void family is the number 1. — 2) We use the symbol Σ for the (cardinal) sum of cardinal numbers and for the sum of sets (in the sense of Section 5); the symbol Σ will also be used for some other purposes in the sequel. Nevertheless, the proper meaning will always be clear from the context. — 3) For any family of sets $\{X_a \mid a \in A\}$, we have $\Sigma \operatorname{card} X_a = \operatorname{card} \Sigma X_a$; if $\{X_a\}$ is disjoint, then $\Sigma \operatorname{card} X_a = \operatorname{card} \bigcup X_a$. — 4) Clearly, $\Sigma\{x_i \mid i \in (1, 2)\} = x_1 + x_2$. — 5) The preceding remarks, with appropriate changes, also apply to the product and to the symbol Π.

9 D.2. *Let φ be a one-to-one relation on a set A onto a set B. Let $\{x_b \mid b \in B\}$ be a family of cardinals. Then $\sum_{b\in B} x_b = \sum_{a\in A} x_{\varphi a}$ and $\prod_{b\in B} x_b = \prod_{a\in A} x_{\varphi a}$.*

This assertion (the "commutativity law" for sums and products of families of cardinals) follows at once from 5 B.5, 5 A.11.

9 D.3. *Let $\{B_a \mid a \in A\}$ be a disjoint family of sets; put $B = \bigcup B_a$. For every $a \in A$, let $\{x_b \mid b \in B_a\}$ be a family of cardinal numbers. Then $\sum_{a\in A} \sum_{b\in B_a} x_b = \sum_{b\in B} x_b$ and $\prod_{a\in A} \prod_{b\in B_a} x_b = \prod_{b\in B} x_b$.*

This proposition follows immediately from 5 B.6 and 5 A.13.

9 D.4. *For any family $\{x_a \mid a \in A\}$ of cardinals and any cardinal number y, $(\Sigma x_a) \,.\, y = \Sigma(x_a \,.\, y)$.*

This assertion is deduced at once from 5 B.7.

9 D.5. *If b is a cardinal, A is a set,* card *$A = a$ and $b_x = b$ for every $x \in A$, then $\sum_{x\in A} b_x = a \,.\, b$, $\prod_{x\in A} b_x = b^a$.*

This is clear since if $B_x = B$ for every $x \in A$, then $\sum_{x\in A} B_x = A \times B$, $\prod_{x\in A} B_x = B^A$.

9 D.6. *If $\{x_a\}$ is a family of cardinals, and y is a cardinal, then $\prod_a x_a^y = (\prod_a x_a)^y$.*

This follows from 5 D.7.

9 D.7. *Let x be a cardinal and let $\{y_a\}$ be a family of cardinals; put $y = \sum_a y_a$. Then $x^y = \prod_a x^{y_a}$.*

This follows from 5 D.8.

9 D.8. *If $\{x_a \mid a \in A\}$, $\{y_a \mid a \in A\}$ are families of cardinals, and $x_a \leqq y_a$ for every $a \in A$, then*

$$\sum_a x_a \leqq \sum_a y_a, \quad \prod_a x_a \leqq \prod_a y_a.$$

9 D.9. Theorem. *If* $\{x_a \mid a \in A\}$, $\{y_a \mid a \in A\}$ *are non-empty families of cardinals and* $x_a < y_a$ *for every* $a \in A$, *then*

$$\sum_a x_a < \prod_a y_a .$$

Proof. Clearly $\Sigma x_a \leqq \Pi y_a$. There exist families of sets $\{X_a\}$, $\{Y_a\}$ such that card $X_a = x_a$, card $Y_a = y_a$. Put $X = \Sigma X_a$, $Y = \Pi Y_a$. We have to prove that card $X <$ $<$ card Y. Let f be a single-valued relation on X into Y. For any $a \in A$ and $x \in X_a$ put $f_a x = \mathrm{pr}_a (f\langle a, x\rangle)$; then f_a is a single-valued relation on X_a into Y_a, and therefore $Y_a - f_a[X_a] \neq \emptyset$ since card $X_a <$ card Y_a. Hence there exists a $z = \{z_a\} \in Y$ such that $z_a \in Y_a - f_a[X_a]$ for every $a \in A$. It is easy to see that $z \notin f[X]$. Thus, for any $f \in Y^X$, the set $f[X]$ is different from Y. This proves that card $X <$ card Y.

10. ORDER

The concept of quasi-order (in particular, of order) has already been introduced in Section 1, and many examples of orders occur in the preceding sections, some of the most important being the following: the order $\subset$ on the class of all sets, the order $\leqq$ on the set N and the order $\leqq$ on the class of all cardinals. As yet, however, we have not discussed this notion in a systematic manner. This will be done in the present section.

A. QUASI-ORDER

First we shall give basic definitions (some of which have already been introduced).

10 A.1. Definition. A transitive relation ϱ is called a *quasi-order in* A if $\mathbf{D}\varrho \cup \cup \mathbf{E}\varrho \subset A$, a *quasi-order on* A if $\mathbf{D}\varrho \cup \mathbf{E}\varrho = A$. A quasi-order ϱ is called an *order* if it is reflexive and $x\varrho y$, $y\varrho x$ implies $x = y$. A quasi-order ϱ is called *strict* if never $x\varrho x$.

Remark. Clearly, if ϱ is a strict quasi-order, then $x\varrho y$, $y\varrho x$ hold simultaneously for no x, y.

10 A.2. Definition. A quasi-order ϱ is called *distinguishing* if $x\varrho y$, $y\varrho x$ implies $x = y$.

Clearly, every order and every strict quasi-order is distinguishing.

To denote a reflexive quasi-order (usually, an order) which is fixed during an exposition, we shall often use the symbol $\leqq$ (whereas $<$ will be used to denote a strict quasi-order). Sometimes, given two classes X, Y endowed with (reflexive) quasi-orders ϱ, σ, we shall even use the same symbol $\leqq$ to denote both ϱ and σ.

10 A.3. *Let ϱ be a quasi-order. Then the class ϱ_1 of all $\langle x, y\rangle \in \varrho$ such that $\langle y, x\rangle \notin \varrho$ is a strict quasi-order, the class ϱ_2 of all $\langle x, y\rangle$ such that either $\langle x, y\rangle \in \in \varrho_1$ or $x = y \in \mathbf{D}\varrho \cup \mathbf{E}\varrho$ is an order, and the class $\bar{\varrho}$ of all $\langle x, y\rangle$ such that either $x = y$ or both $\langle x, y\rangle \in \varrho$ and $\langle y, x\rangle \in \varrho$ is an equivalence.*

The easy proof is omitted.

Definition. We shall call the relations ϱ_1, ϱ_2 and $\bar{\varrho}$ described above, respectively, the *strict quasi-order associated with* ϱ, the *order associated with* ϱ and the *equivalence associated with* ϱ.

Remark. The above concepts and terms will be used chiefly for the case where ϱ is a distinguishing quasi-order. In such a case $\varrho_1 \subset \varrho \subset \varrho_2$, ϱ_1 is the largest strict quasi-order contained in ϱ, and ϱ_2 is the smallest order containing ϱ.

Example. If $<$ and $\leqq$ are the usual relations on $\mathbf{Q}$, then $<$ is the strict quasi-order associated with $\leqq$, and $\leqq$ is the order associated with $<$.

Convention. If $\leqq$ (respectively, $\geqq$) denotes an order, then $<$ (respectively, $>$) is used to denote the associated strict quasi-order, and conversely (see 10 A.2, convention).

10 A.4. *Let ϱ be a quasi-order on A. Let φ be a single-valued relation such that $\mathbf{D}\varphi = A$ and $\{x \to y \mid \varphi x = \varphi y\}$ coincides with the equivalence associated with ϱ. Then $\sigma = \varphi \circ \varrho \circ \varphi^{-1}$ is a distinguishing quasi-order on $\mathbf{E}\varphi = \varphi[A]$.*

Proof. Clearly, σ consists of $\langle a, b\rangle$ such that, for some x, y, we have $x\varrho y$, $\varphi x = a$, $\varphi y = b$. It is clear that σ is transitive.

If $a\sigma b$, $b\sigma a$, then there are x, y, x', y' such that $\varphi x = \varphi x' = a$, $\varphi y = \varphi y' = b$, $x\varrho y$, $y'\varrho x'$. We have $x\vartheta x'$, $y\vartheta y'$ where $\vartheta = \varphi^{-1} \circ \varphi$. Since ϑ coincides with the equivalence associated with ϱ, we find that (1) either $x = x'$ or $x\varrho x'$, $x'\varrho x$, (2) either $y = y'$ or $y\varrho y'$, $y'\varrho y$. This implies $y\varrho x$, hence $x\vartheta y$; therefore $\varphi x = \varphi y$, which proves $a = b$.

Definition. We shall say that the σ described above is the *distinguishing quasi-order induced by ϱ (under φ)*; the order (strict quasi-order) associated with σ will be called the *order (strict quasi-order) induced by ϱ (under φ)*.

Examples. (A) The order induced by the quasi-order $\{X$ is equipollent with a part of $Y\}$ under the relation card coincides with the relation $\leqq$ on the class of all cardinals. – (B) Consider a semi-group structure μ on a class X. Put $x\sigma y$ if and only if $y = z\mu x$ for some $z \in X$, $x\bar{\sigma}y$ if and only if $y^p = z\mu x$ for some $z \in X$ and some $p \in \mathbf{N}$, $p \geqq 1$, y^p being defined, as usual, by $y^1 = y$, $y^{n+1} = y^n\mu y$. If $\langle X, \mu\rangle =$ $= \langle \mathbf{N}, +\rangle$, then σ coincides with the natural order on $\mathbf{N}$, $\bar{\sigma}$ is a reflexive quasi-order and its associated equivalence has two fibres: (0) and $\mathbf{N} - (0)$. If $A = \mathbf{N} - (0)$, $\langle X, \mu\rangle = \langle A, +\rangle$, then σ coincides with the natural quasi-order $<$ on A; on the other hand, $\bar{\sigma} = A \times A$. If $\langle X, \mu\rangle = \langle A, .\rangle$, then it can be shown that the relation $\subset$ restricted to the set of all finite sets $X \subset P$ (where P is the set of all prime numbers) coincides with the order induced by $\bar{\sigma}$ under the relation which assigns to every $n \in A$ the set of all those $p \in P$ which divide n.

10 A.5. Definition. If ϱ is a quasi-order on A and $X \subset A$, then the relation $\varrho \cap$ $\cap (X \times X)$, which is a quasi-order, will be termed the *restriction* of ϱ to a quasi-order (order, strict quasi-order, etc.) on X and will be denoted by ϱ_X.

Remark. 1) The terminology and notation just introduced may be ambiguous since ϱ_X is also used to indicate the domain-restriction of an (arbitrary) relation ϱ or the restriction of a composition ϱ. It will be used only if its meaning is sufficiently clear from the context. – 2) It is clear that the restriction of an order is an order, etc.

10 A.6. *Let ϱ be a quasi-order on X; let σ be the distinguishing quasi-order induced by ϱ under a mapping relation φ. Let $Y \subset X$. Then $\sigma_{\varphi[Y]}$ coincides with the distinguishing quasi-order induced by ϱ_Y under φ_Y.*

10 A.7. *Let ϱ be a quasi-order in a class A. Let $a \in A$, $b \in A$, and suppose that $b\varrho a$ does not hold. Let σ be the smallest transitive relation* (see 1 C.4) *such that $\varrho \subset \sigma$, $\langle a, b\rangle \in \sigma$.*

If ϱ is distinguishing, then σ is also distinguishing; in particular, if ϱ is an order, then σ is an order.

Proof. Put $\varrho' = \varrho \cup (\langle a, b\rangle)$. The relation σ consists (see 3 F.4) of all $\langle u, v\rangle$ such that there exists a finite sequence $\{z_k \mid k \in \mathsf{N}_{n+1}\}$ with $u = z_0$, $v = z_n$, $\langle z_k, z_{k+1}\rangle \in \in \varrho'$ for all $k \in \mathsf{N}_n$. Let ϱ be distinguishing. Suppose that u, v are such that $u\sigma v$, $v\sigma u$.

Then there exists a sequence $\{z_k \mid k \in \mathsf{N}_{p+1}\}$ and a number p, $0 \leqq p \leqq n$, such that $z_k\varrho' z_{k+1}$ for $k \in \mathsf{N}_n$, $z_0 = u$, $z_p = v$, $z_n = u$. If there is no k such that $\langle z_k, z_{k+1}\rangle = = \langle a, b\rangle$, then $z_k\varrho z_{k+1}$ for all $k \in \mathsf{N}_n$, hence $u\varrho v$, $v\varrho u$ which implies $u = v$. If there exist such k, choose the least one. If there is some $l > k$ with $\langle z_l, z_{l+1}\rangle = = \langle a, b\rangle$, then choosing the least such l we obtain $\langle z_i, z_{i+1}\rangle \in \varrho$ for all i with $k < i < l$, hence either $z_{k+1}\varrho\, z_l$ (in the case $k + 1 < l$) or $z_{k+1} = z_l$ (if $k + 1 = l$). Since the first case gives $b\varrho a$, which is excluded, the second case must take place and therefore $b = a$, $\varrho' = \varrho$, $\sigma = \varrho$, from which $u = v$ follows.

Consider now the case that there is exactly one k with $\langle z_k, z_{k+1}\rangle = \langle a, b\rangle$. Then either $0 \leqq k < p$ or $p \leqq k < n$; it is sufficient to consider one of these cases, e.g. $0 \leqq k < p$. Then, clearly, $z_0\varrho z_k$, $z_{k+1}\varrho z_p$, $z_p\varrho z_n$, i.e. $u\varrho a$, $b\varrho v$, $v\varrho u$; it follows that $b\varrho a$, which is impossible.

Thus we have proved that $u = v$, and therefore the first assertion is established. The second is an immediate corollary.

10 A.8. Definition. A quasi-order ϱ is called *monotone* if, for any $x \in \mathbf{D}\varrho \cup \mathbf{E}\varrho$, $y \in \mathbf{D}\varrho \cup \mathbf{E}\varrho$, $x \neq y$, either $x\varrho y$ or $y\varrho x$.

Example. If $\mathscr{A}$ is a class of sets, then clearly $\subset_{\mathscr{A}}$ is monotone if and only if $\mathscr{A}$ is a monotone class of sets.

10 A.9. Definition. An order ϱ on X is called *maximal* if there is no order σ on X such that $\sigma \supset \varrho$, $\sigma \neq \varrho$.

10 A.10. Theorem. *An order is maximal if and only if it is monotone.*

"If" is clear. "Only if" follows at once from 10 A.7.

10 A.11. *Let A be a set. Then each of the following sets is completely multiplicative and monotonically additive: the set of all quasi-orders in A; the set of all reflexive quasi-orders in (or on) A; the set of all distinguishing quasi-orders in (or on) A; the set of all orders in (or on) A.*

Proof. We prove only the last assertion, leaving the rest to the reader. If $\mathscr{M}$ is a non-void system of orders on A, then it is clear that $\bigcap\mathscr{M}$ is also an order on A. Now let $\mathscr{M}$ be a monotone non-void system of orders on A. Consider $\mu = \bigcup\mathscr{M}$.

Clearly μ is reflexive, and it is easy to see that μ is a quasi-order. Now if $x\mu y$, $y\mu x$, then there exist $\varrho \in \mathscr{M}$, $\sigma \in \mathscr{M}$ such that $x\varrho y$, $y\sigma x$. Since $\mathscr{M}$ is monotone, either $\varrho \subset \sigma$ or $\sigma \subset \varrho$. Suppose $\varrho \subset \sigma$; then $x\sigma y$ which, together with $y\sigma x$, gives $x = y$.

10 A.12. Theorem. *Let A be a set; let ϱ be an order in A. Then there exists a monotone order σ on A such that $\sigma \supset \varrho$.*

Proof. By 10 A.11 and 4 C.3, there exists an order σ in A which is maximal in the set of all orders in A and contains ϱ. This order is monotone by 10 A.10 (and clearly is on A).

Corollary. *Every set can be monotonically ordered (more precisely: if A is a set, there exists a monotone order on A).*

B. ORDERED CLASSES

10 B.1. Definition. Let X be a class, ϱ a quasi-order in X. Then $\langle X, \varrho \rangle$ is called a *quasi-ordered class* (a *quasi-ordered set*, if X and hence ϱ are comprisable). If $\langle X, \varrho \rangle$, $\langle Y, \sigma \rangle$ are quasi-ordered classes, we shall say that $\langle Y, \sigma \rangle$ is *embedded in* $\langle X, \varrho \rangle$ or that $\langle Y, \sigma \rangle$ is a *quasi-ordered subclass of* $\langle X, \varrho \rangle$ if $Y \subset X$ and $\sigma = \varrho_Y$.

If ϱ is an order on X, then $\langle X, \varrho \rangle$ is called an *ordered class*; if ϱ is a strict quasi-order in X, then $\langle X, \varrho \rangle$ is called a *strictly quasi-ordered class*; if ϱ is monotone, then $\langle X, \varrho \rangle$ is called a *monotone quasi-ordered class* (or a *monotonically quasi-ordered class*), etc.

Remark. If σ is a quasi-order and σ_X is the restriction of σ to a quasi-order on a class X, then we shall often write $\langle X, \sigma \rangle$ instead of $\langle X, \sigma_X \rangle$ and say, for convenience, that X is endowed with σ when we mean that X is endowed with σ_X.

Examples. (A) For any class X, $\langle X, \mathsf{J}_X \rangle$ is an ordered class. – (B) For any class X, $\langle X, X \times X \rangle$ is a quasi-ordered class (observe that any order induced by $X \times X$ is of the form $(\langle a, a \rangle)$, whereas the associated order (see 10 A.3) on X is equal to J_X. – (C) Let ϱ be the least quasi-order containing all pairs $\langle x, (x) \rangle$ and $\langle x, x \rangle$. Within the framework of the theory of classes and sets presented in this book, it is hardly possible to prove or disprove that ϱ is an order. Nevertheless, the restriction of ϱ to the least ϱ-saturated class containing $\emptyset$ is an order which is isomorphic to the natural order on N. – In the following three examples, the classes under consideration are endowed with a restriction of the inclusion. – (D) The class $\exp(X \times X)$, i.e. the class of all comprisable relations in a given class X. – (E) The subclass of the preceding class consisting of all equivalences in X. – (F) For a given algebraic structure α on a set X, the class of all congruences (see 8 C.10) under α.

10 B.2. Conventions. A quasi-ordered class $\langle X, \varrho \rangle$ will often be denoted, in accordance with convention 7 A.2, simply by X provided its structure ϱ is clear from the context. The distinction between $\langle X, \varrho \rangle$ and X will be disregarded in various expressions and notations; e.g. we shall speak, as usual, of elements of a quasi-ordered class $\mathscr{X}$ instead of elements of its underlying class.

10 B.3. Definition. Let a quasi-ordered class $\langle X, \varrho \rangle$ be given. Let $a \in X$, $b \in X$. Then, as already defined in Section 1, we put $]\!] \leftarrow, a]\!] = \varrho^{-1}[(a)] \cup (a)$; $[\![a, \rightarrow [\![= \varrho[(a)] \cup (a)$; $]\!] \leftarrow, a [\![=]\!] \leftarrow, a]\!] - [\![a, \rightarrow [\![$; $]\!] a, \rightarrow [\![= [\![a, \rightarrow [\![-]\!] \leftarrow, a]\!]$; $[\![a, b]\!] = [\![a, \rightarrow [\![\cap]\!] \leftarrow, b]\!]$, etc. (see 1 C.6).

Moreover, sometimes (only rarely) the symbol $]\!] \leftarrow, \rightarrow [\![$ will be used to denote the whole class X.

Remarks and conventions. 1) The relation ϱ^{-1} occurring above is the inverse relation to ϱ in the sense of 1 B.8; if ϱ is a quasi-order (an order, etc.), ϱ^{-1} will be termed the *inverse quasi-order* (*order*, etc.). – 2) It is easy to see that $]\!] \leftarrow, a]\!] \cap [\![a, \rightarrow [\![$ coincides with the class of elements equivalent with a under the equivalence associated (see 10 A.3) with the given quasi-order; thus, if this quasi-order is either strict or an order, then $]\!] \leftarrow, a]\!] \cap [\![a, \rightarrow [\![= (a)$. – 3) We do not introduce special terms for the various classes described above and refer to them all indifferently as *intervals* (of a given quasi-ordered class).

10 B.4. Definition. Let a quasi-ordered class $\langle X, \varrho \rangle$ be given. A class $Y \subset X$ is called *left-saturated* (*right-saturated*) if it is saturated under ϱ^{-1} (respectively, under ϱ).

Remark. Let $\mathscr{A}$ be a class of sets. If $\mathscr{A}$ is left-saturated relative to the order $\subset$ (i.e., if $X \in \mathscr{A}$, $Y \subset X$ imply $Y \in \mathscr{A}$), then $\mathscr{A}$ is called *hereditary*.

10 B.5. *Let ϱ be an arbitrary relation. Then the class of all ϱ-saturated sets is completely additive and completely multiplicative. In particular, the class of all left-saturated (or right-saturated) subsets of a given quasi-ordered class is completely additive and completely multiplicative.*

Remark. Every left-saturated subset of a quasi-ordered class is a union of a family of intervals of the form $]\!] \leftarrow, a]\!]$, and similarly for right-saturated subsets.

10 B.6. Definition. Let $\langle X, \varrho \rangle$ be a quasi-ordered class. A class $Y \subset X$ is called *interval-like* (*under* ϱ) if, for any $a \in Y$, $b \in Y$, we have $[\![a, b]\!] \subset Y$.

Example. Let A be an uncountable set; let exp A be ordered by inclusion. Then the set of all countably infinite $X \subset A$ is interval-like.

Remark. We shall show later (10 C.3, 10 C.4) that, roughly speaking, a non-void set is interval-like if and only if it is an inverse fibre of an order-preserving mapping onto an ordered class.

10 B.7. *The class of all interval-like subsets of a given quasi-ordered class is completely multiplicative and monotonically additive.*

We only indicate the proof for monotone additivity. If $\mathscr{Y}$ is a monotone system of interval-like subsets, put $Y_0 = \bigcup \mathscr{Y}$. If $a \in Y_0$, $b \in Y_0$, then, for some $Y_1 \in \mathscr{Y}$, $Y_2 \in \mathscr{Y}$, we have $a \in Y_1$, $b \in Y_2$. Thus either $Y_1 \subset Y_2$ or $Y_2 \subset Y_1$; in the first case $a \in Y_2$, hence $[\![a, b]\!] \subset Y_2 \subset Y_0$.

10 B.8. Definition. If $\{\mathcal{X}_a \mid a \in A\} = \{\langle X_a, \sigma_a \rangle \mid a \in A\}$ is a family of quasi-ordered sets, then its *cartesian* (or *cardinal*) *product*, denoted by $\Pi\{\mathcal{X}_a \mid a \in A\}$ or $\Pi\{\mathcal{X}_a\}$, etc., is by definition the set $\Pi\{X_a\}$ endowed with the quasi-order $\Pi\{\sigma_a\}$ (that is, $\Pi_{\mathrm{rel}}\{\sigma_a \mid a \in A\}$; see 5 C.2). In other words, the quasi-order of $\Pi\mathcal{X}_a$ consists of all $\langle\{x_a\}, \{y_a\}\rangle$ such that $x_a\sigma_a y_a$ for each a. – If $\mathcal{A} = \langle A, \varrho\rangle$, $\mathcal{B} = \langle B, \sigma\rangle$ are quasi-ordered classes, then we denote by $\mathcal{A} \times \mathcal{B}$ their product $\langle A \times B, \varrho \times \sigma\rangle$ where $\varrho \times \sigma$ is taken in the sense of 5 C.1.

Remark. We write $\mathcal{X}^A$ instead of $\Pi\mathcal{X}_a$ if all $\mathcal{X}_a$ are equal to $\mathcal{X}$.

Examples. (A) Consider the two-element set (0, 1), endowed with the natural order. Then it is easy to prove that, for any set B, the ordered set $(0, 1)^B$ is isomorphic (see 10 C.1) with $\langle \exp B, \subset\rangle$. – (B) Let $\{X_a \mid a \in A\}$ be a family of sets; let every $\exp X_a$ be ordered by inclusion. Then $\Pi \exp X_a$ is isomorphic (see 10 C.1) with $\exp(\Sigma X_a)$ ordered by inclusion.

10 B.9. *Let $\{X_a \mid a \in A\}$ be a family of quasi-ordered sets. Then ΠX_a is ordered if and only if every X_a is ordered; the quasi-order of ΠX_a is strict if and only if the quasi-order of every X_a is strict.*

We omit the obvious proof. – Observe, however, that the quasi-order of ΠX_a is not monotone, as a rule, even if the quasi-order of every X_a is monotone; if σ_a are orders, and σ'_a are the associated strict quasi-orders, then $\Pi\sigma'_a$ is not, in general, the strict quasi-order associated with $\Pi\sigma_a$.

10 B.10. Definition. Let $\mathcal{A} = \langle A, \varrho\rangle$, $\mathcal{B} = \langle B, \sigma\rangle$ be quasi-ordered classes, ϱ being a distinguishing quasi-order (see 10 A.2). Their *lexicographical product* denoted $\mathcal{A} \times_{\mathrm{lex}} \mathcal{B}$ or $\langle A \times B, \varrho \times_{\mathrm{lex}} \sigma\rangle$, or simply $\mathcal{A} \times \mathcal{B}$ if there is no danger of misunderstanding, is the class $A \times B$ endowed with the quasi-order μ defined as follows: $\langle a, b\rangle \mu \langle a', b'\rangle$ if and only if either $a\varrho a'$, $a \neq a'$ or $a = a'$, $b\sigma b'$.

Remark. The extension of the above definition to the product of finitely many quasi-ordered classes is immediate.

Example. Consider the set $\mathcal{F}$ of linear real-valued functions on the real interval $[\![0, \rightarrow [\![$; for $f \in \mathcal{F}$, $g \in \mathcal{F}$ put $f\sigma g$ if $fx \leqq gx$ for large x. Then σ is an order on $\mathcal{F}$, and it is easy to see that $\langle \mathcal{F}, \sigma\rangle$ is isomorphic with $R \times_{\mathrm{lex}} R$, R being the set of all real numbers endowed with its natural order.

10 B.11. *The lexicographical product of finitely many ordered (strictly quasi-ordered, monotonically ordered) classes is an ordered (strictly quasi-ordered, monotonically ordered) class.*

C. ORDER-PRESERVING MAPPINGS

It is intuitively clear how those mappings are defined for quasi-ordered classes which will play a role analogous to that of homomorphisms for algebraic structs or continuous mappings for topological spaces; for quasi-ordered classes, such mappings will preserve, in some sense, the quasi-order. On the other hand, in contradistinction to the situation for algebraic structs, there are different ways of introducing "order-preserving" correspondences. For this reason, we shall give a definition of order-preserving correspondences only for the "single-valued case"; the corresponding property e.g. of equivalences will be described explicitly whenever it will be needed.

10 C.1. Definition. Let $\mathscr{A} = \langle A, \varrho \rangle$, $\mathscr{B} = \langle B, \sigma \rangle$ be quasi-ordered classes. A single-valued relation f for A and B is called *order-preserving* (*under* ϱ *and* σ) or (ϱ, σ)*-preserving* if $x \in A$, $y \in A$, $x\varrho y$ imply $(fx)\, \sigma\, (fy)$, *order-reversing* (*under* ϱ *and* σ) or (ϱ, σ)*-reversing* if it is order-preserving under ϱ and σ^{-1} (or, which is the same, under ϱ^{-1} and σ), *order-isomorphic* (*under* ϱ *and* σ) or (ϱ, σ)*-isomorphic* or simply *isomorphic* if it is one-to-one and $x\varrho y$ if and only if $(fx)\, \sigma (fy)$.

A single-valued correspondence f for $\mathscr{A}$ and $\mathscr{B}$ will be called *order-preserving*, *order-reversing*, *order-isomorphic* (or simply *isomorphic* or else an *order-isomorphism*) if its graph gr f has the corresponding property (under the structures ϱ and σ of $\mathscr{A}$ and $\mathscr{B}$). An order-preserving (order-reversing) relation or correspondence is called also *increasing* (*decreasing*).

Finally, $\mathscr{A}$ and $\mathscr{B}$ are said to be *order-isomorphic* (or simply *isomorphic*) if there exists an order-isomorphism of $\mathscr{A}$ onto $\mathscr{B}$.

Examples. (A) The relation card (see 9 A.1) is order-preserving (under $\subset$ and $\leqq$). – (B) As in 10 A.4, example (B), let μ be an associative composition on a class X. Put $\nu = \{x \to y \mid y = z\mu x$ for some $z \in X\}$. Then ν is a quasi-order on X. Every relation $\{x \to x\mu a\}$, a fixed, is (ν, ν)-preserving whereas a relation $\{x \to b\mu x\}$ is, in general, not (ν, ν)-preserving. – (C) Let A be a set; for $X \subset A$ put $\gamma X = A - X$. Then $\gamma : \langle \exp A, \subset \rangle \to \langle \exp A, \subset \rangle$ is order-reversing, $\gamma : \langle \exp A, \subset \rangle \to \langle \exp A, \supset \rangle$ is isomorphic. – (D) Denote by σ the quasi-order on N obtained in the manner described in the example (B) above, if μ is the usual multiplication on N. Let P denote the set of all prime natural numbers. For any $p \in P$ and any $n \in \mathsf{N}$, $n \geqq 1$, let $\varphi_p(n)$ be the largest r such that $p^r \sigma n$, and let $\varphi(n) = \{\varphi_p(n) \mid p \in P\}$. Put $A = \mathsf{N} - (0)$. Then it is easy to see that φ is an order-isomorphic correspondence on $\langle A, \sigma \rangle$ into $\langle \mathsf{N}, \leqq \rangle^P$.

Remark. It may happen that a mapping $f : \mathscr{X} \to \mathscr{Y}$ is simultaneously increasing and decreasing without being constant (e.g. any mapping of $\langle A, J_A \rangle$ into a reflexive quasi-ordered class is increasing and decreasing); however, if $\mathscr{X}$ is monotone and the quasi-order of $\mathscr{Y}$ is distinguishing, then every such mapping is constant.

10 C.2. *Let* $\langle A, \varrho \rangle$, $\langle B, \sigma \rangle$, $\langle C, \tau \rangle$ *be quasi-ordered classes. Let* ψ *be a* (ϱ, σ)*-preserving mapping relation for* A *and* B, *and let* φ *be a* (σ, τ)*-preserving mapping relation for* B *and* C. *Then* $\varphi \circ \psi$ *is a* (ϱ, τ)*-preserving mapping relation for* A *and* C.

The proof is straightforward and therefore omitted.

Remark. There is a certain analogy between order-preserving mappings and homomorphisms. However, there are also striking differences. For instance, if f is a bijective order-preserving mapping, then it may happen that f^{-1} is not order-preserving. A trivial example: $f = \mathsf{J} : \langle \mathsf{N}, \mathsf{J}_{\mathsf{N}} \rangle \to \langle \mathsf{N}, \leqq \rangle$.

10 C.3. Consider an ordered class $\langle A, \varrho \rangle$ and an equivalence λ on A. It is natural to ask under what conditions is there an order-preserving mapping f of $\langle A, \varrho \rangle$ onto an ordered class such that $x\lambda x' \Leftrightarrow fx = fx'$. We shall not give a complete answer, but only a necessary condition (see below) and a sufficient one (see 10 C.4).

Let $\mathscr{A} = \langle A, \varrho \rangle$, $\mathscr{B} = \langle B, \sigma \rangle$ be quasi-ordered classes and let σ be distinguishing. Let $f : \mathscr{A} \to \mathscr{B}$ be an order-preserving single-valued correspondence. Then every inverse fibre $f^{-1}[y]$, $y \in \mathscr{B}$, is interval-like.

Proof. Let $fx = y$, $fx' = y$, $x\varrho z$, $z\varrho x'$. Then $y\sigma(fz)$, $(fz)\,\sigma y$ and therefore $fz = y$, since σ is distinguishing.

10 C.4. *Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered class. Let f be a mapping of $\mathscr{A}$ onto a class B; let λ be an equivalence on A such that $x\lambda x' \Leftrightarrow fx = fx'$. Let every fibre of λ (i.e. every inverse fibre of f) be interval-like; suppose that if $x\lambda x'$, $y\lambda y'$, $x\varrho y$, then either $x'\varrho y'$ or $x\lambda y$. Let σ consist of all $\langle \xi, \eta \rangle$ such that for some x, y we have $\xi = fx$, $\eta = fy$, $x\varrho y$. Then (1) σ is a distinguishing quasi-order on B, (2) $f : \mathscr{A} \to \langle B, \sigma \rangle$ is order-preserving, and (3) $\sigma \subset \sigma'$ whenever σ' is a quasi-order on B such that $f : \mathscr{A} \to \langle B, \sigma' \rangle$ is order-preserving.*

Proof. Let $\xi\sigma\eta$, $\eta\sigma\zeta$. Choose x, y, t, z with $fx = \xi$, $fy = \eta$, $x\varrho y$, $ft = \eta$, $fz = \zeta$, $t\varrho z$. Then $y\lambda t$, hence either $y\varrho z$ and therefore $x\varrho z$, $\xi\sigma\zeta$, or $y\lambda z$ and therefore $\eta = \zeta$, $\xi\sigma\zeta$. Thus σ is a quasi-order. If $\xi\sigma\eta$, $\eta\sigma\xi$, choose x, y, x', y' with $fx = fx' = \xi$, $fy = fy' = \eta$, $x\varrho y$, $y'\varrho x'$. By the suppositions made, either $y\lambda x$ or $y\varrho x$ which implies $x\lambda y$ since $f^{-1}[\xi]$ is interval-like; in each case $\xi = fx = fy = \eta$.

Clearly, $f : \mathscr{A} \to \langle B, \sigma \rangle$ is order-preserving. If $f : \mathscr{A} \to \langle B, \sigma' \rangle$ is order-preserving, then let $\xi\sigma\eta$ and choose x, y with $fx = \xi$, $fy = \eta$, $x\varrho y$. Since f is order-preserving, $\xi\sigma'\eta$. This completes the proof.

Remarks. 1) The conditions put on λ above are not necessary for σ to possess the properties in questions. For a trivial example, let A consist of three elements a, b, c; $\varrho = \mathsf{J}_A \cup (\langle a, b \rangle)$; $\lambda = \mathsf{J}_A \cup (\langle a, c \rangle, \langle c, a \rangle)$. However, it can be shown that the conditions in the above proposition are necessary and sufficient in order that the assertions concerning σ should hold "hereditarily" (in a specifiable sense). – 2) Let the conditions described in the proposition be fulfilled. In addition, suppose, for convenience, that ϱ is reflexive. Then, for every cross-section g of f, the mapping $g : \langle B, \sigma \rangle \to \langle A, \varrho \rangle$ is an order-isomorphism.

10 C.5. Observe that if a set $M \subset \mathsf{N}$ is considered as endowed with a quasi-order σ, then, unless the contrary is explicitly stated, we shall suppose that σ is the restriction of the natural order (on N) to M.

Theorem. *Every ordered set is order-isomorphic, for some set P, with a subset of the ordered set $(0, 1)^P$.*

Proof. Consider an ordered set $\mathscr{A} = \langle A, \leqq \rangle$. Let P be the set of all order-preserving mappings $f : \mathscr{A} \to (0, 1)$. Let F be the mapping of $\mathscr{A}$ into $(0, 1)^P$ which assigns to any $x \in A$ the element $\{fx \mid f \in P\}$ from $(0, 1)^P$. It is easy to see that F is order-preserving.

Let x be an arbitrary element of A; we are going to construct an order-preserving mapping f_x of $\mathscr{A}$ into $(0, 1)$. Put $X = \mathbf{E}\{z \mid z\varrho x\}$. Let f_x be a mapping of $\mathscr{A}$ into $(0, 1)$ such that $f_x z = 0$ if $z \in X$, $f_x z = 1$ if $z \in A - X$. If $u\varrho v$, $f_x v = 0$, then $v\varrho x$, $u\varrho x$ which implies $f_x u = 0$; this proves that f_x is order-preserving.

Clearly, if $x \in A$, $y \in A$, and $x\varrho y$ does not hold, then $f_y y = 0$, $f_y x = 1$, $f_y \in P$ and therefore $Fx \neq Fy$. Thus, F is injective. Let $x \in A$, $y \in A$; suppose that $Fx \leqq Fy$ in $(0, 1)^P$. Then $x\varrho y$; for otherwise $f_y x = 1$, $f_y y = 0$ which contradicts $Fx \leqq Fy$. We have proved that $F^{-1} : F[\mathscr{A}] \to \mathscr{A}$ is an order-preserving mapping.

D. BOUNDEDNESS

10 D.1. Definition. Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered class. Let $X \subset A$, $Y \subset A$. If for every $y \in Y$ there exists an $x \in X$ such that either $x = y$ or $y\varrho x$, then we shall say that X *bounds* Y *from the right* or that X *right-bounds* Y *in* $\mathscr{A}$ (or *under* ϱ) or that X *majorizes* Y *in* $\mathscr{A}$ (or *under* ϱ). If for every $y \in Y$ there exists an $x \in X$ such that either $x = y$ or $x\varrho y$, then we shall say that X *bounds* Y *from the left* or that X *left-bounds* or *minorizes* Y *in* $\mathscr{A}$ (or *under* ϱ). If a singleton (x) right-bounds Y in $\mathscr{A}$, we shall also say that x is a *right bound* of Y *in* $\mathscr{A}$ (or *under* ϱ) or that x *right-bounds* (or *majorizes*) Y *in* $\mathscr{A}$ (or *under* ϱ); for a left-bounding (x), the definitions are analogous.

Finally, if X left-bounds Y under ϱ, we shall sometimes say that X *refines* Y *under* ϱ. – The words "in $\mathscr{A}$", "under ϱ", etc., will often be omitted.

Examples. (A) Let $\langle A, \varrho \rangle$ be a monotone ordered set. Then the collection of all finite sets $X \subset A$ is majorized, under $\subset$, by the collection of all $]\!] \leftarrow, x]\!]$, $x \in A$. – (B) Let $\mathscr{A}$ be the collection of all infinite $X \subset \mathsf{N}$ such that $\mathsf{N} - X$ is infinite. Then $\mathscr{A}$ is not majorized by any countable $\mathscr{B} \subset \mathscr{A}$ under $\subset$.

10 D.2. Intuitively, elements of a quasi-ordered class are usually thought of as ordered either "from left to right" or "upwards". The terminology just introduced corresponds to the idea of a "horizontal" arrangement of elements. However, traditionally, the terminology corresponding to the idea of elements arranged "upwards" is used in many cases. Therefore, we shall also use terminology such as an "*upper bound*" (instead of, and more often than, "a right bound"), a "*lower bound*", etc. We will not list all terms of this kind here, using them freely instead of terms connected with the idea of a "left-right" arrangement.

10 D.3. *Let* $\mathscr{A} = \langle A, \varrho \rangle$, $\mathscr{B} = \langle B, \tau \rangle$ *be quasi-ordered classes. Let* X, Y, Z *be subclasses of* A. *If* X *majorizes* Y *and* Y *majorizes* Z, *then* X *majorizes* Z.

If f is an order-preserving mapping of $\mathscr{A}$ into $\mathscr{B}$ and X majorizes Y, then $f[X]$ majorizes $f[Y]$ in $\mathscr{B}$. If X majorizes Y in $\mathscr{A}$, U majorizes V in $\mathscr{B}$, then $X \times U$ majorizes $Y \times V$ in $\mathscr{A} \times \mathscr{B}$.

The proof, as well as the formulation of the corresponding proposition involving minorization, is left to the reader.

10 D.4. Definition. Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered class. A class $X \subset A$ is said to be *right-cofinal* or *cofinal from the right* (*left-cofinal* or *cofinal from the left*) *in* $\mathscr{A}$ if it right-bounds (left-bounds) A, i.e. if for any $z \in A$ there exists $x \in X$ such that either $x = z$ or $z\varrho x$ (respectively $x\varrho z$).

If it is clear from the context whether left-cofinality or right-cofinality is being considered, then we shall speak simply of a *cofinal class.*

Examples. (A) Every infinite subset of N is cofinal in $\langle \mathsf{N}, \leqq \rangle$. – (B) Let M be the collection of all finite subsets of N. Then the collection of all N_p, $p \in \mathsf{N}$, is cofinal in $\langle M, \subset \rangle$. – (C) There is no countable right-cofinal set in $\langle \mathsf{N}, \leqq \rangle^{\mathsf{N}}$.

10 D.5. *Let $\mathscr{A}$, $\mathscr{B}$ be quasi-ordered classes. Let f be an order-preserving mapping of $\mathscr{A}$ into $\mathscr{B}$. Let $X \subset |\mathscr{A}|$, $Y \subset |\mathscr{A}|$. If X is left-cofinal (respectively, right-cofinal) in $\mathscr{A}$, then $f[X]$ is left-cofinal (respectively, right-cofinal) in $f[\mathscr{A}]$.*

10 D.6. Definition. Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered set. The least cardinality of a left-cofinal (right-cofinal) set in $\mathscr{A}$ (i.e. the smallest cardinal x such that $x = \text{card}\, X$ for some cofinal set X) will be called the *left* (*right*) *cofinal character* of $\mathscr{A}$.

Remark. If it is clear which (whether left or right) character is considered, we shall speak simply of the *cofinal character* of $\mathscr{A}$.

10 D.7. *Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered set. Let a be the right cofinal character of $\mathscr{A}$. Then every right-cofinal set X contains a subset Y with* card $Y = a$.

Proof. Let $B \subset A$ be cofinal, $\text{card}\, B = a$. Put $x\sigma y$ whenever $x \in X$, $y \in B$, $x\varrho y$ or $x = y$; put $y\tau x$ whenever $y \in B$, $x \in X$, $y\varrho x$. Since B, X are cofinal, we have $\mathbf{D}\sigma = X$, $\mathbf{D}\tau = B$. Let $\sigma^* \subset \sigma$, $\tau^* \subset \tau$ be single-valued, $\mathbf{D}\tau^* = B$, $\mathbf{D}\sigma^* = X$. Then clearly $Y = \mathbf{E}(\tau^* \circ \sigma^*)$ is cofinal, $\text{card}\, Y \leqq \text{card}\, B = a$.

10 D.8. Definition. Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered class. A class $X \subset A$ will be called *left-bounded* (*bounded from below*) *in* $\mathscr{A}$ (or *under* ϱ) if it is left-bounded (in the sense of 10 D.1) by a singleton, i.e. if there exists an element $x_0 \in A$ such that $x_0 \varrho x$ for every $x \in X$, $x \neq x_0$; the definition of a *right-bounded* (or *bounded from above*) class is analogous. A class which is both left- and right-bounded in $\mathscr{A}$ will be called *bounded.*

Let $X \subset A$. We shall say that $a \in X$ is a *smallest* (*least*) *element* (or a *first element*) *in* X *under* ϱ (or a *ϱ-smallest element in* X) if $a\varrho x$ for every $x \in X$ distinct from a. If $b \in X$ and $x\varrho b$ for every $x \in X$ distinct from b, we shall say that a is a *largest* (*greatest*) *element* (or a *last element*) *in* X *under* ϱ (or a *ϱ-largest element in* X).

Remarks. 1) In general, there may be many smallest (respectively, largest) elements in a quasi-ordered class (example: a class A endowed with $A \times A$). Thus, if $X \subset A$, it may happen that there are many elements which are smallest (least) in the class of all upper bounds of X; each such element is called a least upper bound of X, and similarly for greatest lower bounds. We shall return to these questions in 10 F. However, if the quasi-order ϱ of A is distinguishing, then there is at most one smallest (respectively, largest) element in every class $X \subset A$. – 2) Terms such as "largest set" (in a class of sets) which have been already used are in accordance with the above definition.

10 D.9. *Let $\mathscr{A} = \langle A, \varrho \rangle$, $\mathscr{B} = \langle B, \sigma \rangle$ be quasi-ordered classes. If $X \subset A$, $Y \subset A$, X left-bounds Y and is left-bounded, then Y is left-bounded. If $S \subset A$, $T \subset B$ are left-bounded, then $S \times T$ is left-bounded in $\mathscr{A} \times \mathscr{B}$. If $f: \mathscr{A} \to \mathscr{B}$ is an order-preserving mapping and $X \subset A$ is left-bounded, then $f[X]$ is left-bounded in $\mathscr{B}$.*

We omit the proof. – Observe that the proposition also holds for right-bounded classes and, with appropriate changes, for bounded classes.

10 D.10. Definition. Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered class. An element $u \in A$ will be called *maximal* (respectively, *minimal*) *in* $\mathscr{A}$ (or *under* ϱ) if $x \in A$, $u\varrho x$ imply $x\varrho u$ (respectively, if $x \in A$, $x\varrho u$ imply $u\varrho x$).

Remarks. 1) A largest (a smallest) element in A is maximal (minimal), but not conversely, in general. – 2) We have already considered maximal (minimal) sets in a class of sets. Their definition given in 3 B.3 is in accordance with the present one.

10 D.11. Theorem. *Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered set. If every monotone $M \subset A$ (i.e. any $M \subset A$ such that ϱ_M is monotone) is right-bounded, then, for any $x \in A$, there exists an element m such that $x\varrho m$ or $m = x$ and m is maximal in $\mathscr{A}$.*

Proof. Choose $x \in A$. Let $\mathscr{M}$ be the collection of all monotone sets $M \subset A$ such that $x \in M$. Clearly,[1] $\mathscr{M}$ is non-empty and monotonically additive. Therefore, by 4C.3, there exists a maximal set $M_0 \in \mathscr{M}$. By the suppositions made, M_0 is right-bounded; let $m \in A$ be such that $x\varrho m$ or $x = m$ for any $x \in M_0$. We shall show that m is maximal. Indeed, let $m' \in A$, $m\varrho m'$, $m' \neq m$. Then $M_0 \cup (m')$ is monotone; since M_0 is maximal in $\mathscr{M}$, we obtain $m' \in M_0$ and therefore $m'\varrho m$.

E. DIRECTED CLASSES

10 E.1. Definition. Let $\mathscr{A} = \langle A, \sigma \rangle$ be a quasi-ordered class. If A is non-void and every two-element set $X \subset A$ is left-bounded (respectively, right-bounded) in $\mathscr{A}$, then we shall say that $\mathscr{A}$ is *left-directed* (respectively, *right-directed*). We shall often simply say that $\mathscr{A}$ is *directed* if it is clear from the context which type of directedness is considered, or in those statements in which "directed" is meant as an abbreviation for both "left-directed" or "right-directed". If $X \subset A$ and $\langle X, \sigma_X \rangle$ is directed, we shall say that X is *directed in* $\mathscr{A}$ (or *under* σ).

Remark. Instead of speaking of a "left-directed quasi-ordered class" we shall speak simply of a "*left-directed class*", etc., instead of "a quasi-order σ such that $\langle A, \sigma \rangle$ is a left-directed class" we shall say briefly "a *left-directed quasi-order*" (or "a *left direction*").

Examples. (A) The class of all monotone collections of sets is left-directed but is not right-directed under $\subset$. – (B) The class of all finite sets is left-directed as well as right-directed under $\subset$. – (C) If a quasi-ordered class has a smallest (a largest) element, then it is left-directed (right-directed). – (D) If $\mathscr{G}_1$, $\mathscr{G}_2$ are groups, put $\mathscr{G}_1 \varrho \mathscr{G}_2$ whenever there exists a homomorphism of $\mathscr{G}_1$ onto $\mathscr{G}_2$. Then ϱ is a left- and right-directed quasi-order. If $\mathbf{G}$ is the set of all subgroups of a fixed group $\mathscr{G}$, then ϱ restricted to $\mathbf{G}$ is, in general, right-, but not left-directed.

10 E.2. *Let* $\mathscr{A}$, $\mathscr{B}$ *be quasi-ordered classes. If* f *is an order-preserving mapping of* $\mathscr{A}$ *onto* $\mathscr{B}$ *and* $\mathscr{A}$ *is directed, then* $\mathscr{B}$ *is directed. If* $\mathscr{A}$ *and* $\mathscr{B}$ *are directed, then* $\mathscr{A} \times \mathscr{B}$ *and* $\mathscr{A} \times_{\text{lex}} \mathscr{B}$ *are directed. If* $\mathscr{A}$ *is right-directed and has no greatest element,* $\mathscr{B}$ *is arbitrary non-void, then* $\mathscr{A} \times_{\text{lex}} \mathscr{B}$ *is right-directed.*

The proof is easy and therefore omitted.

Remark. It is evident that a subclass of a directed class need not be a directed class. According to 10 C.5, it is even possible to embed every ordered set into an ordered set which is left- and right-directed.

10 E.3. *Let* $\langle A, \varrho \rangle$ *be a quasi-ordered class. Then the class of all left-directed (or of all right-directed) sets* $X \subset A$ *is monotonically additive.*

The proof is straightforward and proceeds along well-known lines.

10 E.4. Definition. Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered class. A non-void class $X \subset$ $\subset A$ will be called a *left filter in* $\mathscr{A}$ (or *under* ϱ, or a *left* ϱ*-filter*) if X is left-directed and right-saturated in $\mathscr{A}$ (i.e. if (i) for any $x \in X$, $y \in X$ there exists a $z \in X$ such that x and y belong to $[\![z, \rightarrow [\![$, and (ii) $[\![x, \rightarrow [\![\subset X$ whenever $x \in X$). The definition of a *right filter* (*in* $\mathscr{A}$ or *under* ϱ, or a *right* ϱ*-filter*) is similar (a right filter under ϱ may also be defined as a left filter under ϱ^{-1}).

If X is a left (respectively, right) filter under ϱ, then we shall also say that $\langle X, \varrho_X \rangle$ is a *left* (respectively, *right*) *filter*.

Examples. (A) The class of all finite sets is a right $\subset$-filter. More generally, a class of sets is a right $\subset$-filter if and only if it is additive and hereditary. – (B) In a monotone quasi-ordered class, every left-saturated (right-saturated) non-void class is a right (left) filter. – (C) For $m \in \mathsf{N}$, $n \in \mathsf{N}$ put $m \varrho n$ if and only if m divides n. Then every left ϱ-filter is equal to some $\mathbf{E}\{kn \mid k \in \mathsf{N}\}$ where $n \in \mathsf{N}$.

10 E.5. Convention. If it is clear from the context whether left or right filters are considered, and also in those statements in which "filter" can be replaced by "left filter" as well as by "right filter", we shall often speak simply of filters, etc.

10 E.6. *Let $\mathscr{A} = \langle A, \sigma \rangle$ be a quasi-ordered class. Then the class of all comprisable filters is monotonically additive.*

Proof. The union of a monotone collection of left-directed right-saturated sets is left-directed by 10 E.3 and right-saturated by 10 B.5.

10 E.7. Definition. A filter M in a quasi-ordered class $\mathscr{A} = \langle A, \varrho \rangle$ is called *proper* if $M \neq A$, *maximal* if it is proper and, for any proper filter M_1 in A, $M \subset \subset M_1$ implies $M_1 = M$.

10 E.8. Theorem. *Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered set. Suppose that there is a largest (smallest) element in A. Let $F \subset A$ be a proper right (respectively, left) filter in A. Then there exists a maximal right (respectively, left) filter M in $\mathscr{A}$ such that $M \supset F$.*

Proof. Let b be a largest element in A. Let $\mathbf{F}$ be the collection of all right filters X in $\mathscr{A}$ such that $X \supset F$, $b \notin X$. By 10 E.7, $\mathbf{F}$ is monotonically additive. Therefore, by 4 C.3, there exists a filter which is maximal in $\mathbf{F}$. Since every right filter containing b coincides with A, M is a maximal filter.

F. JOIN AND MEET

Recall that, given a quasi-ordered class $\mathscr{A} = \langle A, \varrho \rangle$, an element $u \in A$ is called a left or lower bound of a class $X \subset A$ if $X \subset [\![u, \rightarrow [\![$, and an element v is called a right or upper bound of $X \subset A$ if $X \subset]\!] \leftarrow, v]\!]$. If a class $X \subset A$ is bounded from below (above), then the class of all lower (upper) bounds of X may contain a greatest (a least) element, i.e. a greatest lower bound (respectively, a least upper bound) of X. In this subsection, we consider some properties of such bounds and in particular orders under which every finite non-empty set has a greatest lower bound and a least upper bound.

10 F.1. Definition. Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered class; let $X \subset A$. An element $u \in A$ will be called a *greatest lower bound* or a *meet of X in $\mathscr{A}$* if the following holds: (1) $X \subset [\![u, \rightarrow [\![$, i.e., for any $x \in X$, either $u \varrho x$ or $u = x$, (2) if $u' \in A$, $X \subset [\![u', \rightarrow [\![$, then $u' \in]\!] \leftarrow, u]\!]$, i.e. either $u' \varrho u$ or $u' = u$. An element $v \in A$ will be called a *least upper bound* or a *join of X in $\mathscr{A}$* if the following holds: (1) $X \subset \subset]\!] \leftarrow, v]\!]$, (2) if $v' \in A$, $X \subset]\!] \leftarrow, v']\!]$, then $v' \in [\![v, \rightarrow [\![$. The class of all joins of X in $\mathscr{A}$ will be denoted by $\mathrm{Sup}_{\mathscr{A}} X$ (or simply $\mathrm{Sup}\, X$), that of all meets of X in $\mathscr{A}$ will be denoted by $\mathrm{Inf}_{\mathscr{A}} X$ (or simply $\mathrm{Inf}\, X$).

Conventions. 1) If $\mathscr{A} = \langle A, \varrho \rangle$ is a quasi-ordered class and $\mathscr{X} = \{x_b \mid b \in B\}$ is an indexed class of elements of A, then provided there is no danger of misunderstanding, we shall speak of a greatest lower bound, etc., of $\{x_b\}$ instead of that of $\mathbf{E}\{x_b\}$, and we shall write $\mathrm{Sup}\, \{x_b\}$ instead of $\mathrm{Sup}\, \mathbf{E}\{x_b\}$. — 2) We shall sometimes use the abbreviations l.u.b. for "least upper bound", g.l.b. for "greatest lower bound".

Examples. (A) Let S be the class of all sets; consider the ordered class $\langle S, \subset \rangle$.

Then every non-empty class $X \subset S$ has exactly one meet, namely $\bigcap X$, whereas $\operatorname{Inf} \emptyset = \emptyset$ since there is no largest element in S. A class $X \subset S$ has a join if and only if X is comprisable; in this case the join is unique and equal to $\bigcup X$. – (B) Let M be a fixed set. Then in $\langle \exp M, \subset \rangle$ every set $X \subset \exp M$ has exactly one join $\bigcup X$ and exactly one meet $\bigcap_M X$ (see 2.11). – (C) In $\langle \mathsf{N}, \leqq \rangle$, $\operatorname{Sup} \emptyset = (0)$, $\operatorname{Inf} \emptyset = \emptyset$; if $\emptyset \neq X \subset \mathsf{N}$, then $\operatorname{Sup} X = \emptyset$ if and only if X is infinite; otherwise $\operatorname{Sup} X = (w)$, where w is the greatest number in X. – (D) If m, n are integers, put $m \varrho n$ if and only if m divides n. Then $\operatorname{Inf} \emptyset = (0)$, $\operatorname{Sup} \emptyset = (1, -1)$; if $\emptyset \neq X \subset Z$, X is finite, then $\operatorname{Sup} X$ consists of all least common multiples of elements of X. – (E) Let M be an infinite class. Put $A = \exp M$; for $x \in A$, $y \in A$, put $x \varrho y$ if and only if either $x = y$ or the set $y - x$ is infinite whereas $x - y$ is finite. Then $\mathscr{A} = \langle A, \varrho \rangle$ is an ordered class. If $s \in A$, $t \in A$ are disjoint infinite sets, then (s, t) is bounded from above, but there is no join of (s, t).

10 F.2. As shown by the above examples, the classes $\operatorname{Inf}_{\mathscr{A}} X$, $\operatorname{Sup}_{\mathscr{A}} X$ may contain many elements or a single element or may be void. It is easy to give a condition (see 10 F.4) necessary and sufficient for all $\operatorname{Inf} X$, $\operatorname{Sup} X$ to be singletons or void. Questions relating to the existence of meets and joins are far more complicated (and more important). We shall consider some aspects of these briefly, with a view to applications mainly in Chapter VI and VII.

We add some minor remarks. – 1) Clearly, a join (meet) of X in $\langle A, \varrho \rangle$ is a meet (join) of X in $\langle A, \varrho^{-1} \rangle$. – 2) It is easy to see that if ϱ_1, ϱ_2 are quasi-orders on A and $\varrho_1 \cup \mathsf{J}_A = \varrho_2 \cup \mathsf{J}_A$, then joins and meets of classes $X \subset A$ under ϱ_1 and ϱ_2 coincide. – 3) Every non-empty $\operatorname{Sup} X$ or $\operatorname{Inf} X$ is equal to some $[\![a, a]\!]$; conversely, $\operatorname{Sup}(a) = \operatorname{Inf}(a) = [\![a, a]\!]$.

10 F.3. Definition. Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered class. Let $X \subset A$. If there exists exactly one element of A which is a greatest lower bound of X, then this element is called the *infimum* (or *the greatest lower bound* or *the meet*) of X in $\mathscr{A}$, and is denoted by $\inf_{\mathscr{A}} X$ (or simply by $\inf X$) or by $\bigwedge_{\mathscr{A}} X$ (or simply by $\bigwedge X$). If there exists exactly one element of A which is a least upper bound of X, then this element is called the *supremum* (or *the least upper bound* or *the join*) of X in $\mathscr{A}$, and is denoted by $\sup_{\mathscr{A}} X$ (or simply by $\sup X$) or by $\bigvee_{\mathscr{A}} X$ (or simply by $\bigvee X$).

If X has either more than one or no greatest lower bound, we shall often say, for convenience, although not quite correctly, that "$\inf X$ does not exist" and similarly "$\sup X$ does not exist".

Convention. If $\mathscr{X} = \{x_b \mid b \in B\}$ is an indexed class of elements of A, we shall often denote $\sup_{\mathscr{A}} \mathbf{E}\{x_b \mid b \in B\}$ simply by $\sup_{\mathscr{A}} \{x_b \mid b \in B\}$ or $\bigvee_{\mathscr{A}}\{x_b \mid b \in B\}$ or also by abbreviated symbols such as $\sup x_b$, etc. A similar convention is adopted for $\inf_{\mathscr{A}} \mathbf{E}\{x_b \mid b \in B\}$.

If $x \in A$, $y \in A$, then we shall write $x \vee_{\mathscr{A}} y$ or simply $x \vee y$ instead of $\bigvee_{\mathscr{A}}(x, y)$, and $x \wedge_{\mathscr{A}} y$ or simply $x \wedge y$ instead of $\bigwedge_{\mathscr{A}}(x, y)$.

Examples. Consider the examples from 10 F.1. In (A), we have $\bigvee X = \bigcup X$ for any comprisable class of sets and $\bigwedge X = \bigcap X$ for any non-empty class of sets X; there exists no infimum of $\emptyset$, i.e. "$\bigwedge \emptyset$ does not exist". In (C), for any $m \in \mathsf{N}$, $n \in \mathsf{N}$ $m \vee n$ (respectively $m \wedge n$) is the greater (the smaller) of the numbers m, n. In (D) $\sup X$ exists if and only if $0 \in X$ or X is infinite (for otherwise $\operatorname{Sup} X$ consists of two elements m and $-m$).

Remark. We point out that the terms "supremum", "infimum", will be used only if there is a unique l.u.b. or g.l.b.

10 F.4. *Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered class. Then the following conditions are equivalent:* (1) *for any $X \subset A$, $\operatorname{Sup} X$ contains one element at most,* (2) *for any $X \subset A$, $\operatorname{Inf} X$ contains one element at most,* (3) *ϱ is a distinguishing quasi-order.*

Proof. Let (1) be satisfied. Let $x \in A$, $y \in A$ and let $x\varrho y$, $y\varrho x$. Then $x \in \operatorname{Sup}(x)$, $y \in \operatorname{Sup}(x)$, hence, by (1), $x = y$. Therefore, (1) $\Rightarrow$ (3), and similarly (2) $\Rightarrow$ (3). – Let (3) be satisfied. If $M \subset A$, $x \in \operatorname{Sup} M$, $y \in \operatorname{Sup} M$, then $x\varrho y$ (or $x = y$) and $y\varrho x$ (or $y = x$); since ϱ is distinguishing we get $x = y$. Thus (3) $\Rightarrow$ (1), and similarly (3) $\Rightarrow$ (2).

The above proposition answers the problem of the uniqueness of joins and meets. As for their existence, we are going to consider the important case where every finite non-empty set has a join or a meet or both. Complete quasi-ordered classes, defined by the condition that every non-empty subset has a join and a meet, will be considered in the following subsection.

As for "finitely complete" classes, they will be examined in the present subsection, as already indicated.

10 F.5. Definition. Let there be given a quasi-ordered class $\mathscr{A} = \langle A, \varrho \rangle$. If every finite non-empty $X \subset A$ has a join (respectively, a meet) in $\mathscr{A}$, then we shall say that $\mathscr{A}$ is *finitely join-complete* (*meet-complete*). If every finite non-empty $X \subset A$ has both a join and a meet we shall say that $\mathscr{A}$ is *finitely complete* (or *finitely order-complete*). Finally, if $\mathscr{A}$ is finitely complete and its quasi-order ϱ is distinguishing, i.e. if $\sup X$ and $\inf X$ exist for any finite non-void $X \subset A$, we shall say that $\mathscr{A}$ is *lattice-quasi-ordered*; if, in addition, ϱ is an order, we shall say that $\mathscr{A}$ is *lattice-ordered*; a comprisable lattice-ordered non-void class will be called a *lattice* (see, however, 10 F.16).

Examples. (A) Let M be an uncountable set. Let A consist of all finite sets $X \subset M$; let B consist of all finite and all uncountable sets $X \subset M$; let C consist of all $X \subset M$ containing n elements at most, with $n \in \mathsf{N}$ fixed, $n > 0$. Then $\langle A, \subset_A \rangle$ is finitely complete (even a lattice), $\langle B, \subset_B \rangle$ is finitely join-complete, but not finitely meet-complete, $\langle C, \subset_C \rangle$ is finitely meet-complete, but not finitely join-complete. (B) Any of the following collections is a lattice (if endowed with $\subset$): the collection of all subgroups of a given group, of all subrings of a given ring, etc.

Remark. Clearly, $\mathscr{A}$ is finitely join-complete if and only if $x \vee y$ exists for any x, y from $\mathscr{A}$.

10 F.6. Before passing to join-stable classes, join-preserving mappings, etc., some remarks are added concerning the relationship of $\mathrm{Sup}_{\mathscr{A}}$ and $\mathrm{Sup}_{\mathscr{B}}$, where $\mathscr{B}$ is a subclass of $\mathscr{A}$, and other related questions.

Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered class. Let $B \subset A$, $\mathscr{B} = \langle B, \varrho_B \rangle$. Let $X \subset B$. It is clear that $B \cap \mathrm{Sup}_{\mathscr{A}} X \subset \mathrm{Sup}_{\mathscr{B}} X$ and that if $x \in \mathrm{Sup}_{\mathscr{A}} X$, $y \in \mathrm{Sup}_{\mathscr{B}} X$, then $x \varrho y$ or $x = y$; in particular, if $\sup_{\mathscr{A}} X$, $\sup_{\mathscr{B}} X$ exist, then either they are equal or $(\sup_{\mathscr{A}} X) \varrho (\sup_{\mathscr{B}} X)$. Similar assertions hold for Inf. It may happen that $\sup_{\mathscr{A}} X$ is actually distinct from $\sup_{\mathscr{B}} X$; for an example see 10 ex. 14.

For these reasons it is useful to find conditions under which a subclass of a finitely complete class is finitely complete.

10 F.7. Definition. Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered class. A class $B \subset A$ is called, respectively, (1) *join-stable*, (2) *meet-stable*, (3) *lattice-stable in* $\mathscr{A}$ (or *under* ϱ) if, for any finite non-empty $X \subset B$, (1) every join of X belongs to B; (2) every meet of X belongs to B; (3) both (1) and (2) hold.

Remark. It may happen that a class is join-stable without being meet-stable, and also conversely; see 10 ex. 15.

10 F.8. *If a quasi-ordered class $\mathscr{A} = \langle A, \varrho \rangle$ is finitely join-complete (finitely meet-complete, finitely complete) and $B \subset A$ is join-stable (meet-stable, lattice-stable respectively) in $\mathscr{A}$, then $\langle B, \varrho_B \rangle$ is finitely join-complete (finitely meet-complete, finitely complete respectively).*

Proof. Consider the case of finite join-completeness. If $X \subset B$ is finite, $X \neq \emptyset$, then $\mathrm{Sup}_{\mathscr{A}} X \neq \emptyset$ and since B is join-stable, $\mathrm{Sup}_{\mathscr{A}} X \subset B$ which proves the assertion. The other cases are analogous.

Remark. The above conditions are not necessary, however, for $\langle B, \varrho_B \rangle$ to be finitely join-complete, etc. For instance, let $\langle G, \sigma \rangle$ be a group and let B be the collection of all sets $H \subset G$ which are subgroups (under σ). Consider the ordered set $\mathscr{A} = \langle \exp G, \subset \rangle$, which is clearly a lattice. The ordered subset $\mathscr{B} = \langle B, \subset_B \rangle$ is a lattice. However, in general, it is not join-stable in $\langle \exp G, \subset \rangle$ since the union of two subgroups is not, in general, a subgroup; if H, H' are subgroups, then $H \vee_{\mathscr{A}} H' = H \cup H'$ is distinct, in general, from $H \vee_{\mathscr{B}} H'$.

10 F.9. *Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered class. Then the intersection of any class, as well as the union of any monotone class of join-stable (in $\mathscr{A}$) subsets is join-stable in $\mathscr{A}$, and similarly for meet-stable and lattice-stable subsets.*

The proof is left to the reader.

10 F.10. An order-preserving mapping does not necessarily preserve joins and meets. For instance, consider the ordered sets $\mathscr{A}$, $\mathscr{B}$ from 10 F.8, remark. Let F be the identical embedding of $\mathscr{B}$ into $\mathscr{A}$; then F is order-preserving and meet-preserving, but, in general, $F(x \vee_{\mathscr{B}} y) \neq (Fx) \vee_{\mathscr{A}} (Fy)$.

In view of these facts, it is convenient to consider mappings which preserve joins or meets or both.

Definition. Let $\mathscr{A} = \langle A, \varrho \rangle$, $\mathscr{B} = \langle B, \tau \rangle$ be quasi-ordered classes. A mapping F of $\mathscr{A}$ into $\mathscr{B}$ is called *join-preserving* (*meet-preserving*) if the following holds: if $X \subset A$ is finite non-empty and x is a join (a meet) of X, then Fx is a join (a meet) of $F[X]$. If F is both join- and meet-preserving, then it is called *lattice-preserving.*

10 F.11. *Let $\mathscr{A} = \langle A, \varrho \rangle$, $\mathscr{B} = \langle B, \tau \rangle$ be quasi-ordered classes. If F is a join-preserving or meet-preserving mapping of $\mathscr{A}$ into $\mathscr{B}$, then $F : \langle A, \varrho \rangle \to \langle B, \tau \cup J_B \rangle$ is order-preserving.*

Proof. If $x \varrho y$, then y is a join of (x, y), hence Fy is a join of (Fx, Fy), and therefore either $(Fx)\, \tau\, (Fy)$ or $Fx = Fy$.

Remark. Clearly, if ϱ is a strict quasi-order on a class A containing more than one element, then $J : \langle A, \varrho \cup J_A \rangle \to \langle A, \varrho \rangle$ is lattice-preserving, but is not order-preserving.

10 F.12. *Let $\mathscr{A} = \langle A, \varrho \rangle$, $\mathscr{B} = \langle B, \tau \rangle$ be quasi-ordered classes. Let $F = F : \mathscr{A} \to \mathscr{B}$ be bijective. Then the following conditions are equivalent:* (1) *both F and F^{-1} are join-preserving*; (2) *both F and F^{-1} are meet-preserving*; (3) *if $x \in A$, $y \in A$, $x \neq y$, then $x \varrho y \Leftrightarrow (Fx)\, \tau\, (Fy)$.*

Proof. By 10 F.11, (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3). – If $X \subset A$ is finite non-empty and a is a join of X, then Fa is an upper bound of $f[X]$ in B. Also let b be an upper bound of $f[X]$ in B. Then $F^{-1}b$ is an upper bound of X in $\mathscr{A}$, hence $F^{-1}b = a$ or $a \varrho (F^{-1}b)$. If $F^{-1}b = a$, then $b = Fa$; if not, then $(Fa)\, \tau b$. This proves that Fa is a join of $F[X]$.

Definition. Let $\mathscr{A}$, $\mathscr{B}$ be quasi-ordered classes. A bijective mapping F of $\mathscr{A}$ onto $\mathscr{B}$ is called a *join-isomorphism* or a *join-isomorphic mapping* if it satisfies the above conditions (1)–(3).

Remarks. 1) Clearly, if $\mathscr{A}$, $\mathscr{B}$ are ordered classes, then a bijective mapping F: $\mathscr{A} \to \mathscr{B}$ is join-isomorphic if and only if it is order-isomorphic. – 2) Let ϱ, σ be quasi-orders on A. Then J: $\langle A, \varrho \rangle \to \langle A, \sigma \rangle$ is a join-isomorphism if and only if $\varrho \cup J_A = \sigma \cup J_A$.

10 F.13. *Let $\mathscr{A} = \langle A, \varrho \rangle$, $\mathscr{B} = \langle B, \tau \rangle$ be quasi-ordered classes. Let F be a mapping of $\mathscr{A}$ onto $\mathscr{B}$. If $\mathscr{A}$ is finitely join-complete (respectively, finitely meet-complete, finitely complete) and F is join-preserving (respectively, meet-preserving, lattice-preserving), then $\mathscr{B}$ is finitely join-complete (respectively, finitely meet-complete, finitely complete).*

Proof. Let $x \in B$, $y \in B$; choose u, v such that $Fu = x$, $Fv = y$. Let w be a join of u and v in $\mathscr{A}$; then Fw is a join of x and y in $\mathscr{B}$. In the remaining cases, the proof is analogous.

Remark. In particular, if $\mathscr{A}$ is lattice-ordered, $\mathscr{B}$ is ordered and F is lattice-preserving, then $\mathscr{B}$ is lattice-ordered.

As shown above (see 10 F.8), an identical embedding of an ordered class into an ordered class is not necessarily join-preserving. It is convenient to give a special name to classes for which this embedding is join-preserving or meet-preserving, etc.

10 F.14. Definition. Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered class. A class $B \subset A$, (as well as the quasi-ordered class $\langle B, \varrho_B \rangle$) is called *join-preserving* (*meet-preserving, lattice-preserving*) *in* $\mathscr{A}$ (or *under* ϱ) if the mapping $\mathsf{J} : \langle B, \varrho_B \rangle \to \mathscr{A}$ has the property in question.

Thus $B \subset A$ is join-preserving in $\mathscr{A} = \langle A, \varrho \rangle$ if and only if, for any finite non-empty $X \subset B$, every join of X in $\langle B, \varrho_B \rangle$ is a join of X in $\mathscr{A}$.

Examples. (A) Every disjoint class of sets is lattice-preserving (in a trivial manner) under $\subset$. — (B) If $\langle B, \tau \rangle$ is an ordered class, then the class of all interval-like sets $X \subset B$ is meet-preserving, but not join-preserving under $\subset$.

Observe that, under a quite natural additional condition, every finitely join-complete join-preserving class is join-stable (see below, 10 F.15). Therefore, join-preserving but not join-stable classes are often somewhat artificial, and similarly for meet-preserving and lattice-preserving classes. Only one proposition on join-preserving classes is given here. Further examples and some propositions concerning join-preserving classes are deferred to 10 ex. 16 – 19.

10 F.15. *Let* $\mathscr{A} = \langle A, \varrho \rangle$ *be a quasi-ordered class; let* $B \subset A$, $\mathscr{B} = \langle B, \varrho_B \rangle$. *If* $\mathscr{A}$ *is finitely join-complete and* B *is join-stable in* $\mathscr{A}$, *then* $\mathscr{B}$ *is finitely join-complete and join-preserving in* $\mathscr{A}$. *If the quasi-order* ϱ *is distinguishing,* $\mathscr{B}$ *is finitely join-complete and join-preserving in* $\mathscr{A}$, *then* $\mathscr{B}$ *is join-stable in* $\mathscr{A}$.

Proof. By 10 F.8, B is finitely join-complete. Let $X \subset B$ be finite non-empty. If w is a join of X in $\mathscr{B}$, and v is a join of X in $\mathscr{A}$, then $v \in B$ (since B is join-stable) and therefore either $w = v$ or $w\varrho v$, $v\varrho w$; in both cases, clearly, w is a join of X in $\mathscr{A}$; thus $\mathscr{B}$ is join-preserving in $\mathscr{A}$. We turn to the second assertion. Let $X \subset B$ be finite non-empty. Let v be a join of X in $\mathscr{A}$. There exists a join w of X in $\mathscr{B}$, and since $\mathscr{B}$ is join-preserving, w is also a join of X in A. Since ϱ is distinguishing, every $X \subset A$ has one join at most; hence $v = w$, $v \in B$.

10 F.16. We are going to show that if $\mathscr{A}$ is lattice-quasi-ordered, then $\bigvee_{\mathscr{A}}$, $\bigwedge_{\mathscr{A}}$ are commutative semi-group structures on $|\mathscr{A}|$ satisfying certain further conditions; conversely, starting from a pair of associative commutative compositions of a special kind to be described below, a lattice-order is obtained.

Definition. Let σ, μ be commutative associative compositions on a class A. If $x\mu(x\sigma y) = x$, $x\sigma(x\mu y) = x$ for any $x \in A$, $y \in A$, then we shall say that $\langle \sigma, \mu \rangle$ is a *lattice structure* on X, and $\langle X, \sigma, \mu \rangle$ will be called a *lattice-structured class*; if X is a non-void set, then $\langle X, \sigma, \mu \rangle$ will be termed a *lattice* (thus we use the word "lattice" in a twofold sense, cf. 10 F.5).

Remark. It is easy to show that $x\sigma x = x$, $x\mu x = x$ holds in every lattice-structured class.

Examples. (A) $\langle \bigcup, \bigcap \rangle$ is a lattice structure (as well as a semi-ring structure). – (B) Let $\mathscr{G}$ be a group. Consider the set Γ of all subgroups of $\mathscr{G}$. If $H_1 \in \Gamma$, $H_2 \in \Gamma$, let $H_1 \sigma H_2$ be the smallest subgroup containing both H_1 and H_2; put $H_1 \mu H_2 = H_1 \cap H_2$. Clearly, $\langle \Gamma, \sigma, \mu \rangle$ is a lattice; it is easy to show that $\langle \Gamma, \sigma, \mu \rangle$ is not a semi-ring, in general (since the distributive law does not hold).

10 F.17. Theorem. *Let $\mathscr{A} = \langle A, \varrho \rangle$ be a lattice-quasi-ordered class; let ϱ be distinguishing. Then $\langle \bigvee_{\mathscr{A}}, \bigwedge_{\mathscr{A}} \rangle$ is a lattice-structure on A.*

Let σ be a commutative associative composition on a class A such that $x \sigma x = x$ for any $x \in A$; denote by ϱ the class of all pairs $\langle x, y \rangle$ such that $x \sigma y = y$. Then ϱ is an order, $\langle A, \varrho \rangle$ is finitely join-complete and, for any $x \in A$, $y \in A$, $x \sigma y$ is the join of x and y under ϱ.

Let $\langle \sigma, \mu \rangle$ be a lattice structure on a class A. Then there exists exactly one order ϱ on A such that, for any $x \in A$, $y \in A$, $x \sigma y$ is the join and $x \mu y$ is the meet of x and y under ϱ.

Proof. I. Let $\mathscr{A}$ possess the properties indicated in the first assertion; let $\bigvee, \bigwedge$ be written instead of $\bigvee_{\mathscr{A}}, \bigwedge_{\mathscr{A}}$, and $\leqq$ instead of ϱ. It is clear that $\bigvee$ and $\bigwedge$ are commutative associative compositions. If $x \in A$, $y \in A$, then $x \leqq x \vee y$, $x \wedge y \leqq x$, and hence $x \wedge (x \vee y) = x$, $x \vee (x \wedge y) = x$ which proves the first assertion. – II. Let σ, ϱ possess the properties indicated in the second assertion. If $\langle x, y \rangle \in \varrho$, $\langle y, z \rangle \in \varrho$, then $x \sigma y = y$, $y \sigma z = z$, and hence $x \sigma z = x \sigma (y \sigma z) = (x \sigma y) \sigma z = y \sigma z = z$; thus ϱ is transitive. Clearly $\langle x, x \rangle \in \varrho$ for any $x \in A$; if $\langle x, y \rangle \in \varrho$, $\langle y, x \rangle \in \varrho$, then $x \sigma y = y$, $y \sigma x = x$ and therefore $x = y$. We have proved that ϱ is an order; we shall now write $\leqq$ instead of ϱ. It is clear that $x \leqq x \sigma y$, $y \leqq x \sigma y$ for any $x \in A$, $y \in A$; if $z \in A$, $x \leqq z$, $y \leqq z$, then $x \sigma z = z$, $y \sigma z = z$, hence $(x \sigma y) \sigma z = z$ and therefore $x \sigma y \leqq z$. This proves that $x \sigma y$ is the join of x and y under ϱ. We have proved that every two-element set $(x, y) \subset A$ has a join (namely, $x \sigma y$). This implies that $\langle A, \varrho \rangle$ is finitely join-complete. – III. We are going to prove the third assertion. It is easy to deduce from II that there exists exactly one order ϱ (respectively, τ) on A such that, for any $x \in A$, $y \in A$, the join of x and y under ϱ (under τ) is equal to $x \sigma y$ (respectively, $x \mu y$). To complete the proof it is sufficient to show that $\tau = \varrho^{-1}$. Now, if $x \varrho y$, then $x \sigma y = y$, hence $x \mu y = x \mu (x \sigma y) = x$ and therefore $y \tau x$; conversely, if $y \tau x$, then $y \mu x = x$, hence $y \sigma x = y \sigma (y \mu x) = y$ and therefore $x \varrho y$.

G. COMPLETENESS

10 G.1. Definition. A quasi-ordered class $\langle A, \varrho \rangle$ is called *complete* (or *order-complete*) if every non-empty set $X \subset A$ has a join and a meet, *boundedly complete* (or *boundedly order-complete*) if every non-empty right-bounded set $X \subset A$ has a join and every non-empty left-bounded set $X \subset A$ has a meet.

Remarks. 1) Observe that we do not require the existence of a join or a meet of the void set or of a non-comprisable class $X \subset A$, and that joins and meets are not required to be unique. – 2) We do not define join-complete, meet-complete, etc., classes since they differ only slightly from complete classes (see 10 C.3). – 3) Properties of complete and boundedly complete quasi-ordered classes are closely related. It is easy to see that if we add two further elements to a boundedly complete class, "the first element" and "the last element", then we obtain a complete quasi-ordered class (see 10 G.5). For these reasons we shall often give statements concerning either complete or boundedly complete classes only, leaving to the reader the task of stating the corresponding propositions for the other property. – 4) Two important related properties, namely countable completeness and monotone completeness, are dealt with briefly in the exercises (see 10 ex. 20, 22). Observe that e.g. countable join-completeness and countable meet-completeness are essentially different.

Examples. (A) Let S be the class of all sets. Then $\mathscr{S} = \langle S, \subset \rangle$ is complete. The class $\mathscr{S} \times \mathscr{S}$ is also complete whereas $\mathscr{S} \times_{\text{lex}} \mathscr{S}$ is not (e.g. if A_n, $n = 1, 2, \ldots$, are non-empty sets, $\bigcap A_n = \emptyset$, then the set of all $\langle A_n, \emptyset \rangle$ has no meet). – (B) $\langle \mathsf{N}, \leqq \rangle$ is boundedly complete, but not complete, for the set N has no join.

10 G.2. *Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered class; let $B \subset A$. Suppose that (1) every non-empty set $X \subset B$ (every non-empty $X \subset B$ bounded from above in $\mathscr{A}$) has a join in $\mathscr{A}$, (2) every $x \in A$ is a join in $\mathscr{A}$ of some set $X \subset B$.*

Then every non-empty set $X \subset A$ (every non-empty $X \subset A$ bounded from above in $\mathscr{A}$) has a join in $\mathscr{A}$. If $\mathscr{A}$ is comprisable, then $\mathscr{A}$ is boundedly complete.

Analogous assertions hold if "above" is replaced by "below", "join" by "meet".

Proof. I. If $\emptyset \neq X \subset A$ and the set X is bounded from above in $\mathscr{A}$, let $\{Y_x \mid x \in X\}$ be a family of subsets of B such that, for any $x \in X$, the element x is a join of Y_x in $\mathscr{A}$. Put $Y = \bigcup Y_x$. Let b be an upper bound of X in $\mathscr{A}$; then, clearly, b is an upper bound of Y; we may suppose that $Y \neq \emptyset$; therefore, by supposition (1), the set Y has a join y^*. Since y^* is an upper bound of Y_x for each x, we have $x \varrho y^*$ or $x = y^*$ for every $x \in X$. Let y' be an upper bound of X. Then, clearly, y' is an upper bound of every Y_x, hence also of $Y = \bigcup Y_x$, and therefore $y^* \varrho y'$ or $y = y'$. We have proved that y^* is a join of X.

II. Let $\mathscr{A}$ be comprisable; let $X \subset A$ be bounded from below. Let Z be the class of all lower bounds of X. Then, by supposition, Z is a non-empty set and therefore Z has a join. If z is a join of Z, then clearly z is a meet of X.

Remark. If we replace the words "bounded ... in $\mathscr{A}$" by "bounded ... in $\langle B, \varrho \rangle$" in condition (1), then we obtain a false statement. Example: A is the class of all finite and all uncountable sets, $\mathscr{A} = \langle A, \subset_A \rangle$, B consists of all finite sets.

10 G.3. Theorem. *Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered set. If every non-empty $X \subset A$ bounded from above (respectively, from below) has a join (a meet) in $\mathscr{A}$, then $\mathscr{A}$ is boundedly complete. If every non-empty $X \subset A$ has a join (a meet) in $\mathscr{A}$, and there is a least (greatest) element in $\mathscr{A}$, then $\mathscr{A}$ is complete.*

This follows at once from 10 G.2.

Remark. If $\mathscr{A}$ is not comprisable, the assertion does not hold in general; see 10 G.1, example (A): every non-empty subset of $\mathscr{S} \times_{\text{lex}} \mathscr{S}$ has a join, but $\mathscr{S} \times_{\text{lex}} \mathscr{S}$ is not complete.

10 G.4. In connection with the above theorem and remark, it seems natural (provided we are interested not only in sets but also in non-comprisable classes) to investigate quasi-ordered classes every subclass of which has a join and a meet. However, we have chosen completeness in the sense of 10 G.1 as the basic property to be examined here. The reason lies in the fact that various classes of structs and other related classes ordered (or quasi-ordered) in a natural way are complete but do not satisfy the stronger condition that every subclass is to possess a join and a meet.

10 G.5. In 10 G.1 we have roughly indicated the connection between completeness and bounded completeness. Now we give a precise statement.

Let $\mathscr{A} = \langle A, \varrho \rangle$ *be a quasi-ordered class; let a be a least, and b be a greatest element in* $\mathscr{A}$. *Then the following condition are equivalent*: (1) $\mathscr{A}$ *is complete,* (2) $]\!]\, a, b\, [\![$ *is boundedly complete under* ϱ.

The proof is immediate.

10 G.6. We intend to show that it is possible to embed any quasi-ordered set $\mathscr{A}$ into a complete one, say $\mathscr{A}^*$. In a striking difference with analogous propositions concerning, for instance, metric spaces (see 41 ex. 1), uniform spaces (see 41 B), etc., such a completion is by no means unique if we require only that $\mathscr{A}$ should be "dense" in $\mathscr{A}^*$, in an appropriate sense. Of course, if more stringent requirements are imposed, then uniqueness (naturally up to an isomorphism) can be achieved.

Before stating propositions concerning completion (see 10 G.12), it is useful to introduce "complete" properties corresponding to join-stability, join-preservation, etc.

10 G.7. Definition. Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered class. A class $B \subset A$ is called, respectively, (1) *completely join-stable*, (2) *completely meet-stable*, (3) *completely lattice-stable in* $\mathscr{A}$ (or *under* ϱ) if, for any non-empty set $X \subset B$: (1) every join of X belongs to B; (2) every meet of X belongs to B; (3) both (1) and (2) hold.

We defer to 10 ex. 15, 21, 23 some remarks and examples concerning these concepts.

Proofs of propositions below are omitted and the reader is invited to carry them out as an exercise.

10 G.8. *If a quasi-ordered class* $\mathscr{A} = \langle A, \varrho \rangle$ *is complete (respectively, boundedly complete) and* $B \subset A$ *is completely join- (or meet-) stable, then* $\langle B, \varrho_B \rangle$ *is complete (respectively, boundedly complete).*

10 G.9. *Let* $\mathscr{A} = \langle A, \varrho \rangle$ *be a quasi-ordered class. Then the intersection of any class of completely join- (or meet- or lattice-) stable (in* $\mathscr{A}$*) subsets is join- (or meet- or lattice-) stable in* $\mathscr{A}$.

10 G.10. Definition. Let $\mathscr{A} = \langle A, \varrho \rangle$, $\mathscr{B} = \langle B, \tau \rangle$ be quasi-ordered classes; let F be a mapping of $\mathscr{A}$ into $\mathscr{B}$. Then F is called *completely join-* (*meet-*) *preserving* if the following holds: if $X \subset A$ is a non-empty set and x is a join (meet) of X in $\mathscr{A}$, then Fx is a join (meet) of $F[X]$ in $\langle F[A], \tau \rangle$. If F is both completely join-preserving and completely meet-preserving, then F is called *completely lattice-preserving.*

If $\mathscr{A} = \langle A, \varrho \rangle$ is a quasi-ordered class, then a class $M \subset A$ will be called *completely join-* (or *meet-*, or *lattice-*) *preserving* in $\mathscr{A}$ if the identical mapping J: $: \langle M, \varrho_M \rangle \to \mathscr{A}$ has the property in question.

Clearly, M is completely join-preserving in $\mathscr{A}$ if and only if, for any non-empty set $X \subset M$, $\operatorname{Sup}_M X \subset \operatorname{Sup}_{\mathscr{A}} X$, i.e. every join of X in M is a join of X in $\mathscr{A}$.

10 G.11. *Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered class. Let $\mathbf{M}$ be a class of subsets of A such that* (1) *$\mathbf{M}$ contains every $]\!] \leftarrow, a [\![\cup (a)$, $a \in A$;* (2) *every $X \in \mathbf{M}$ which is not of the form indicated in* (1) *is left-saturated and completely join-stable in $\mathscr{A}$, and has no join in $\mathscr{A}$. Put $X \tau Y$ if $X \in \mathbf{M}$, $Y \in \mathbf{M}$ and either $X \subset Y$, $X \neq Y$ or for every $x \in X$ there is an element $y \in Y$ with $x \varrho y$.*

Then $\langle \mathbf{M}, \tau \rangle$ is a quasi-ordered class. Let F be the mapping of $\mathscr{A}$ into $\langle \mathbf{M}, \tau \rangle$ which assigns $]\!] \leftarrow, x [\![\cup (x)$ to $x \in A$. Then F is a completely join-preserving mapping, $F : \mathscr{A} \to F[\mathscr{A}]$ is an order-isomorphism.

If $Z \subset A$, $X \in \mathbf{M}$, $Z \subset X$ and Z is cofinal in X, then X is a join of $F[Z]$ in $\langle \mathbf{M}, \tau \rangle$; in particular, every $X \in \mathbf{M}$ is a join of $F[X]$ in $\langle \mathbf{M}, \tau \rangle$. Finally, if $X \in \mathbf{M}$, $Y \in \mathbf{M}$ and $X \tau Y$, $Y \tau X$, then either $X = Y$ or $X \in F[A]$, $Y \in F[A]$.

Proof. I. Let $X \tau Y$, $Y \tau Z$. If $X \subset Y$, $X \neq Y$, $Y \subset Z$, $Y \neq Z$, then clearly $X \tau Z$; if $X \subset Y$, $X \neq Y$ and for each $y \in Y$ there exists a $z \in Z$ such that $y \varrho z$, then for each $x \in X$ there exists a $z \in Z$ with $x \varrho z$, hence $X \tau Z$; the other cases are similar. Thus, τ is a quasi-order.

II. Let a, b be elements of A, $a \varrho b$. Then $x \varrho b$ for every $x \in Fa$, hence $(Fa) \tau (Fb)$; conversely, if $(Fa) \tau (Fb)$, then either $a \varrho y$ for some $y \in Fb$, hence $a \varrho b$, or $Fa \subset Fb$, $Fa \neq Fb$, hence $a \in Fb$, $a \neq b$, $a \varrho b$. Thus $F : \mathscr{A} \to F[\mathscr{A}]$ is an order-isomorphism. Let $X \subset A$ be a non-empty set and let $a \in A$ be a join of X in $\mathscr{A}$. Clearly, Fa is an upper bound of $F[X]$ in $\langle \mathbf{M}, \tau \rangle$. Let $Z \in \mathbf{M}$ be an upper bound of $F[X]$ in $\langle \mathbf{M}, \tau \rangle$. If $Z = Fz$ for some $z \in A$, then z is an upper bound of X, $a \varrho z$ or $a = z$ and therefore $(Fa) \tau Z$ or $Fa = Z$. If not, then Z is left-saturated and completely join-stable. Since, for any $x \in X$, $(Fx) \tau Z$ or $Fx = Z$, we get $Fx \subset Z$ for every $x \in X$, $X \subset Z$ and therefore, Z being completely join-stable, $a \in Z$, from which it follows that either $Fa = Z$ or $(Fa) \tau Z$. Thus Fa is a join of $F[X]$ which proves that F is completely join-preserving.

III. Let $X \in \mathbf{M}$. Clearly, $(Fx) \tau X$ or $Fx = X$ for every $x \in X$. Now let $Y \in \mathbf{M}$ be an upper bound of $F[X]$ in $\langle \mathbf{M}, \tau \rangle$. If $Y \notin F[A]$, then, for any $x \in X$, either $x \varrho y$ for some $y \in Y$, hence $x \in Y$ (for Y is left-saturated) or $Fx \in Y$; in any case, $X \subset Y$ and therefore either $X \tau Y$ or $X = Y$. If $Y = Fb$, then b is an upper bound of X in $\mathscr{A}$, from which it follows that $X \tau (Fb)$ or $X = Fb$. This proves that X is a join of $F[X]$ in $\langle \mathbf{M}, \tau \rangle$. Let $Z \subset A$, $X \in \mathbf{M}$, $Z \subset X$ and let Z be cofinal in X. Clearly, X is an upper bound of $F[Z]$. If Y is an upper bound of $F[Z]$, then it is easy to see that Y is also an upper bound of $F[X]$, hence, as shown above, $X\tau Y$ or $X = Y$. This proves that X is a join of $F[Z]$.

IV. If $X\tau Y$, $Y\tau X$, then it is easy to prove that either $X = Fa$, $Y = Fb$ for some $a \in A$, $b \in B$, or $X \subset Y$, $Y \subset X$, hence $X = Y$.

Remarks. 1) Clearly, an analogous proposition is true with joins replaced by meets, sets $]\!] \leftarrow, a [\![\cup (a)$ by sets $]\!] a, \rightarrow [\![\cup (a)$, etc. – 2) Observe that a class $\boldsymbol{M}$ with the properties described above exists if and only if every $]\!] \leftarrow, a [\![$ is comprisable. However, the above proposition can also be extended, with appropriate changes, to more general cases.

10 G.12. Theorem. *Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered set. Then there exists a complete quasi-ordered set $\mathscr{A}^* = \langle A^*, \varrho^* \rangle$ such that (1) A is identically embedded in $\mathscr{A}^*$, i.e. $A \subset A^*$, $\varrho = \varrho^*_A$; (2) if $x\varrho^* y$, $y\varrho^* x$, then either $x = y$ or $x \in A$, $y \in A$; (3) if an element $a \in A$ is a join (a meet) of a set $X \subset A$ in $\mathscr{A}$, then a is a join (a meet) of X in $\mathscr{A}^*$, (4) every $x \in A^*$ is a join of $A \cap]\!] \leftarrow, x]\!]$ and a meet of $A \cap [\![x, \rightarrow [\![$.*

Proof. In 10 G.11, let $\boldsymbol{M}$ consist of all $]\!] \leftarrow, a [\![\cup (a)$, $a \in A$, as well as of all those sets $X \subset A$ which are join-stable, have no join and fulfil the condition (*) X consists of all lower bounds of the set X^* of all upper bounds of X.

Then there exists a completely join-preserving mapping F of $\mathscr{A}$ into $\langle \boldsymbol{M}, \tau \rangle$ such that $F : \mathscr{A} \rightarrow F[\mathscr{A}]$ is an isomorphism. There exists a one-to-one relation g with $\mathbf{D}g \supset A$, $\mathbf{E}g = \boldsymbol{M}$ such that $gx = Fx$ for $x \in A$. Put $A^* = \mathbf{D}g$, $\varrho^* = g^{-1} \circ \circ \tau \circ g$. Then assertion (1) is evident since $F : \mathscr{A} \rightarrow F[\mathscr{A}]$ is an order-isomorphism. Since $g : \mathscr{A}^* \rightarrow \langle \boldsymbol{M}, \tau \rangle$ is an order-isomorphism, assertions (2) and (3) follows easily from 10 G.11.

We are going to prove assertion (4). For any $X \in \boldsymbol{M}$, let $P(X) =]\!] \leftarrow, X]\!]$, $Q(X) = = [\![X, \rightarrow [\![$ (intervals in $\langle \boldsymbol{M}, \tau \rangle$). Clearly, $F[A] \cap P(X) = F[X]$; by 10 G.11, X is a join of $F[X]$ in $\langle \boldsymbol{M}, \tau \rangle$. It is easy to see that $F[A] \cap Q(X) = F[X^*]$ where X^* is the set of all upper bounds of the set X in A. Clearly, X is a lower bound of the set $F[X^*]$ in $\langle \boldsymbol{M}, \tau \rangle$. On the other hand, if $Y \in \boldsymbol{M}$ is a lower bound of $F[X^*]$ in $\langle \boldsymbol{M}, \tau \rangle$, then, for every $z \in X^*$, either $Y\tau(]\!] \leftarrow, z [\![\cup (z))$ or $Y =]\!] \leftarrow, z [\![\cup (z)$. Therefore, for any $y \in Y$, $z \in X^*$, either $y\varrho z$ or $y = z$. By property (*) of $\boldsymbol{M}$, this implies that $Y \subset X$, hence $Y\tau X$. This proves that X is a meet of $F[X^*]$. We have shown that, for any $X \in \boldsymbol{M}$, X is a join of $F[A] \cap P(X)$ and a meet of $F[A] \cap Q(X)$. This implies at once the assertion (4).

If $X \subset A$ is a non-empty set without a join in $\mathscr{A}$, then let Y be the intersection of all sets containing X and satisfying condition (*) indicated above. It is easy to see that Y satisfies condition (*). If z is an upper bound of X in $\mathscr{A}$, then $X \subset]\!] \leftarrow, z]\!]$, hence $Y \subset]\!] \leftarrow, z]\!]$; thus, upper bounds of X and Y coincide, and therefore Y has no join. We have shown that $Y \in \boldsymbol{M}$; it is easy to prove that the element $Y \in A^*$ is a join of X in $\mathscr{A}^*$. We have proved that every non-empty $X \subset A$ has a join in $\mathscr{A}^*$. From this and 10 G.3 it follows that $\mathscr{A}^*$ is complete.

Remark. It can be proved that $\mathscr{A}^*$ is essentially unique. On the other hand, it is easy to see that a complete quasi-ordered set with properties (1)–(3) described in the theorem is not uniquely determined. Taking another appropriate set $\boldsymbol{M}$, we may obtain an essentially different quasi-ordered set with properties (1)–(3) as indicated above and property (4) replaced by an appropriate substitute.

10 G.13. For monotone quasi-ordered classes, a sharper proposition holds. Before stating this proposition, we shall present, for convenience, some other statements concerning monotone quasi-order.

10 G.14. *A quasi-ordered class* $\mathscr{A} = \langle A, \varrho \rangle$ *is monotone if and only if* $]\!] \leftarrow, a]\!] \cup]\!] a, \rightarrow [\![= A$ *(or, equivalently,* $]\!] \leftarrow, a [\![\cup [\![a, \rightarrow [\![= A$*) for any* $a \in A$.

Proof. "Only if" is clear. If the condition holds, let $x \in A$, $y \in A$, $x \neq y$. Then either $y \in]\!] \leftarrow, x]\!]$, hence $y \varrho x$, or $y \in]\!] x, \rightarrow [\![$, hence $x \varrho y$.

10 G.15. *Let* $\mathscr{A} = \langle A, \varrho \rangle$, $\mathscr{B} = \langle B, \sigma \rangle$ *be quasi-ordered classes. Let* ϱ *be monotone reflexive, let* σ *be distinguishing. If* f *is a one-to-one order-preserving mapping of* $\mathscr{A}$ *into* $\mathscr{B}$, *then* $f^{-1} : f[\mathscr{A}] \rightarrow \mathscr{A}$ *is also order-preserving.*

Proof. Let $(fx) \sigma (fy)$. Then $y \varrho x$ implies $(fy) \sigma (fx)$, hence $fx = fy$, $x = y$, $x \varrho y$. This proves the assertion.

10 G.16. *Let* $\langle A, \varrho \rangle$ *be a monotone quasi-ordered class. Then* $B \subset A$ *is cofinal from the left (respectively, from the right) if and only if either* (i) *B contains an element which is a smallest (largest) one in A, or* (ii) *B is not left- (right-) bounded in A.*

10 G.17. *Let* $\mathscr{A} = \langle A, \varrho \rangle$ *be a monotone quasi-ordered class. Let* $x \in A$ *be maximal (minimal). Then x is a largest (smallest) element in* $\mathscr{A}$.

Remark. It is easy to see that if, in every subclass $B \subset A$, every maximal (minimal) element is a largest (smallest) one, then $\mathscr{A}$ is monotone.

10 G.18. *Every monotone quasi-ordered class is lattice-quasi-ordered (hence, left- and right-directed).*

10 G.19. *Let* ϱ *be a monotone distinguishing quasi-order on a class A. Then any* $X \subset A$ *is join-stable and join-preserving.*

This is clear. Observe, however, that a set $X \subset A$ need be neither completely join-stable nor completely join-preserving.

10 G.20. Theorem. *Let* $\mathscr{A} = \langle A, \varrho \rangle$ *be a monotone quasi-ordered set. Then there exists a monotone quasi-ordered set* $\mathscr{A}^* = \langle A^*, \varrho^* \rangle$ *such that* (1) $\mathscr{A}$ *is a quasi-ordered subclass of* $\mathscr{A}^*$; (2) *if* $\xi \in A^*$, $\eta \in A^*$ *do not belong to A, then* $\xi = \eta$ *if and only if* $\xi \varrho^* \eta$, $\eta \varrho^* \xi$; (3) *for any* $\xi \in A^*$, ξ *is a join (in* $\mathscr{A}^*$*) of* $]\!] \leftarrow, \xi]\!] \cap A$ *as well as a meet (in* $\mathscr{A}^*$*) of* $[\![\xi, \rightarrow [\![\cap A$; (4) $\mathscr{A}^*$ *is complete.*

If $\mathscr{A}' = \langle A', \varrho' \rangle$ *is a set with the properties indicated above, then there exists exactly one order-isomorphism f of* $\mathscr{A}^*$ *onto* $\mathscr{A}'$ *such that* $fx = x$ *for every* $x \in A$.

Proof. Consider the quasi-ordered set $\mathscr{A}^* = \langle A^*, \varrho^* \rangle$ described in 10 G.12. To prove that $\mathscr{A}^*$ possesses all properties required, we have only to show that it is monotone, for the remaining properties are asserted in 10 G.12. Now, by 10 G.12, every $x \in A^*$ is a join of $P(x) = A \cap]\!] \leftarrow, x]\!]$. It is easy to see that every $P(x)$ is left-saturated in $\mathscr{A}$. Since $\mathscr{A}$ is monotonically ordered, the collection of all left-saturated sets is clearly monotone. Thus for any $x \in A^*$, $y \in A^*$ either (1) $P(x) \subset P(y)$, hence $x\varrho^* y$ or $x = y$, or (2) $P(y) \subset P(x)$, hence $y\varrho^* x$ or $y = x$. This proves the monotonicity of $\mathscr{A}^*$.

The proof of the uniqueness of $\mathscr{A}^*$ (up to an isomorphism) is left to the reader.

10 G.21. The following assertions are immediate consequences of 10 G.20.

Let $\mathscr{A} = \langle A, \varrho \rangle$ be a monotone ordered set. Then there exists a monotone ordered set $\mathscr{A}^* = \langle A^*, \varrho^* \rangle$ such that (i) *$\mathscr{A}$ is an ordered subclass of $\mathscr{A}^*$,* (ii) *for any $\xi \in A^*$, ξ is a join (in $\mathscr{A}^*$) of $]\!] \leftarrow, \xi]\!] \cap A$ as well as a meet (in $\mathscr{A}^*$) of $[\![\xi, \rightarrow [\![\cap A$,* (iii) *$\mathscr{A}^*$ is complete.*

Let $\mathscr{A} = \langle A, \varrho \rangle$ be a monotone quasi-ordered set possessing neither a smallest nor a largest element. Then there exists a monotone quasi-ordered set $\mathscr{A}^* = \langle A^*, \varrho^* \rangle$ such that (1)–(3) *from* 10 G.20 *are satisfied, and $\mathscr{A}^*$ has neither a smallest nor a largest element.*

In both cases, $\mathscr{A}^*$ is unique in the sense indicated in 10 G.20.

H. ORDERED ALGEBRAIC SYSTEMS

This last part of the present section is concerned with the examination of some ordered algebraic structs. Such structs, in particular the ordered groups, ordered modules, etc., are of considerable importance. However, we do not investigate them in this book and they will occur in the sequel only rarely, in examples and exercises. On the other hand, the concept of an ordered group (or ring) is quite useful for a proper introduction of real numbers. Therefore, we limit ourselves to two topics here: first, we give exact definitions of basic concepts such as "a quasi-ordered algebraic struct" (in a sense not quite general, but more than sufficient for our purposes) along with some examples but virtually no propositions; secondly, we consider monotonically ordered groups briefly in order to prepare the way for the introduction of real (as well as complex) numbers which concludes the section.

10 H.1. Consider the ring of rationals $\langle \mathrm{Q}, +, . \rangle$, also denoted by $\langle \mathrm{Q}, \sigma, \mu \rangle$ here, and the natural order on Q, denoted by $\leqq$. Then compositions σ and μ are both compatible with this order, though in a somewhat different sense. Namely, we have (1) $x \leqq y \Rightarrow (x\sigma z) \leqq (y\sigma z)$, but it is not true that (2) $x \leqq y \Rightarrow (x\mu z) \leqq (y\mu z)$. However, the following assertion holds: (3) $x \leqq y$, $z \geqq 0 \Rightarrow (x\mu z) \leqq (y\mu z)$; observe that assertion (3), as it stands, is meaningful only if there is a zero for the struct in question.

Thus we may try to distinguish two kinds of "compatibility" of an order (or quasi-order) with a composition; first, a compatibility not involving other compositions and expressed by condition (1) above; secondly, a "compatibility" involving two compositions and similar to that expressed by (3); however, if the zero is not to appear explicitly, condition (3) has to be expressed in another form, e.g. (4) $x \leqq y$, $u \leqq v \Rightarrow (x\mu v) + (y\mu u) \leqq (x\mu u) + (y\mu v)$ (we write $+$ instead of σ here).

Now we are going to state the definition for internal structs limited to two special cases: that of one composition and that of a module-like struct with a commutative basic constituent.

10 H.2. Definition. If ϱ is an associative composition on a class X, τ is a quasi-order on X, $\boldsymbol{r}^* = \langle \varrho, \tau \rangle$ and the compatibility condition indicated below is satisfied, then we shall say that ϱ and τ are *compatible* (more explicitly *additively compatible*) and that $\boldsymbol{r}^*$ is a *quasi-ordered semi-group structure on* X (observe that, properly speaking, this expression is not correct, since, of course, τ is not a quasi-order on ϱ); τ will be termed the *order-constituent of* $\boldsymbol{r}^*$. If, in addition, X is a non-empty set, we shall say that $\mathscr{X} = \langle X, \varrho, \tau \rangle$ is a *quasi-ordered semi-group.* The compatibility condition in question is as follows:

if $x\tau y$, $z \in X$, then $(x\varrho z)\, \tau\, (y\varrho z)$ and $(z\varrho x)\, \tau\, (z\varrho y)$.

Remarks. 1) The above condition implies that $(x\varrho x')\, \tau\, (y\varrho y')$ whenever $x\tau y$, $x'\tau y'$. — 2) We do not define explicitly expressions such as "an ordered semi-group", "a monotone quasi-ordered group" (or, equivalently, "a monotonically ordered group"), etc., since their meaning is sufficiently clear.

Examples. (A) $\langle \mathsf{N}, +, \leqq \rangle$ is a monotone ordered semi-group, $\langle \mathsf{Q}, +, \leqq \rangle$ is a monotone ordered group. — (B) $\langle \mathsf{N}^{\mathsf{N}}, +^{\mathsf{N}}, \leqq^{\mathsf{N}} \rangle$ is an ordered semi-group (of course, it is not monotonically ordered). — (C) $\langle \cup, \subset \rangle$ is an ordered commutative semi-group structure. Observe that $X \cup Z \subset Y \cup Z$ does not imply $X \subset Y$.

10 H.3. Definition. Let X be a class. Let ϱ be a commutative associative composition on X. Let μ be a composition on X distributive with respect to ϱ. Let τ be a quasi-order on X. Suppose that ϱ and τ are additively compatible and that the following condition (in which it is written $a \leqq b$ instead of "$a\tau b$ or $a = b$") is satisfied:

if $x \leqq y$, $u \leqq v$, then $(x\mu v)\, \varrho\, (y\mu u) \leqq (x\mu u)\, \varrho\, (y\mu v)$,

then we shall say that μ and τ are *multiplicatively compatible relative to* ϱ (or that μ is *compatible with* τ *relative to* ϱ).

Remark. We consider, for convenience, only the case of a commutative ϱ. The general case of a multiplicative compatibility of μ and τ with respect to an arbitrary associative ϱ is somewhat complicated; on the other hand, the case of a (possibly non-commutative) group structure ϱ can be dealt with easily (as concerns a proper introduction of basic concepts).

10 H.4. Definition. Let $\boldsymbol{r}$ be an internal algebraic structure on a class X, $\boldsymbol{r} = \varrho_0$ or

$\boldsymbol{r} = \langle \varrho_0, \ldots, \varrho_m \rangle$, $m \geqq 1$. Suppose that $\boldsymbol{r}$ is module-like and ϱ_0 is commutative. If (1) ϱ_0 is additively compatible with τ and (2) every φ_k, $k \geqq 1$, is (multiplicatively) compatible with τ relative to ϱ_0 (observe that if $m \geqq 1$, then this condition implies, according to 10 H.3, that ϱ_0 is commutative), then we shall say that $\boldsymbol{r}$ and τ are *compatible*; $\boldsymbol{r}' = \langle \varrho_0, \ldots, \varrho_m, \tau \rangle$ will be called a *quasi-ordered internal algebraic structure on* X and τ will be termed the *order-constituent of* $\boldsymbol{r}'$ (or also of $\langle X', \boldsymbol{r} \rangle$); $\langle X, \boldsymbol{r}' \rangle$ will be called a *quasi-ordered internal algebraic struct.*

Convention. If $\langle \varrho, \sigma \rangle$ is a semi-ring structure and $\langle \varrho, \sigma, \tau \rangle$ is a quasi-ordered internal algebraic structure, then we shall say that $\langle \varrho, \sigma, \tau \rangle$ is a *quasi-ordered semi-ring structure* and similarly in other analogous cases. Thus, we shall use expressions like "an *ordered semi-group*", "a *quasi-ordered group*", "a *monotonically ordered ring*", "a *quasi-ordered module structure*" (see 10 H.8), etc.

Examples. (A) $\langle \cup, \cap, \subset \rangle$ is an ordered internal algebraic structure. – (B) $\langle \mathsf{N}, +, ., \leqq \rangle$ is a monotone ordered semi-ring, $\langle \mathsf{Z}, +, ., \leqq \rangle$ is a monotone ordered ring, $\langle \mathsf{Q}, +, ., \leqq \rangle$ is a monotone ordered field.

10 H.5. *Let X be a class. Let $\langle \varrho, \mu \rangle$ be a ring structure on X and let o denote the neutral element under ϱ. Let τ be a quasi-order on X. Then μ is multiplicatively compatible with τ relative to ϱ if and only if the following holds:*
if $o\tau x$ and $o\tau y$, then $o\tau(x\mu y)$ or $o = x\mu y$.

The proof is easy and therefore omitted. In most cases, the basic constituent of a module-like algebraic structure $\boldsymbol{r}$ is a commutative group-structure and therefore, to verify that $\boldsymbol{r}$ is compatible with a quasi-order τ, the above condition will be used (this is simpler than that in 10 H.3).

10 H.6. Definition. Let X be a class. Let ϱ be a commutative associative composition on X. Let $\mathscr{A}$ be a quasi-ordered internal algebraic struct (or else a quasi-ordered class); let us denote its underlying class by A, its order-constituent (respectively, if A is a quasi-ordered class, its structure) by α. Let $\boldsymbol{m}$ be an external composition over $\mathscr{A}$ acting on X; let $\boldsymbol{m}$ be action-distributive with respect to ϱ. Let τ be a quasi-order on X. Suppose that ϱ and τ are additively compatible and that the following condition (in which $a \leqq b$ is written instead of "$a\alpha b$ or $a = b$", $x \leqq y$ instead of "$x\tau y$ or $x = y$") is satisfied:

if $a \in A$, $b \in A$, $x \in X$, $y \in X$, and $a \leqq b$, $x \leqq y$, then $(a\boldsymbol{m}y)\,\varrho\,(b\boldsymbol{m}x) \leqq$ $\leqq (a\boldsymbol{m}x)\,\varrho\,(b\boldsymbol{m}y)$.

Then we shall say that $\boldsymbol{m}$ and τ are (*multiplicatively*) *compatible relative to* ϱ.

Example. Let Z be a set, $X = \exp \mathsf{Z}$, $A = \mathsf{Z}^{\mathsf{Z}}$. Let μ consist of all $\langle Y, \varphi, \varphi[Y] \rangle$ where $Y \subset \mathsf{Z}$, $\varphi \in A$. Clearly, $\mathscr{A} = \langle A, \circ, \subset_A \rangle$ is a quasi-ordered semi-group. Put $\boldsymbol{m} = \langle \mu, \circ, \subset_A \rangle$. Then $\boldsymbol{m}$ is an external composition over A on X. It is easy to see that $\boldsymbol{m}$ and $\subset_X$ are compatible relative to $\bigcup_X$.

10 H.7. *Let X be a class. Let ϱ be a commutative group structure on X and let o denote the neutral element under ϱ. Let τ be a quasi-order on X.*

If m *is a composition on* X *over* $\mathscr{A} = \langle A, \dots, \alpha \rangle$, $\mathscr{A}$ *being a quasi-ordered internal algebraic struct (or a quasi-ordered class), then* m *is compatible with* τ *relative to* ϱ *if and only if the following condition is satisfied:*

if $u\alpha v$ *and* $o\tau x$, *then* $(umx)\ \tau\ (vmx)$ *or* $umx = vmx$.

We omit the proof and observe that a remark similar to that in 10 H.5 applies here.

10 H.8. Definition. Let r be a module-like algebraic structure on a class X; let $r = r_0$ or $r = \langle r_0, \dots, r_m \rangle$, r_k being compositions (internal or external). For internal compositions r_k, put $r'_k = r_k$; if r_k is external, let r'_k be the external composition (over a quasi-ordered internal algebraic struct or a quasi-ordered class) obtained by enriching r_k with a quasi-order τ_k. Let τ be a quasi-order on X. Suppose that r'_0 (which is an internal composition) is additively compatible with τ and that every r'_k, $k \geqq 1$, is multiplicatively compatible (in the sense of 10 H.3 or of 10 H.6) with τ relative to r'_o. Put $r' = \langle r'_0, \dots, r'_m, \tau \rangle$.

Then r' will be called a *module-like quasi-ordered algebraic structure on* X, and τ will be called the *order-constituent* of r' (or of $\langle X, r' \rangle$); the struct $\langle X, r' \rangle$ will be called a *module-like quasi-ordered algebraic struct.*

Conventions. 1) We have defined two kinds of "quasi-ordered algebraic structures" (cf. 10 H.4). We shall not introduce any more general types of such structures; therefore, the structures (and structs) just defined will be termed, for short, *quasi-ordered algebraic structures* (respectively, *structs*). — 2) Let $\langle X, \varrho, \sigma, \kappa, \lambda \rangle$ be a module over a ring $\langle A, \kappa, \lambda \rangle$, $A = \mathbf{D}\sigma$. If ν is a quasi-order on A, τ is a quasi-order on X and $\mathscr{X} = \langle X, \varrho, \langle \sigma, \kappa, \lambda, \nu, \tau \rangle \rangle$ is a quasi-ordered algebraic struct, then $\mathscr{X}$ will be called a *quasi-ordered module (over the quasi-ordered ring* $\langle A, \kappa, \lambda, \nu \rangle$), and similarly in other analogous cases (see 10 H.4, remark). — 3) Without giving a detailed exact definition, we observe that if $r = \langle \varrho_0, \dots, \varrho_m, \tau \rangle$ is a quasi-ordered algebraic structure on X, then the structure r^* obtained by deleting all quasi-orders from r (i.e. by deleting τ and the order-constituents of the external compositions ϱ_k) will be termed the *underlying algebraic structure* of r, and $\langle X, r^* \rangle$ will be called the *underlying algebraic struct of* $\langle X, r \rangle$. — 4) If $\mathscr{X}$ and $\mathscr{Y}$ are quasi-ordered algebraic structs, $\mathscr{X}^*$ and $\mathscr{Y}^*$ are their underlying algebraic structs and f is a homomorphism of $\mathscr{X}^*$ into $\mathscr{Y}^*$, then $f : \mathscr{X} \to \mathscr{Y}$ will be also called a *homomorphism.*

10 H.9. The investigation of general properties of quasi-ordered algebraic structs could now proceed along similar lines as in Section 8 for algebraic structs. We restrict ourselves to some basic facts (sometimes using notions introduced for algebraic structs provided their definition can be easily extended to quasi-ordered algebraic structs); in particular, we omit the consideration of "exceptional" cases (e.g. of the possibility that the order-constituent coincides with some composition) similar to that in 8 A.16.

10 H.10. *Let* r *be an algebraic structure on a class* X; *let* τ *be a quasi-order on* X *and let* r *and* τ *be compatible. If* $Y \subset X$ *and there is a restriction* r_Y *of* r *to an algebraic structure on* Y, *then* r_Y *and* τ_Y *are compatible.*

Definition. If $\boldsymbol{r} = \langle \boldsymbol{r}^{(0)}, \ldots, \boldsymbol{r}^{(m)}, \tau \rangle$ or $\boldsymbol{r} = \langle \boldsymbol{r}^{(0)}, \tau \rangle$ is a quasi-ordered algebraic structure on X, $\boldsymbol{r}' = \langle \boldsymbol{r}^{(0)}, \ldots, \boldsymbol{r}^{(m)} \rangle$ (or $\boldsymbol{r}' = \boldsymbol{r}^{(0)}$), $Y \subset X$ and there exists a restriction $\boldsymbol{r}'_Y$ of $\boldsymbol{r}'$ to a structure on Y, then $\langle \boldsymbol{r}_Y^{(0)}, \ldots, \boldsymbol{r}_Y^{(m)}, \tau_Y \rangle$ will be denoted by $\boldsymbol{r}_Y$ and called the *restriction of* $\boldsymbol{r}$ *to a structure on* Y.

10 H.11. Definition. Let $\mathscr{A} = \langle A, \ldots, \sigma \rangle$, $\mathscr{B} = \langle B, \ldots, \tau \rangle$ be quasi-ordered algebraic structs; let f be a mapping of $\mathscr{A}$ into $\mathscr{B}$. If $f : \langle A, \sigma \rangle \to \langle B, \tau \rangle$ is order-preserving (respectively, order-reversing, see 10 C.1) and $f : \mathscr{A}^* \to \mathscr{B}^*$ is a homomorphism, $\mathscr{A}^*$ and $\mathscr{B}^*$ being the underlying algebraic structs of $\mathscr{A}$ and $\mathscr{B}$, then f will be called an *order-preserving* (respectively, *order-reversing*) *homomorphism.* In particular, if both f and f^{-1} are order-preserving homomorphisms, then f is called an *isomorphism.* If $\mathscr{A}, \mathscr{B}$ are quasi-ordered algebraic structs and there exists an isomorphism of $\mathscr{A}$ onto $\mathscr{B}$, then we shall say that $\mathscr{A}$ and $\mathscr{B}$ are *isomorphic.*

Examples. (A) Consider the group $\langle \mathrm{Q}, + \rangle$. For any $x \in \mathrm{Q}$, $y \in \mathrm{Q}$ put $f_x y = x \,.\, y$. Then every $f_x : \langle \mathrm{Q}, + \rangle \to \langle \mathrm{Q}, + \rangle$ is an endomorphism and every endomorphism of $\langle \mathrm{Q}, + \rangle$ can be uniquely expressed in this form; clearly, f_x is bijective unless $x = 0$. A mapping $f_x : \langle \mathrm{Q}, +, \leqq \rangle \to \langle \mathrm{Q}, +, \leqq \rangle$ is an order-preserving homomorphism if and only if $x \geqq 0$. A mapping $f_x : \mathscr{Q} \to \mathscr{Q}$, where $\mathscr{Q} = \langle \mathrm{Q}, +, ., \leqq \rangle$, is a homomorphism if and only if $x = 0$ or $x = 1$; in such a case it is also order-preserving. – (B) Let H denote the set of all endomorphisms of $\langle \mathrm{Q}, + \rangle$; for $h_1 \in H$, $h_2 \in H$ put $h_1 \sigma h_2$ if and only if $h_1 x \leqq h_2 x$ for every $x \in \mathrm{Q}$, $x \geqq 0$. Then $\langle H, +, \circ, \sigma \rangle$ is isomorphic with $\mathscr{Q}$ (see example (A)).

10 H.12. *Let* $\mathscr{A} = \langle A, \sigma, \mu, \tau \rangle$ *be a quasi-ordered ring. Let* μ^* *consist of all* $\langle a, b, a\mu b \rangle$. *Then* $\langle A, \sigma, \langle \mu^*, \sigma, \mu, \tau \rangle, \tau \rangle$ *is a quasi-ordered module over* $\mathscr{A}$.

This is clear.

10 H.13. Proposition und definition. *Let* $\{\mathscr{A}_b \mid b \in B\}$ *be a non-empty family of quasi-ordered algebraic structs of the same type,* $\mathscr{A}_b = \langle A_b, \boldsymbol{r}_b^{(0)}, \ldots, \boldsymbol{r}_b^{(m)}, \tau_b \rangle$. *Put* $A = \Pi\{A_b\}$, $\boldsymbol{r}^{(k)} = \Pi\{\boldsymbol{r}_b^{(k)} \mid b \in B\}$, $\tau = \Pi \tau_b$ (see 8 B.8 and 10 B.8). *Then* $\langle A, \boldsymbol{r}^{(0)}, \ldots, \boldsymbol{r}^{(m)}, \tau \rangle$ *is a quasi-ordered algebraic struct which will be called the* cartesian product *of the family* $\{\mathscr{A}_b\}$ *and denoted by* $\Pi\{\mathscr{A}_b \mid b \in B\}$ *or* $\Pi\{\mathscr{A}_b\}$, *etc.*

If all $\mathscr{A}_b$ are equal to a struct $\mathscr{A}$, then we write $\mathscr{A}^B$ instead of $\Pi\{b \to \mathscr{A} \mid b \in B\}$.

Convention. Let B be an arbitrary non-empty set. Let $\mathscr{A}$ be an ordered ring. Let $\mathscr{A}^*$ be the ordered module defined in 10 H.12. Then $(\mathscr{A}^*)^B$ is often called the ordered module of families (indexed by elements from B) of elements from $\mathscr{A}$.

Now, we proceed to questions concerning monotone ordered groups.

10 H.14. Definition. If $\mathscr{A}$ is a quasi-ordered algebraic struct, $\mathscr{A} = \langle A, \varrho, \ldots, \tau \rangle$, then an element $x \in A$ is called *positive in* $\mathscr{A}$ if $z\tau(z\varrho x)$, $z\tau(x\varrho z)$ for every $z \in A$; *negative in* $\mathscr{A}$ if $(z\varrho x)\,\tau z$, $(x\varrho z)\,\tau z$ for every $z \in A$; *strictly positive in* $\mathscr{A}$ if it is positive, but not negative; *strictly negative in* $\mathscr{A}$ if it is negative, but not positive.

Observe that if τ is distinguishing, then there exists at most one element which is both positive and negative, and such an element is neutral under ϱ.

10 H.15. *Let $\mathscr{G} = \langle G, \sigma, \tau \rangle$ be a quasi-ordered group. If $a \in G$ and the set of all n-th σ-powers a^n, $n \geqq 1$, of a (i.e. the set of all elements a, aσa, aσaσa, ...) has a meet, then a is positive in $\mathscr{G}$.*

Proof. Denote by e the neutral element of $\mathscr{G}$. Let ξ be a meet of the set of all a^n, $n \geqq 1$. Then $(\xi \sigma a^{-1}) \tau a^n$ for $n = 1, 2, \ldots$ and therefore $(\xi \sigma a^{-1}) \tau \xi$, hence $e \tau a$.

10 H.16. *Let $\mathscr{G}$ be a boundedly complete ordered group. If $a \in G$ and the set of all n-th powers, $n \geqq 1$, of a is bounded, then a is the neutral element of $\mathscr{G}$.*

Proof. The set in question is bounded, hence has a meet and therefore, by 10 H.15, a is positive. Similarly, it can be shown that a is negative.

10 H.17. Definition. Let $\mathscr{G} = \langle G, \sigma, \tau \rangle$ be a monotone ordered group. We shall say that $\mathscr{G}$ is *archimedean* if, for every strictly positive $x \in G$ and every $y \in G$, there exists an $n \in \mathrm{N}$ such that $y \tau x^n$.

Examples. $\langle \mathrm{Q}, +, \leqq \rangle$ is archimedean, whereas the following ordered group is not: $\langle \mathrm{Q}, + \rangle \times \langle \mathrm{Q}, + \rangle$ endowed with the lexicographical product (see 10 B.10) of $\leqq$ and $\leqq$.

Remark. Let $\langle G, \sigma, \leqq \rangle$ be archimedean. If $a \in G$, $a > 0$, $x \in G$, then there exists exactly one $n \in \mathrm{Z}$ such that $a^n \leqq x < a^{n+1}$.

10 H.18. Theorem. *Every boundedly complete monotone ordered group is archimedean.*

This follows at once from 10 H.16.

10 H.19. Theorem. *Every archimedean monotone ordered group $\mathscr{G} = \langle G, \sigma, \leqq \rangle$ can be embedded into a boundedly complete monotone ordered commutative group.*

Proof. If there exists a smallest strictly positive element, say a, of $\mathscr{G}$, then it is easy to see that $\{n \to a^n\} : \langle \mathrm{Z}, +, \leqq \rangle \to \mathscr{G}$ is an isomorphism; $\langle \mathrm{Z}, +, \leqq \rangle$ is clearly a boundedly complete commutative monotone ordered group. Therefore, we consider only the case when there is no smallest strictly positive element. Let 0 denote the neutral element of $\mathscr{G}$. For convenience, we first prove that $\mathscr{G}$ is commutative. Suppose the contrary; choose a, b such that $a \sigma b < b \sigma a$ and put $c = b \sigma a \sigma b^{-1} \sigma a^{-1}$; then $c > 0$. Choose $z \in G$ with $0 < z < c$; choose $x \in G$ such that $0 < x < z$, $x < c \sigma z^{-1}$. Then clearly $0 < x < x^2 < c$. Since $\mathscr{G}$ is archimedean, there exist (see 10 H.17, remark) integers m, n such that $x^m \leqq a < x^{m+1}$, $x^n \leqq b < x^{n+1}$. Hence, $x^{m+n} \leqq a \sigma b < x^{m+n+2}$, $x^{m+n} \leqq b \sigma a < x^{m+n+2}$, $c \leqq x^2$, which is a contradiction.

Now we shall write $x + y$ instead of $x \sigma y$, $x - y$ instead of $x \sigma y^{-1}$, etc. It follows from 10 G.21 that there exists a boundedly complete ordered set $\langle G^*, \leqq \rangle$ such that $\langle G, \leqq \rangle$ is an ordered subset of $\langle G^*, \leqq \rangle$ and every $\xi \in G^*$ is a join of $]\!] \leftarrow, \xi \,]\!] \cap G$ as well as a meet of $[\![\, \xi, \to [\![\cap G$.

Now, for any $\xi \in G^*$, $\eta \in G^*$, let $\xi \sigma^* \eta$ be the supremum of the set of all $x + y$, $x \in G$, $y \in G$, $x \leqq \xi$, $y \leqq \eta$. Clearly, σ^* is a composition on G^* and σ is its restriction to G. It is also clear that $\xi \leqq \xi'$, $\eta \leqq \eta' \Rightarrow (\xi \sigma^* \eta) \leqq (\xi' \sigma^* \eta')$.

Since, evidently, σ^* is a commutative associative composition and 0 is neutral under σ^*, it remains to prove that, for any $\xi \in G^*$, there exist a $\eta \in G^*$ such that $\xi \sigma^* \eta = 0$. Now, if $\xi \in G^*$, let X consist of all $x \in G, x \leqq \xi$, and let Y consist of all elements $y = -u$, $u \in G$, $u \geqq \xi$. Put $\eta = \sup Y$. It is clear that $x \in X$, $y \in Y \Rightarrow x + y \leqq 0$; hence, $\xi + \eta \leqq 0$. For any $t \in G$, $t > 0$, there exist $x \in X$, $u \in G$, $u \geqq \xi$ such that $u - x < t$; indeed, if not, we would obtain that, for any $x \in X$, $x + t \leqq \xi$, hence $x + t \in X$, and therefore $x + nt \in X$ which contradicts the fact that G is archimedean. Thus, for any $z < 0$ there exist $x \in X$, $y \in Y$ such that $x + y > z$. This shows that $\xi + \eta = 0$ and completes the proof.

10 H.20. Theorem. *Let $\mathscr{G}, \mathscr{R}$ be archimedean monotone ordered groups. Let $\mathscr{R}$ be boundedly complete and possess no smallest strictly positive element. Let a be a strictly positive element of $\mathscr{G}$. Let x be an element of $\mathscr{R}$. If $x \geqq 0$, then there exists exactly one order-preserving homomorphism f_x of $\mathscr{G}$ into $\mathscr{R}$ such that $f_x a = x$. If $x < 0$, then there exists exactly one order-reversing homomorphism f_x of $\mathscr{G}$ into $\mathscr{R}$ such that $f_x a = x$.*

Proof. I. We shall denote the underlying sets of $\mathscr{G}$ and $\mathscr{R}$ by G and R; the compositions (of both $\mathscr{G}$ and $\mathscr{R}$) will be denoted by $+$, the orders by $\leqq$; observe that $\mathscr{G}$ and $\mathscr{R}$ are commutative, by 10 H.19. The set of all homomorphisms of $\langle G, + \rangle$ into $\langle R, + \rangle$ will be denoted by H and will be considered as endowed with the usual composition $+$ and the order denoted by $\leqq$ and determined by $h_1 \leqq h_2 \Leftrightarrow (h_1 x \leqq \leqq h_2 x$ for all positive $x)$. – II. The uniqueness of the homomorphisms f_x is shown quite easily; indeed, if f, f' possess the properties in question, then $h = f - f' \in H$, $ha = 0$, $h(na) = 0$ for each $n \in \mathrm{Z}$, $hx = 0$ for each $x \in G$ such that $0 \leqq x \leqq na$ for some $n \in \mathrm{Z}$, hence, $\mathscr{G}$ being archimedean, for all $x \in G$. – III. We are now going to show that, for any $x \in R$ and any $n \in \mathrm{Z}$, $n \neq 0$, there exists exactly one $y \in R$ such that $ny = x$. It is sufficient to consider the case $x > 0$, $n > 0$. Since $\mathscr{R}$ is archimedean and there exist no smallest strictly positive $z \in R$, there is an element $u \in R$, $u > 0$, such that $nu \leqq x$. Let y be the supremum of all such u. It is easy to prove (using the fact that $\mathscr{R}$ is archimedean) that $ny = x$. – IV. To prove the existence of f_x it is sufficient to examine the case $x > 0$. First consider the set Q of all $z \in G$ such that, for some $m \in \mathrm{Z}$, $n \in \mathrm{Z}$, $m \neq 0$, we have $mz = na$; clearly, Q is a subgroup of $\mathscr{G}$, and, for every $\zeta \in G$, ζ is the join of the set of all $z \in Q$, $z \leqq \zeta$ (the easy proof of this fact is left to the reader). If $z \in Q$, $mz = na$, $m \neq 0$, then there exists, according to part III of this proof, exactly one element $y \in R$ such that $my = nx$; we put $\varphi_x z = y$. It is easy to see that the mapping $\varphi_x : \langle Q, +, \leqq \rangle \to \mathscr{R}$ is an order-preserving homomorphism. If $Q = G$, the proof is complete. If not, we put, for any $\zeta \in G$, $\varphi_x \zeta = \sup \{\varphi_x z \mid z \in Q, z \leqq \zeta\}$. It is easily proved that $f_x = \varphi_x : \mathscr{G} \to \mathscr{R}$ has the properties in question.

10 H.21. Theorem. *Any two boundedly complete monotone ordered groups without a smallest strictly positive element are isomorphic provided both contain more than one element.*

Proof. Let $\mathscr{R}_1, \mathscr{R}_2$ be the ordered groups in question. By 10 H.18, they are archimedean. Choose strictly positive elements a_i of $\mathscr{R}_i$. By 10 H.20 there exist order-preserving homomorphisms f_1 of $\mathscr{R}_1$ into $\mathscr{R}_2$, f_2 of $\mathscr{R}_2$ into $\mathscr{R}_1$ such that $f_1 a_1 = a_2$, $f_2 a_2 = a_1$. Then $g = f_2 \circ f_1$ is an order-preserving homomorphism of $\mathscr{R}_1$ into $\mathscr{R}_1$ such that $ga_1 = a_1$. Hence, by 10 H.20, $gx = x$ for every $x \in \mathscr{R}_1$. Thus $f_2 \circ f_1$ (and similarly, $f_1 \circ f_2$) coincides with the identical mapping, and therefore f_1, f_2 are isomorphisms.

10 H.22. Proposition and definition. *Let $\mathscr{G} = \langle G, +, \leqq \rangle$ be a commutative ordered group. Let $\langle H, +, \circ \rangle$ be the ring of endomorphisms of the group $\langle G, + \rangle$ (cf. 6 E.13). For $h_1 \in H$, $h_2 \in H$ put $h_1 \leqq h_2$ if and only if $h_2 - h_1$ is order-preserving. Then $\mathscr{H} = \langle H, +, \circ, \leqq \rangle$ is an ordered ring; it will be called the* ordered ring of endomorphisms of $\mathscr{G}$. *The set H^* of all $h \in H$ of the form $h = h_1 - h_2$, where $h_1 \in H$, $h_2 \in H$ are order-preserving, is an ordered subring of $\mathscr{H}$.*

Proof. Clearly, if $h_1 \leqq h_2$, then $h_2 + h \leqq h_2 + h$ for every $h \in H$, and if $h_1 \geqq 0$, $h_2 \geqq 0$, then $h_2 \circ h_1 \geqq 0$. Thus $\mathscr{H}$ is an ordered ring. If h, h' belong to H^*, then clearly $h + h'$, $h - h'$ belong to H^*. Let $h = h_1 - h_2$, $h' = h_3 - h_4$, where h_i are order-preserving; then $h \circ h' = (h_1 \circ h_3 + h_2 \circ h_4) - (h_1 \circ h_4 + h_2 \circ h_3)$ and therefore $h \circ h' \in H^*$.

10 H.23. Theorem. *Let $\mathscr{G} = \langle G, +, \leqq \rangle$ be a boundedly complete monotone ordered group possessing no smallest strictly positive element; let $a \in G$ be strictly positive. Then there exists precisely one composition μ on G such that $\langle G, +, \mu, \leqq \rangle$ is an ordered ring and a is a unit under $\langle +, \mu \rangle$. The composition μ is commutative and $\langle G, +, \mu, \leqq \rangle$ is an ordered field.*

Proof. Consider the ordered ring $\mathscr{H}$ of all endomorphisms of G (see 10 H.22) and its subring $\mathscr{H}^* = \langle H^*, +, \circ, \leqq \rangle$ consisting of all $h = h_1 - h_2$ where h_1, h_2 are order-preserving endomorphisms. It is easy to show (using 10 H.20) that, for any $a \in G$, $\{h \to ha\} : \langle H^*, +, \leqq \rangle \to \mathscr{G}$ is an isomorphism. Let h_x denote the endomorphism $h \in H^*$ such that $ha = x$. Put, for $x \in G$, $y \in G$, $x\mu y = z$ where z is such that $h_z = h_x \circ h_y$. Then, clearly, $G' = \langle G, +, \mu, \leqq \rangle$ is an ordered ring (since so is $\langle H, +, \circ, \leqq \rangle$) and a is a unit of G'. If ν is a composition on G such that $\langle G, +, \nu, \leqq \rangle$ is an ordered ring, then, for any $x \in G$ such that $x > 0$, both $\{y \to x\nu y\}$ and $\{y \to x\mu y\}$ are graphs of order-preserving endomorphisms of $\mathscr{G}$. Since $x\nu a = x = x\mu a$, we obtain, by 10 H.20, that these endomorphisms coincide. Hence there follows $\nu = \mu$. Putting $x\nu y = y\mu x$ we get a composition ν such that $\langle G, +, \nu, \leqq \rangle$ is an ordered ring and therefore $\nu = \mu$. This proves that μ is commutative. Finally, it is easy to show, using 10 H.20, that every $h \in H^*$ distinct from 0 is an isomorphism. This implies that $\langle G, +, \mu \rangle$ is a field.

10 H.24. We are now ready to introduce the real numbers. As in other analogous cases (see 3 D.1, 8 E.4, 8 E.5, 8 F.8, 8 F.9), this can be done by choosing one well-determined ordered field with the properties described e.g. in 10 H.23. However, for reasons indicated in 8 F.8, and keeping in line with the procedure adopted in analogous situations, we prefer an axiomatic definition. Theorem 10 H.23 will then serve to justify the axioms.

10 H.25. Axioms for real numbers.

(a) R_{ofd} *is a boundedly complete monotone ordered field*;

(b) $\langle \mathsf{Q}, +, ., \leqq \rangle$ *is an ordered subfield of* R_{ofd}.

Remark. It follows at once from (a) that R_{ofd} is isomorphic with every boundedly complete monotone ordered field.

Conventions. 1) The underlying set of R_{ofd} will be denoted by R; the compositions and the order with which R is endowed to obtain R_{ofd} will be denoted, respectively, by $+_{\mathsf{R}}, \cdot_{\mathsf{R}}, \leqq_{\mathsf{R}}$, or simply by $+, ., \leqq$; thus $\mathsf{R}_{\text{ofd}} = \langle \mathsf{R}, +, ., \leqq \rangle$. The symbol R will usually also denote the structs $\langle \mathsf{R}, + \rangle$, $\langle \mathsf{R}, +, \leqq \rangle$, $\langle \mathsf{R}, \leqq \rangle$, etc.; the exact meaning of the symbol R will usually be clear from the context. – 2) Every element of R will be called a *real number*; R will be called the *set of (all) real numbers*, $\langle \mathsf{R}, + \rangle$ will be called the *additive group of (all) real numbers* and so on. A mapping of an (arbitrary) class M into R will also be called a *real-valued function* (or simply *function*) on M.

We do not develop the theory of the ordered field R. Its elementary properties as well as the basic properties of elementary real-valued functions on R, etc., will be assumed to be known in what follows.

10 H.26. In several places, the so-called extended real line will be needed. To this end we introduce the following

Convention. If $\mathscr{S} = \langle S, \varrho \rangle$ is an ordered set such that (1) $\langle \mathsf{R}, \leqq \rangle$ is embedded in $\mathscr{S}$, (2) $S = (\alpha) \cup \mathsf{R} \cup (\beta)$ with $\alpha \varrho x \varrho \beta$ for all $x \in \mathsf{R}$, then we will occasionally say that $\mathscr{S}$ (and also S) is the *extended real line*; the elements α and β will usually be denoted by $-\infty$ and $+\infty$ (or ∞) respectively, and the extended real line will also be denoted by $\overline{\mathsf{R}}$.

Remarks. 1) In speaking about the extended real line, the real numbers will sometimes be termed its *finite elements*. Thus if $M \subset \overline{\mathsf{R}}$, then "sup M is finite" will mean that $\sup M \in \mathsf{R}$, so that there exists a real number a such that $x \leqq a$ for all $x \in M$. – 2) The absolute value of the elements $-\infty$ and $+\infty$ of the extended real line, to be denoted by $|-\infty|$ and $|+\infty|$, is defined as the element $+\infty$. – 3) For some purposes it is useful to extend the addition in R to a (partial) composition in $\overline{\mathsf{R}}$, and similarly for multiplication.

To conclude the section, we turn now to complex numbers, introducing them also by means of defining axioms. Complex numbers will seldom occur in the sequel; their elementary properties will be assumed to be known whenever needed.

10 H.27. Axioms for complex numbers.

(a) C_{fd} *is a field*;

(b) $\langle \mathsf{R}, +, . \rangle$ *is a subfield of* C_{fd};

(c) *there exists an element i of the field* C_{fd} *such that* (1) $i^2 = -1$, (2) *for any element* ξ *of the field* C_{fd}, *there exist* $x \in \mathsf{R}$, $y \in \mathsf{R}$ *with* $\xi = x + y \,.\, i$.

Conventions. 1) The underlying set of C_{fd} will be denoted by C, the compositions of C_{fd} will be denoted by $+_{\mathsf{C}}$, $._{\mathsf{C}}$ or simply by $+, .$ — 2) Every element of C will be called a *complex number*; the set C will be called the *set of (all) complex numbers*, $\mathsf{C}_{\mathrm{fd}} = \langle \mathsf{C}, +, . \rangle$ will be called the *field of (all) complex numbers* and so on; any mapping of a class M into C will be called a *complex-valued function* on M. As a rule, C will also be used to denote $\langle \mathsf{C}, +, . \rangle$, etc.

Remark. The following field possesses all the properties required of C_{fd} in the above axioms. Consider the set $\mathsf{R} \times \mathsf{R}$; if $\xi = \langle x_1, x_2 \rangle$, $\eta = \langle y_1, y_2 \rangle$ belong to $\mathsf{R} \times \mathsf{R}$, put $\xi + \eta = \langle x_1 + y_1, x_2 + y_2 \rangle$, $\xi \,.\, \eta = \langle x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1 \rangle$. It is easy to see that $\langle \mathsf{R} \times \mathsf{R}, +, . \rangle$ is a field; replacing every $\langle x, 0 \rangle$ by x we obtain a field as required. — The fact just shown justifies the axioms 10 H.27.

10 H.28. Clearly, there exists only one isomorphism of $\langle \mathsf{R}, +, . \rangle$ onto itself, namely the identical isomorphism. On the other hand, if $i \in \mathsf{C}$, $i^2 = -1$, and a mapping f assigns $x - y \,.\, i$ to $x + y \,.\, i$, where x and y are real numbers, then f is a non-identical isomorphism of $\langle \mathsf{C}, +, . \rangle$ onto itself. However, it is sometimes necessary to have a fixed "imaginary unit", i.e. an element i such that $i^2 = 1$; this is the case e.g. if we have to define the "real part" and the "imaginary part" of every complex number. The choice of a fixed imaginary unit may be suitably performed by adding the following statement as an axiom: "i is a complex number, $i^2 = -1$."

11. WELL-ORDER

In this section well-ordered classes are considered and ordinal numbers are introduced. Considerations involving well-ordered sets can often be replaced with advantage by arguments based on the maximality principle (see 4 C.1) and related theorems. Nevertheless, well-ordered classes as well as ordinal numbers retain a considerable importance.

A. WELL-ORDERED CLASSES

11 A.1. Definition. Let $\langle A, \sigma \rangle$ be an ordered class. If every non-empty $X \subset A$ has a smallest element (under σ), then we shall say that σ is a *well-order* and $\langle A, \sigma \rangle$ is a *well-ordered class.*

All conventions introduced in Section 10 will be used freely here: e.g. an order will usually be denoted by $\leqq$ (even if several orders are considered at the same time), the strict quasi-order associated with $\leqq$ will be denoted by $<$, A will be written instead of $\langle A, \leqq \rangle$, etc.

Examples. (A) Every finite monotone ordered set is well-ordered. – (B) $\langle \mathsf{N}, \leqq \rangle$ is a well-ordered set. – (C) For any $n \in \mathsf{N}$, let Pn denote the set of all prime numbers p which divide n. Put $m\sigma n$ if and only if either card $Pm <$ card Pn or card $Pm =$ card Pn and $m < n$. It is easy to prove that $\langle \mathsf{N}, \sigma \rangle$ is well-ordered.

11 A.2. Theorem. *A monotone ordered class $\mathscr{X}$ is well-ordered if and only if no subclass of $\mathscr{X}$ is order-isomorphic with $\langle \mathsf{N}, \geqq \rangle$.*

Proof. The necessity of the condition is evident. – Suppose that $\mathscr{X} = \langle X, \leqq \rangle$ is not well-ordered. Let $Y \subset X$ be non-void and contain no smallest element. Then, for any $y \in Y$, there exists a $z \in Y$ such that $z < y$. By 4 C.7, we obtain an infinite sequence $\{y_n\}$ such that $y_{n+1} < y_n$, $n = 1, 2, \ldots$ Clearly, $\langle \mathbf{E}\{y_n\}, \leqq \rangle$ is isomorphic with $\langle \mathsf{N}, \geqq \rangle$.

11 A.3. *Any well-ordered class is boundedly complete and monotone.*

The proof is left to the reader.

11 A.4. Theorem. *Every subclass of a well-ordered class is well-ordered. If $\mathscr{X}$ is well-ordered, $\mathscr{Y}$ is ordered and there is an order-preserving mapping of $\mathscr{X}$ onto*

$\mathscr{Y}$, then $\mathscr{Y}$ is well-ordered. The lexicographical product of two well-ordered classes is well-ordered.

Proof. The first assertion being evident, we prove the second and third. Let F be an order-preserving mapping of $\mathscr{X}$ onto $\mathscr{Y}$. Let $M \neq \emptyset$ be a subclass of $\mathscr{Y}$. There is a smallest element x_0 of $F^{-1}[M]$; clearly, Fx_0 is the smallest element of M. – If $\mathscr{X}$, $\mathscr{Y}$ are well-ordered classes, let $M \subset \mathscr{X} \times_{\text{lex}} \mathscr{Y}$. Since $\mathscr{X}$ is well-ordered, there is a smallest x_0 such that $\langle x_0, y\rangle \in M$ for some y; since $\mathscr{Y}$ is well-ordered, there exists a smallest y_0 such that $\langle x_0, y_0\rangle \in M$. Clearly $\langle x_0, y_0\rangle$ is the smallest element in $\mathscr{X} \times_{\text{lex}} \mathscr{Y}$.

11 A.5. *Let $\mathscr{A} = \langle A, \leqq\rangle$ be a well-ordered class. Let F be an injective order-preserving mapping of $\mathscr{A}$ into $\mathscr{A}$. Then $F[A]$ is right-cofinal in $\mathscr{A}$.*

Proof. Suppose that $F[A]$ is not cofinal; then there exists an element $a \in A$ such that $Fx < a$ for every $x \in A$. In particular $Fa < a$ and therefore, F being injective, $F^{n+1}a < F^n a$, $n = 1, 2, \ldots$. By 11 A.2, this contradicts the fact that $\mathscr{A}$ is well-ordered.

11 A.6. *Let $\mathscr{A} = \langle A, \leqq\rangle$ be a well-ordered class. Let F be an injective order-preserving mapping of $\mathscr{A}$ into $\mathscr{A}$. Let $F[A]$ be interval-like and contain the smallest element of $\mathscr{A}$. Then $F = \mathsf{J} : \mathscr{A} \to \mathscr{A}$. In particular, there exists no order-isomorphic mapping of $\mathscr{A}$ into a segment (see 11 A.8) of $\mathscr{A}$.*

Proof. Suppose $F \neq \mathsf{J} : \mathscr{A} \to \mathscr{A}$. Choose the smallest a with $Fa \neq a$. Then either $Fa < a$ or $a < Fa$, i.e. $F^{-1}(Fa) < Fa$; in both cases we obtain a contradiction.

11 A.7. Theorem. *Let $\mathscr{A}$ and $\mathscr{B}$ be well-ordered classes. Then there exists at most one order-isomorphism of $\mathscr{A}$ onto $\mathscr{B}$.*

Proof. Otherwise there would exist an order-isomorphism of $\mathscr{A}$ onto $\mathscr{A}$ distinct from $\mathsf{J} : \mathscr{A} \to \mathscr{A}$. This contradicts 11 A.6.

11 A.8. Convention. Let $\mathscr{A} = \langle A, \leqq\rangle$ be a well-ordered class; let $x \in A$. Then the class $]\!] \leftarrow, x [\![$ of all elements $y < x$ (and also the class $]\!] \leftarrow, x [\![$ endowed with the restriction of $\leqq$) will be called a *segment of $\mathscr{A}$* or, more explicitly, the *segment of $\mathscr{A}$ determined by* x, and will be denoted by $\mathscr{A}_x$ or A_x (if the order considered is clear from the context).

11 A.9. Theorem. *Let $\mathscr{A}$, $\mathscr{B}$ be well-ordered classes. Then either $\mathscr{A}$ and $\mathscr{B}$ are isomorphic or $\mathscr{A}$ is isomorphic with a segment of $\mathscr{B}$ or $\mathscr{B}$ is isomorphic with a segment of $\mathscr{A}$.*

Proof. Let φ be the relation consisting of all $\langle x, y\rangle, x \in \mathscr{A}$, $y \in \mathscr{B}$, such that $\mathscr{A}_x$ is isomorphic with $\mathscr{B}_y$. Then 11 A.6 implies that φ is one-to-one; clearly, if $x \in \mathscr{A}$, $x' \in \mathscr{A}$, then $x \leqq x' \Rightarrow \varphi x \leqq \varphi x'$, and, if $y \in \mathscr{B}$, $y' \in \mathscr{B}$, then $y \leqq y' \Rightarrow \varphi^{-1}y \leqq \leqq \varphi^{-1}y'$. It is easy to see that (1) either $\mathbf{D}\varphi = \mathscr{A}$ or $\mathbf{D}\varphi = \mathscr{A}_a$ for some $a \in A$, (2) either $\mathbf{E}\varphi = \mathscr{B}$ or $\mathbf{E}\varphi = \mathscr{B}_b$ for some $b \in \mathscr{B}$. Suppose $\mathbf{D}\varphi = \mathscr{A}_a$, $\mathbf{E}\varphi = \mathscr{B}_b$. Then $\mathscr{A}_a$ and $\mathscr{B}_b$ are isomorphic, hence $\langle a, b\rangle \in \varphi$, which is a contradiction. Thus, either $\mathbf{D}\varphi = \mathscr{A}$ or $\mathbf{E}\varphi = \mathscr{B}$ (or both). This proves the theorem.

11 A.10. Theorem. *Let* $\mathscr{A} = \langle A, \leqq \rangle$ *be a well-ordered class. Let* C *be a class such that, for any* $x \in A$, $A_x \subset C$ *implies* $x \in C$. *Then* $C \supset A$.

Proof. Suppose $A - C \neq \emptyset$. Choose the smallest $a \in A - C$. Then $A_a \subset C$, hence $a \in C$, which is a contradiction.

11 A.11. The above theorem as well as each of the two following statements is often called the principle of transfinite induction.

(A) *Let* $\mathscr{A} = \langle A, \leqq \rangle$ *be a well-ordered class. Let* **P** *be a given property. Suppose that* $x \in A$ *possesses property* **P** *whenever every* $y \in A_x$ *possesses this property. Then every* $x \in A$ *possesses property* **P**.

(B) *Let* $\mathscr{A} = \langle A, \leqq \rangle$ *be a well-ordered class; for every* $x \in A$, *let* $\mathbf{S}(x)$ *be a proposition. Suppose that, for any* $x \in A$, $\mathbf{S}(x)$ *holds whenever* $\mathbf{S}(y)$ *holds for every* $y \in A_x$. *Then* $\mathbf{S}(x)$ *holds for every* $x \in A$.

Clearly, (A) is obtained from 11 A.10 if the class C of all x possessing property **P** is considered, and (B) is obtained from 11 A.10 if we consider the class C of all x such that $\mathbf{S}(x)$ holds.

Observe, however, that statements (A) and (B) are, properly speaking, propositions about certain propositions, and not about mathematical objects; their strictly mathematical counterpart is Theorem 11 A.10.

11 A.12. Definition. Let $\mathscr{A} = \langle A, \leqq \rangle$ be a well-ordered class. An element $x \in A$ is called a *limit element* of $\mathscr{A}$ if there is no largest element in A_x, an *isolated element* of $\mathscr{A}$ if A_x contains a largest element. If $x \in A$, $y \in A$, $x < y$ and there is no z such that $x < z < y$, we shall occasionally say that y is the *successor* of x in $\mathscr{A}$.

11 A.13. Theorem. (Transfinite Recursion Theorem, restricted form). *Let* $\mathscr{A} = \langle A, \leqq \rangle$ *be a well-ordered class every segment of which is comprisable. Let* M *be a class. Let* p *be a single-valued relation such that* $\mathbf{E}p \subset M$ *and for any* $x \in A$, $M^{A_x} \subset \mathbf{D}p$.

Then there exists exactly one single-valued relation f *on* A *into* M *such that* $fx = pf_x$ *for every* $x \in A$, *where* f_x *is the restriction of* f *to* A_x.

Proof. For any single-valued relation h with $\mathbf{D}h \subset A$ and any $x \in A$ denote by h_x the restriction of h to A_x. For any $x \in A$ let Φ_x denote the class of all $g \in M^{A_x}$ such that $gy = pg_y$ for every $y < x$. Clearly, if $g \in \Phi_x$, $y < x$, then $g_y \in \Phi_y$.

Suppose that there is an $x \in A$ such that Φ_x contains more than one element. Choose the least such x. Let $h \in \Phi_x$, $h' \in \Phi_x$, $h \neq h'$. For every $y < x$, $h_y \in \Phi_y$, $h'_y \in \Phi_y$, hence $h_y = h'_y$. If x is a limit element, then $h = \bigcup_{y<x} h_y$, $h' = \bigcup_{y<x} h'_y$ and we get a contradiction. If not, let z be the largest element in A_x; then $h'_z = h_z$, $hz = ph_z = ph'_z = h'z$, hence $h = h'$ which is a contradiction. We have proved that every Φ_x is a singleton or void. For any $x \in A$ such that $\Phi_x \neq \emptyset$, we shall put $\Phi_x = (\varphi^{(x)})$.

Suppose there is an $x \in A$ with $\Phi_x = \emptyset$. Choose the least such element x. If x is a limit element, then clearly $\bigcup_{y<x} \varphi^{(y)} \in \Phi_x$; if there is a largest element z in A_x, then

$\varphi^{(z)} \cup (\langle z, p\varphi^{(z)} \rangle) \in \Phi_x$. In both cases, there is a contradiction; this proves that every Φ_x is a singleton.

Now we put $f = \varphi^{(z)} \cup (\langle z, p\varphi^{(z)} \rangle)$ or $f = \bigcup_{x \in A} \varphi^{(x)}$ according as there is or not a largest element z in A. The rest of the proof may be left to the reader.

Remarks. The above theorem (also called the principle of definition by transfinite induction) includes, as a special case for $A = \mathbf{N}$, the usual recursion theorem (principle of recursive construction).

11 A.14. Theorem. (Transfinite Recursion Theorem, extended form). *Let $\mathscr{A} = \langle A, \leqq \rangle$ be a well-ordered class every segment of which is comprisable. Let M be a class. Let ϱ be a relation such that $\mathbf{E}\varrho \subset M$ and, for any $x \in A$, $M^{A_x} \subset \mathbf{D}\varrho$.*

Then there exists a single-valued relation f on A into M such that $f_x \varrho (fx)$ for every $x \in A$, where f_x is the restriction of f to A_x.

Proof. By 4 B.2, there exists a single-valued relation $p \subset \varrho$ such that $\mathbf{D}p = \mathbf{D}\varrho$. Now apply 11 A.13.

Remark. This theorem differs quite essentially from 11 A.13: it is not supposed that ϱ is single-valued, and the resulting relation f is not uniquely determined; the use of the Axiom of Choice is quite essential.

11 A.15. Theorem. *Let $\mathscr{A} = \langle A, \leqq \rangle$ be a well-ordered class such that every A_x is comprisable. Let $\mathscr{M}$ be a non-empty class of single-valued relations such that (1) if $g \in \mathscr{M}$, then $\mathbf{D}g$ is equal to A or to some A_x, $x \in A$; (2) if h is a single-valued relation, $\mathbf{D}h = A$ or $\mathbf{D}h = A_x$ for some $x \in A$, and the restriction of h to any A_z, where $z \in \mathbf{D}h$, belongs to $\mathscr{M}$, then $h \in \mathscr{M}$; (3) for any $g \in \mathscr{M}$, $\mathbf{D}g = A_x$ implies the existence of an element ξ such that $g \cup (\langle x, \xi \rangle) \in \mathscr{M}$.*

Then for any $\varphi \in \mathscr{M}$ there exists an $f \in \mathscr{M}$ such that $\mathbf{D}f = A$, $f \supset \varphi$.

Proof. Choose $\varphi \in \mathscr{M}$. Let V denote the universal class. For any relation h with $\mathbf{D}h \subset A$ and any $z \in A$ denote by h_z the restriction of h to A_z.

We are going to define a certain relation $\boldsymbol{r}$. If $h \in V^{A_x}$, $x \in \mathbf{D}\varphi$ and $h \subset \varphi$ does not hold, then $h\boldsymbol{r}y$ for every element y. If $h \in V^{A_x}$, $x \in \mathbf{D}\varphi$, $h \subset \varphi$, then $h\boldsymbol{r}y$ if and only if $y = \varphi x$. If $h \in V^{A_x}$, $x \in A - \mathbf{D}\varphi$, then we put $h\boldsymbol{r}y$ if and only if either $h \in \mathscr{M}$ and y is such that $h \cup (\langle x, y \rangle) \in \mathscr{M}$ or $h \notin \mathscr{M}$ and y is arbitrary.

It is easy to see that the suppositions of 11 A.14 are satisfied (with $M = V$). Therefore there exists a single-valued relation f on A such that $f_x\boldsymbol{r}(fx)$, for every $x \in A$. We are going to prove that $f \in \mathscr{M}$, $f \supset \varphi$.

Suppose that $f \supset \varphi$ does not hold. Choose the smallest $x \in \mathbf{D}\varphi$ such that $fx \neq \varphi x$. Then $f_x\boldsymbol{r}(fx)$; on the other hand, $f_x = \varphi_x$, hence $f_x \subset \varphi$ and therefore, by definition of $\boldsymbol{r}$, $f_x\boldsymbol{r}y$ if and only if $y = \varphi x$. Hence $fx = \varphi x$ which is a contradiction.

Suppose $f \notin \mathscr{M}$. Then, by condition (2) above, there exists an $x \in A$ such that $f_x \notin \mathscr{M}$. Choose the smallest such element x. Clearly, x is isolated (otherwise $\bigcup\{A_z \mid z < x\} = A_x$ which would imply $f_x \in \mathscr{M}$ since all f_z, $z < x$, belong to $\mathscr{M}$). Let u be the largest element in A_x. We have $f_u\boldsymbol{r}(fu)$; on the other hand, since $f_u \in \mathscr{M}$, we have $f_u r y$ if and only if $f_u \cup (\langle u, y\rangle) \in \mathcal{M}$. Thus $f_u \cup (\langle u, fu\rangle) \in \mathcal{M}$, i.e. $f_x \in \mathcal{M}$ which is a contradiction. This completes the proof.

11 A.16. We intend to prove the well-known theorem asserting that every set can be well-ordered. This can be done in various ways. E.g. a rather simple proof is obtained if we consider a non-comprisable well-ordered class and apply 11 A.14.

We give another proof based on the maximality principle. First we shall prove an auxiliary result.

Let A be a class. Let $\mathbf{F}$ *be the class of all one-to-one relations φ such that* (1) $\mathbf{D}\varphi \subset A$, $\mathbf{E}\varphi \subset \exp \mathbf{D}\varphi$, (2) *if $x \in \varphi y$, $y \in \varphi z$, then $x \in \varphi z$,* (3) *if $x \in \mathbf{D}\varphi$, $y \in \mathbf{D}\varphi$, then either $x \in \varphi y$ or $y \in \varphi x$,* (4) *if $B \subset \mathbf{D}\varphi$, $B \neq \emptyset$, then there exists an element $b \in B$ such that $b \in \varphi x$ for every $x \in B$.*

Then (a) $\mathbf{F}$ *is monotonically additive,* (b) *for any $\varphi \in \mathbf{F}$ there exists exactly one well-order σ on a subset of A such that $x \in \varphi y \Leftrightarrow x\sigma y$.*

Proof. Let $\mathcal{M} \subset \mathbf{F}$ be a non-empty monotone collection. Put $\Phi = \bigcup\{\varphi \mid \varphi \in \mathcal{M}\}$. Clearly, $\mathbf{D}\Phi \subset A$; if $X \in \mathbf{E}\Phi$, then $X = \varphi x$ for some $x \in A$ and some $\varphi \in \mathcal{M}$, hence $X \subset \mathbf{D}\varphi \subset \mathbf{D}\Phi$ which proves (1) for Φ. – It is easy to show that (2) and (3) are satisfied for Φ. – Let $B \subset \mathbf{D}\Phi$, $B \neq \emptyset$. Choose $\varphi \in \mathcal{M}$ such that $B \cap \mathbf{D}\varphi \neq \emptyset$; choose $b \in B \cap \mathbf{D}\varphi$ such that $b \in \varphi x$, hence $b \in \Phi x = \varphi x$, for every $x \in B \cap \mathbf{D}\varphi$. Suppose that there is a $z \in B$ such that $b \notin \Phi z$; then $z \notin \mathbf{D}\varphi$. Since $z \in \mathbf{D}\Phi$, $b \in \mathbf{D}\Phi$, we get, by (3), $z \in \Phi b = \varphi b$ and therefore, by (1), $z \in \mathbf{D}\varphi$ which is a contradiction. – We have proved (a); assertion (b) follows immediately from (1)–(4).

11 A.17. Theorem. *On every set there exists a well-order.*

Proof. Let A be a set. Consider the class $\mathbf{F}$ described in the above lemma. By 4 C.3 and 11 A.16, there is a maximal set μ in $\mathbf{F}$. Suppose $\mathbf{D}\mu \neq A$; then, for any $z \in A - \mathbf{D}\mu$, $\mu' = \mu \cup (\langle z, \mathbf{D}\mu \cup (z)\rangle)$ is a relation satisfying conditions (1)–(4). Hence $\mu' \in \mathbf{F}$, $\mu' \supset \mu$, $\mu' \neq \mu$ which is a contradiction. Therefore, $\mathbf{D}\mu = A$ and $\sigma = \{x \to y \mid x \in \mu y\}$ is a well-order on A (see 11 A.16).

11 A.18. Proposition and definition. *If A is a set, then there exists a well-order ϱ on A, which will be called a* minimal well-order on A, *such that, for any well-order σ on A, $\langle A, \varrho\rangle$ is isomorphic either to $\langle A, \sigma\rangle$ or to a segment of $\langle A, \sigma\rangle$.*

Proof. There exists a well-order τ on A. If $\operatorname{card} A_x < \operatorname{card} A$ for every $x \in A$, put $\varrho = \tau$. If not, let x be the least (under τ) element of A such that $\operatorname{card} A_x = \operatorname{card} A$; choose a bijective relation φ for A and A_x, and put $\varrho = \varphi^{-1} \circ \tau \circ \varphi$. It is easy to see that ϱ possesses the properties required.

B. ORDINALS

Now we are ready to proceed to the introduction of ordinal numbers, briefly called ordinals. In essence, ordinals will serve for a characterization of a well-ordered set irrespective of the particular nature of its elements, similarly as the cardinals serve to characterize the size or the "number of elements" of a set. It is possible to introduce ordinals as objects assigned to well-ordered sets in such a manner that two such objects coincide if and only if the corresponding ordered sets are isomorphic. For example, we could, as in the treatment of cardinal numbers, "assign a certain element" to each class of mutually isomorphic well-ordered sets and declare it to be ordinal number of every well-ordered set from this class. It is also possible to characterize ordinal numbers by means of axioms in a manner similar to that adopted for cardinal numbers.

We shall choose a different procedure more similar to that used for the introduction of natural numbers (Section 3). The starting point is the fact that any two non-comprisable well-ordered classes every segment of which is comprisable are isomorphic. It is a corollary of theorems in Section 4 that such a well-ordered class exists, and it is easy to see that, as soon as ordinal numbers are introduced, it can be proved that the class of all ordinals ordered "by magnitude" is non-comprisable whereas every segment of it is comprisable. Thus we can choose an "arbitrary but fixed" well-ordered class with the property in question and declare it to be the class of all ordinals. Technically, this is done by adopting appropriate axioms for ordinal numbers. It will then be proved that to every well-ordered set there is assigned, in a natural way, an ordinal number.

Since finite well-ordered sets are isomorphic if and only if they have the same number of elements, it is natural to identify finite ordinals with natural numbers. This will be done by means of an additional axiom.

We now proceed to perform the steps indicated above.

11 B.1. Theorem. *There exists a non-comprisable well-ordered class every segment of which is a set.*

Proof. It is easy to see that the class $\mathscr{R}$ of theorem 4 D.1 ordered by inclusion has the required property.

11 B.2. Theorem. *For* $i = 1, 2$, *let* A_i *be a non-comprisable well-ordered class every segment of which is comprisable. Then* A_1 *and* A_2 *are isomorphic.*

This follows immediately from theorem 11 A.9.

11 B.3. Axioms for the class of ordinal numbers.

(a) Ord *is a non-comprisable well-ordered class every segment of which is comprisable.*

(b) $\langle \mathsf{N}, \leqq \rangle$ *is a segment of* Ord.

Definition. Every element of the underlying class $|\mathrm{Ord}|$ of Ord will be called an *ordinal number* or an *ordinal*; the class $|\mathrm{Ord}|$ itself will be called the *class of (all) ordinal numbers.* The structure (the order) of Ord will be denoted simply by $\leqq$. If ξ and η are ordinals and $\xi < \eta$, we shall say that ξ is less than η, η is greater than ξ.

11 B.4. Theorem and definition. *If X is a well-ordered set, then there exists exactly one ordinal ξ such that X is isomorphic to the segment* Ord_ξ. *We put* $\xi = \operatorname{ord} X$ *and say that ξ is the* ordinal *of the well-ordered set X. In accordance with* 1 D.1, *the single-valued relation assigning* $\operatorname{ord} X$ *to X, X being a well-ordered set, will be denoted by* ord.

Proof. The existence of ξ follows from 11 A.9, the uniqueness from 11 A.6.

11 B.5. Theorem. *The relation* ord *possesses the following properties*:

(a) ord *is a single-valued relation whose domain is the class of all well-ordered sets*;

(b) *if X and Y are well-ordered sets, then* $\operatorname{ord} X = \operatorname{ord} Y$ *if and only if X and Y are isomorphic*;

(c) *if X is a finite well-ordered set,* $\operatorname{card} X = n$, *then* $\operatorname{ord} X = n$.

Proof. Assertion (a) follows immediately from 11 A.9. If $\operatorname{ord} X = \operatorname{ord} Y = \xi$, then both X and Y are isomorphic to Ord_ξ and therefore X is isomorphic to Y; conversely, if X and Y are isomorphic and $\operatorname{ord} X = \xi$, then Y is isomorphic to Ord_ξ, hence $\operatorname{ord} Y = \xi$. Finally, if $\mathscr{X} = \langle X, \varrho\rangle$ is a finite well-ordered class and $\operatorname{card} X = n$, then $\langle X, \varrho\rangle$ is isomorphic to $\langle \mathrm{N}_n, \leqq\rangle$; by Axiom (b), $\langle \mathrm{N}_n, \leqq\rangle$ coincides with the segment Ord_n and therefore $n = \operatorname{ord} X$.

Remark. Clearly $\mathbf{E} \operatorname{ord} = |\mathrm{Ord}|$, i.e. the range of ord coincides with the class of all ordinals.

11 B.6. Definition. An ordinal ξ will be called *finite* if $\xi = \operatorname{ord} X$ for some finite well-ordered set X (i.e. if Ord_ξ is finite); otherwise it will be called *infinite.* The smallest infinite ordinal (which is clearly equal to $\operatorname{ord} \mathrm{N}$) will be denoted by ω_0. If ξ is an ordinal and no Ord_η, $\eta < \xi$, is equipollent with Ord_ξ, then ξ will be called an *initial ordinal number.*

Remarks. 1) Clearly, an ordinal ξ is finite if and only if $\xi \in \mathrm{N}$. – 2) It is easy to see that for every cardinal x there exists exactly one initial ordinal number ξ such that $\operatorname{card} \mathrm{Ord}_\xi = x$ (it is sufficient to prove that, for some ordinal η, $\operatorname{card} \mathrm{Ord}_\eta = x$; this follows from the non-comprisability of Ord).

11 B.7. *For every ordinal ξ,* $\xi = \operatorname{ord}(\mathrm{Ord}_\xi)$.

11 B.8. *If X and Y are sets and* $\operatorname{ord} X \leqq \operatorname{ord} Y$, *then* $\operatorname{card} X \leqq \operatorname{card} Y$.

The proof of these two assertions is immediate.

The following theorem asserts that the class of all infinite cardinals as well as the class of all cardinals (as the reader may easily verify) ordered "by magnitude" is isomorphic to the class Ord, so that infinite cardinals can be "numbered" by means or ordinals.

11 B.9. Theorem and definition. *There exists exactly one order-isomorphic mapping of* Ord *onto the class of infinite cardinal numbers endowed with the natural order. The value of this mapping at an ordinal α is a cardinal which will be denoted by $\aleph_\alpha$ and called the* aleph of index α.

Proof. Let K be the class of all infinite cardinals; let $\leqq$ denote its natural order (see 9 B.1). According to 9 B.14, $\langle K, \leqq \rangle$ is well-ordered. Clearly, K is non-comprisable and every segment of $\langle K, \leqq \rangle$ is a set. Therefore, by 11 B.2, there exists exactly one order-isomorphism of Ord onto $\langle K, \leqq \rangle$.

Remarks. 1) In view of the above theorem and definition, infinite cardinals are also termed *alephs*. – 2) The notation $\aleph_0$ introduced in 9 B.6 for the cardinality of a countable set is obviously in agreement with the above definition. – 3) It is easy to see (cf. 11 A.18) that, for any ordinal α, there exists exactly one ordinal number β, usually denoted by ω_α, such that $\text{card}(\mathsf{Ord}_\beta) = \aleph_\alpha$, but $\text{card}(\mathsf{Ord}_\xi) < \aleph_\alpha$ for $\xi < \beta$. Every ω_α is an initial number, and every infinite initial ordinal number β is equal to some ω_α. – 4) The notation just indicated is in agreement with that introduced in 11 B.6 for the least infinite ordinal. – 5) Clearly, the class of all cardinals, as well as that of all ordinals, is minimally non-comprisable.

11 B.10. Let $x = \aleph_\alpha$ be an aleph (i.e. an infinite cardinal). Then $x < 2^x$ (see 9 B.2); on the other hand, clearly, $\aleph_{\alpha+1}$ is the least cardinal greater than x. Hence $\aleph_{\alpha+1} \leqq 2^{\aleph_\alpha}$, for any ordinal α. As for the assertion "$\aleph_{\alpha+1} = 2^{\aleph_\alpha}$", called the Generalized Continuum Hypothesis, cf. 9 C.8, remark 3.

11 B.11. Since Ord is a non-comprisable well-ordered class with comprisable segments, propositions 11 A.13, 11 A.14, 11 A.15 hold with $\mathscr{A} = \mathsf{Ord}$; the name "Transfinite Recursion Theorem", etc., is often reserved for this special case.

12. COVERS AND FILTERS

In this section we consider two different though related concepts: covers, already mentioned in Section 1, and filters of sets, which may be conceived either as a special case of filters in the sense of 10 E.4 or as certain $\bigcap$-ideals in the sense of 8 D.4. The role of these concepts will become more clear later in connection with topological questions. In this section, only some basic properties of covers and filters will be examined.

A. COVERS

First, we shall state basic definitions concerning covers and related notions. Some of these have been given in Section 1 (see 1 E.12) and are restated here for convenience.

12 A.1. Definition. Every family of sets and every collection of sets will be called a *cover*. If $\mathscr{X}$ is a cover and $\bigcup\mathscr{X} \supset A$, we shall say that $\mathscr{X}$ *covers* A; if $\bigcup\mathscr{X} = A$, then $\mathscr{X}$ will be called a *cover of* A.

If ϱ is a comprisable relation, then $\mathscr{X} = \{\varrho[(x)] \mid x \in \mathbf{D}\varrho\}$ (occasionally also any $\mathscr{X} = \{\varrho[(x)] \mid x \in A\}$ with $\varrho[A] = \mathbf{E}\varrho$) is called the *family of fibres of* ϱ; $\mathscr{X}$ as well as $\mathbf{E}\mathscr{X}$ is also said to be *associated with* ϱ. A disjoint cover $\mathscr{X} \neq \emptyset$ is called a *decomposition* (of $\bigcup\mathscr{X}$) if $\mathscr{Y} = \mathscr{X}$ whenever $\emptyset \neq \mathscr{Y} \subset \mathscr{X}$, $\bigcup\mathscr{Y} = \bigcup\mathscr{X}$.

Remark. If $\mathscr{A} = \langle A, \alpha \rangle$ is a struct, then a cover of A will also be called a cover of $\mathscr{A}$, and similarly for decompositions, etc.

Examples. (A) For any set A, the collection of all (x), $x \in A$, is a cover of A associated with J_A. – (B) Let $\langle A, \varrho \rangle$ be a quasi-ordered set. Then $\mathscr{X} = \{\rrbracket \leftarrow, x \rrbracket \mid x \in A\}$ is a cover of A. Clearly, if ϱ is reflexive, then $\mathscr{X}$ is the cover associated with the relation ϱ^{-1}. – (C) Let A be a set; let $\mathscr{B}$ be a collection of non-empty subsets of A. For any $x \in A$ put $\mathscr{B}_x = \mathbf{E}\{Y \in \mathscr{B} \mid x \in Y\}$. Then $\{\mathscr{B}_x\}$ is a cover of $\mathscr{B}$. – (D) The preceding examples are rather illustrative whereas the following one plays a certain role in algebra and topology. Let $\mathscr{A} = \langle A, \sigma, \mu \rangle$ be a ring with unit and suppose that A contains more than one element. Consider the set $\mathscr{M}$ of all maximal ideals of $\mathscr{A}$ (see 8 D.19); the set $\mathscr{M}$ is not empty by 8 D.20. For any proper (i.e. distinct from A) ideal T of $\mathscr{A}$ denote by $\mathscr{U}_T$ the set of all those $M \in \mathscr{M}$ which do not contain T. Consider a set $\mathscr{T}$ of proper ideals of $\mathscr{A}$ and the cover $\{\mathscr{U}_T \mid T \in \mathscr{T}\}$. The family

$\{\mathscr{U}_T \mid t \in \mathscr{T}\}$ is a cover of $\mathscr{M}$ if and only if there are $T_k \in \mathscr{T}$, $k = 0, \ldots, n$, and $t_k \in T_k$ such that $t_0 \sigma t_1 \sigma \ldots \sigma t_n$ is equal to 1. Indeed, if $\{\mathscr{U}_T\}$ does not cover $\mathscr{M}$, then $\bigcup \mathscr{T} \subset M$ for some $M \in \mathscr{M}$, and therefore the condition does not hold (since $1 \notin M$). Conversely, let $\{\mathscr{U}_T \mid T \in \mathscr{T}\}$ be a cover of $\mathscr{M}$. It is easy to see that the set T^* of all $t_0\ \sigma \ldots \sigma\ t_n$, where $n \in \mathsf{N}$, $t_k \in \bigcup \mathscr{T}$, is an ideal. Since T^* is contained in no $M \in \mathscr{M}$, we obtain $1 \in T^*$.

12 A.2. Definition. A reflexive relation with domain A will be called a *vicinity of the diagonal of the square of A* or simply a *vicinity on A*; every reflexive relation will be called a *vicinity*. For a motivation of this terminology see **23** A.1. If a family $\mathscr{X} = \{X_a \mid a \in A\}$ is a cover, then $\Sigma\mathscr{X}$ denotes (see 5 B.1) the set of all $\langle a, x \rangle$ such that $x \in X_a$. If $\mathscr{X}$ is a collection of sets (hence a cover), then $\Sigma\mathscr{X}$ will denote the set of all $\langle X, x \rangle$ where $X \in \mathscr{X}$, $x \in X$. In both cases, $\Sigma\mathscr{X}$ will be termed the *sum of* $\mathscr{X}$.

If $\mathscr{X}$ is a cover, then the set $(\Sigma\mathscr{X}) \circ (\Sigma\mathscr{X})^{-1}$ (which is clearly a vicinity) will be called the *vicinity associated with* $\mathscr{X}$ and occasionally denoted by $\mathbf{V}\mathscr{X}$. Observe that a vicinity associated with a cover is symmetric.

Example. Consider the examples from 12 A.1. The associated vicinities are as follows: in (A), the diagonal J_A of A; in (B), the set of all $\langle x, y \rangle$ such that (x, y) is right-bounded; in (C), the set of all $\langle X, Y \rangle$ such that $X \in \mathscr{B}$, $Y \in \mathscr{B}$, $X \cap Y \neq \emptyset$.

12 A.3. A cover, as defined in 12 A.1, may be either a family of sets or a collection of sets. This approach is in line with the current practice of interchanging both kinds of covers. However, it makes necessary twofold definitions of many properties (as in 12 A.2), twofold proofs of various propositions, etc. We shall sometimes give statements referring explicitly to only one of the two cases, leaving to the reader the task of extending them to the other case. For some purposes, covers of both kinds will be considered simultaneously (see e.g. 12 A.4).

We shall occasionally refer to a cover which is a family as a *cover (family)* using the expression "*cover (collection)*" in the other case.

We are now going to introduce a certain natural quasi-order on the class of all covers. It will turn out that the relations assigning $\Sigma\mathscr{X}$ and $\mathbf{V}\mathscr{X}$ to a cover $\mathscr{X}$ are order-preserving. Then some operations on covers will be introduced.

12 A.4. Definition. Let $\mathscr{X}$, $\mathscr{Y}$ be covers. We shall say that $\mathscr{X}$ *refines* $\mathscr{Y}$ if the following holds: if a set X is a member or an element of $\mathscr{X}$ (according as $\mathscr{X}$ is a family or a collection of sets), then there exists a set Y such that $X \subset Y$ and Y is a member or an element of $\mathscr{Y}$.

Example. If A is a set, then J_A refines any cover of A, and (A) is refined by every cover of A.

12 A.5. *The relation* $\{\mathscr{X}$ *refines* $\mathscr{Y}\}$ *is a reflexive quasi-order on the class of all covers. Under this quasi-order,* $\{X_a \cap Y_b\}$ *is a meet of* $\{X_a\}$ *and* $\{Y_b\}$, $\{X_a \mid a \in A\} \cup \cup \{Y_b \mid b \in B\}$ *is a join of* $\{X_a \mid a \in A\}$, $\{Y_b \mid b \in B\}$ *provided A, B are disjoint.*

Convention. Every class of covers will be conceived, unless the contrary is stated, as endowed with the restriction of the quasi-order indicated above.

12 A.6. *The mapping of the class of all covers into the class of all vicinities (respectively, all relations) which assigns* $\mathbf{V}\mathscr{X}$ *(respectively* $\Sigma\mathscr{X}$*) to* $\mathscr{X}$*, is order-preserving and completely lattice-preserving.*

Proof. We consider only the case of $\mathbf{V}\mathscr{X}$, of covers which are families and of preservation of meets; the rest is left to the reader. Let $\{\mathscr{X}^{(b)} \mid b \in B\}$ be a family of covers, $\mathscr{X}^{(b)} = \{X_a^{(b)} \mid a \in A_b\}$. Let $\mathscr{X}$ be a meet of $\{\mathscr{X}^{(b)}\}$. Then clearly, $\mathbf{V}\mathscr{X} \subset \bigcap\{\mathbf{V}\mathscr{X}^{(b)}\}$. If $\langle z, z'\rangle$ belongs to every $\mathbf{V}\mathscr{X}^{(b)}$, then it is clear that the collection consisting of exactly one set (z, z') refines every $\mathscr{X}^{(b)}$. Therefore it also refines $\mathscr{X}$, from which $\langle z, z'\rangle \in \mathbf{V}\mathscr{X}$ follows. We have proved that $\mathbf{V}\mathscr{X} = \bigcap\{\mathbf{V}\mathscr{X}^{(b)}\}$.

12 A.7. Definition. Let $\mathscr{X}$ be a cover; let M be a set. The *$\mathscr{X}$-star* of M (or, the *star of M with respect to $\mathscr{X}$*) is, by definition, the set $(\mathbf{V}\mathscr{X})[M]$, that is (i) if $\mathscr{X}$ is a family $\{X_a\}$, the set of all y such that, for some a, $y \in X_a$ and X_a intersects M, (ii) if $\mathscr{X}$ is a collection of sets, the set of all y such that, for some $X \in \mathscr{X}$, we have $y \in X$, $X \cap M \neq \emptyset$. The $\mathscr{X}$-star of M will be denoted by $\operatorname{st}(\mathscr{X}, M)$ or $\operatorname{st}_{\mathscr{X}} M$. If $\mathscr{X}$ is a cover, then every $\operatorname{st}_{\mathscr{X}}(x)$ with $x \in \bigcup\mathscr{X}$ will be called a *point-star*; the family $\{\operatorname{st}_{\mathscr{X}}(x) \mid x \in \bigcup\mathscr{X}\}$ will be denoted by $\operatorname{St}\mathscr{X}$.

If $\mathscr{X}$, $\mathscr{Y}$ are covers, consider the cover associated with $(\mathbf{V}\mathscr{X}) \circ (\Sigma\mathscr{Y})$, that is (i) if $\mathscr{Y}$ is a family $\{Y_b \mid b \in B\}$, the cover $\{\operatorname{st}(\mathscr{X}, Y_b) \mid b \in B\}$, (ii) if $\mathscr{Y}$ is a collection of sets, the cover $\{\operatorname{st}(\mathscr{X}, Y) \mid Y \in \mathscr{Y}\}$. This cover will be denoted by $\operatorname{St}(\mathscr{X}, \mathscr{Y})$. In particular, every member of $\operatorname{St}(\mathscr{X}, \mathscr{X})$ will be called a *star of $\mathscr{X}$*.

Example. Let A be a non-empty set. Let $\mathscr{B}$ be the set of all finite non-empty $X \subset A$. For every $x \in A$, put $\mathscr{B}_x = \mathbf{E}\{Y \in \mathscr{B} \mid x \in Y\}$; consider the cover $\mathbf{B} = \{\mathscr{B}_x \mid x \in A\}$ of $\mathscr{B}$. Then $\operatorname{St}\mathbf{B}$ is the cover $\{\mathbf{E}\{Y \in \mathscr{B} \mid Y \cap X \neq \emptyset\} \mid X \in \mathscr{B}\}$ and $\operatorname{St}(\mathbf{B}, \mathbf{B})$ is the constant family $\{x \to \mathscr{B} \mid x \in A\}$.

12 A.8. *For any cover* $\mathscr{X}$, $\operatorname{St}\mathscr{X} = \operatorname{St}(\mathscr{X}, \mathsf{J}_X)$, where $X = \bigcup\mathscr{X}$. *If* $\mathscr{X}, \mathscr{Y}$ *are covers, then the sum of* $\operatorname{St}(\mathscr{X}, \mathscr{Y})$ *is equal to* $(\mathbf{V}\mathscr{X}) \circ (\Sigma\mathscr{Y})$*, and the vicinity associated with* $\operatorname{St}(\mathscr{X}, \mathscr{Y})$ *is equal to* $(\mathbf{V}\mathscr{X}) \circ (\mathbf{V}\mathscr{Y}) \circ (\mathbf{V}\mathscr{X})$*. In particular, the sum of* $\operatorname{St}\mathscr{X}$ *is equal to* $\mathbf{V}\mathscr{X}$*, the associated vicinity is equal to* $(\mathbf{V}\mathscr{X}) \circ (\mathbf{V}\mathscr{X})$*; the sum of* $\operatorname{St}(\mathscr{X}, \mathscr{X})$ *is equal to* $(\mathbf{V}\mathscr{X}) \circ (\Sigma\mathscr{X})$*, the associated vicinity is equal to* $(\mathbf{V}\mathscr{X}) \circ (\mathbf{V}\mathscr{X}) \circ (\mathbf{V}\mathscr{X})$.

Proof. We only prove the assertions on $\operatorname{St}(\mathscr{X}, \mathscr{Y})$ for the case when $\mathscr{X} = \{X_a\}$, $\mathscr{Y} = \{Y_b \mid b \in B\}$ are families, leaving the rest to the reader. – The set $\Sigma \operatorname{St}(\mathscr{X}, \mathscr{Y})$ consists, by definition, of all $\langle b, x\rangle$ such that $b \in B$, $x \in \operatorname{st}(\mathscr{X}, Y_b)$, i.e. $x \in X_a$ for some a and some z such that $z \in X_a$, $z \in Y_b$. This implies at once that $\Sigma \operatorname{St}(\mathscr{X}, \mathscr{Y}) = (\mathbf{V}\mathscr{X}) \circ (\Sigma\mathscr{Y})$. The set $\mathbf{V}(\operatorname{St}(\mathscr{X}, \mathscr{Y}))$ is equal (see 12 A.2) to $(\Sigma \operatorname{St}(\mathscr{X}, \mathscr{Y})) \circ (\Sigma \operatorname{St}(\mathscr{X}, \mathscr{Y}))^{-1}$, i.e. to $(\mathbf{V}\mathscr{X}) \circ (\Sigma\mathscr{Y}) \circ (\Sigma\mathscr{Y})^{-1} \circ (\mathbf{V}\mathscr{X})^{-1} = (\mathbf{V}\mathscr{X}) \circ (\mathbf{V}\mathscr{Y}) \circ (\mathbf{V}\mathscr{X})$.

12 A.9. *If* $\mathscr{X}, \mathscr{Y}$ *are covers,* $\mathscr{X}$ *refines* $\mathscr{Y}$*,* R*,* S *are sets, and* $R \subset S$*, then* $\operatorname{st}(\mathscr{X}, R) \subset \operatorname{st}(\mathscr{Y}, S)$*. In particular, if* $\mathscr{X}$ *refines* $\mathscr{Y}$*, then* $\operatorname{St}\mathscr{X}$ *refines* $\operatorname{St}\mathscr{Y}$, $\operatorname{St}(\mathscr{X}, \mathscr{X})$ *refines* $\operatorname{St}(\mathscr{Y}, \mathscr{Y})$*; if* $\mathscr{X}$ *refines* $\mathscr{Y}$ *and* $\mathscr{X}'$ *refines* $\mathscr{Y}'$*, then* $\operatorname{St}(\mathscr{X}, \mathscr{X}')$ *refines* $\operatorname{St}(\mathscr{Y}, \mathscr{Y}')$.

We omit the proof. – Observe that the mapping $\{\mathscr{X} \to \operatorname{St}\mathscr{X}\}$, as well as $\{\mathscr{X} \to \operatorname{St}(\mathscr{X}, \mathscr{X})\}$, does not preserve meets (not even finite); see 12 ex. 4.

12 A.10. Convention. Let $\mathscr{X}$ be a cover; let Z be a set. Then we denote by $Z \cap \mathscr{X}$ (provided there is no danger of misunderstanding) the following cover: (i) if $\mathscr{X} = \{X_a\}$, i.e. if $\mathscr{X}$ is a family, the family $\{Z \cap X_a\}$, (ii) if $\mathscr{X}$ is a collection of sets, the cover (collection) consisting of all $Z \cap X$, $X \in \mathscr{X}$.

We do not consider relationships between $\mathscr{X}$ and $Z \cap \mathscr{X}$ in any detail, deferring such considerations to those sections where properties of coverings will be really needed. Only some statements are given here (without proof):

If $\mathscr{X}$ refines $\mathscr{Y}$, then $A \cap \mathscr{X}$ refines $A \cap \mathscr{Y}$. The mapping $\{\mathscr{X} \to A \cap \mathscr{X}\}$ of the class of all covers into itself is order-preserving and completely lattice-preserving. The sum of $A \cap \mathscr{X}$ is equal to $\mathsf{J}_A \circ (\Sigma\mathscr{X})$, the associated vicinity $\mathbf{V}(A \cap \mathscr{X})$ is equal to $\mathsf{J}_A \circ (\mathbf{V}\mathscr{X}) \circ \mathsf{J}_A$.

For any cover $\mathscr{X}$, $\operatorname{St}(A \cap \mathscr{X}) \subset A \cap \operatorname{St}\mathscr{X}$; namely, $\operatorname{St}(A \cap \mathscr{X})$ is the domain-restriction of $A \cap \operatorname{St}\mathscr{X}$ to A, $\operatorname{St}(A \cap \mathscr{X}) = \{A \cap \operatorname{st}(\mathscr{X}, x) \mid x \in A\}$, $A \cap \operatorname{St}\mathscr{X} = \{A \cap \operatorname{st}(\mathscr{X}, x) \mid x \in \bigcup\mathscr{X}\}$.

12 A.11. Definition. Let $\mathscr{X}$, $\mathscr{Y}$ be covers; suppose, for convenience, that either (i) both $\mathscr{X}$ and $\mathscr{Y}$ are families, $\mathscr{X} = \{X_a \mid a \in A\}$, $\mathscr{Y} = \{Y_b \mid b \in B\}$, or (ii) both $\mathscr{X}$ and $\mathscr{Y}$ are collections of sets. Then we call the *cover-product* of $\mathscr{X}$ and $\mathscr{Y}$ and denote by $\mathscr{X} \times \mathscr{Y}$ (provided it is clear from the context that the cover-product is meant, and not the current product of sets or relations) the family $\{X_a \times Y_b \mid \langle a, b\rangle \in A \times B\}$ (case (i)), or the collection of all $X \times Y$, $X \in \mathscr{X}$, $Y \in \mathscr{Y}$ (case (ii)).

We do not examine the product of covers here, mentioning only some almost evident facts.

Let $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{X}'$, $\mathscr{Y}'$ be covers, and let $\mathscr{X} \times \mathscr{Y}$, $\mathscr{X}' \times \mathscr{Y}'$ exist. If $\mathscr{X}$ refines $\mathscr{X}'$, $\mathscr{Y}$ refines $\mathscr{Y}'$, then $\mathscr{X} \times \mathscr{Y}$ refines $\mathscr{X}' \times \mathscr{Y}'$. The cover $\operatorname{St}(\mathscr{X} \times \mathscr{Y})$ is equal to $\operatorname{St}\mathscr{X} \times \operatorname{St}\mathscr{Y}$ (relational product); $\Sigma(\mathscr{X} \times \mathscr{Y})$ is equal to $(\Sigma\mathscr{X}) \times (\Sigma\mathscr{Y})$ (relational product); $\mathbf{V}(\mathscr{X} \times \mathscr{Y})$ is equal to $(\mathbf{V}\mathscr{X}) \times (\mathbf{V}\mathscr{Y})$ (relational product).

12 A.12. *Let ϱ be a relation or a correspondence. Let $\mathscr{X}$ be a cover (namely, either* (i) *a family $\{X_a \mid a \in A\}$ of sets, or* (ii) *a collection of sets). Then $\{\varrho[X_a] \mid a \in A\}$ (in case* (i)*) or $\mathbf{E}\{\varrho[X] \mid X \in \mathscr{X}\}$ (case* (ii)*) is a cover which will be termed the* ϱ-image *of $\mathscr{X}$.*

The following assertions are clear.

Let ϱ be a relation or a correspondence. Then the mapping which assigns to every cover $\mathscr{X}$ its ϱ-image $\mathscr{Y}$ is order-preserving; the ϱ-image of $\operatorname{St}\mathscr{X}$ *refines* $\operatorname{St}\mathscr{Y}$.

Observe that the mapping which transforms $\mathscr{X}$ into the ϱ-image of $\mathscr{X}$ in general preserves neither joins nor meets (not even finite).

It is clear that if f is a mapping, then we obtain as special cases of ϱ-images, the f-image $\{f[X_a]\}$ of a cover $\{X_a\}$ and the f^{-1}-image $\{f^{-1}[Y_b]\}$ of a cover $\{Y_b\}$.

12 A.13. Definition. Let $\mathscr{X}$, $\mathscr{Y}$ be covers. If $\operatorname{St}\mathscr{X}$ refines $\mathscr{Y}$, then we shall say that $\mathscr{X}$ *star-refines* $\mathscr{Y}$. Clearly, the relation $\{\mathscr{X} \to \mathscr{Y} \mid \mathscr{X}$ star-refines $\mathscr{Y}\}$ is a quasi-order on the class of all covers.

Example. Let A be a non-empty set. Consider the set $\mathscr{B}$ of all finite non-empty $X \subset A$. For any $Y \in \mathscr{B}$ put $\mathscr{M}_X = \mathsf{E}\{Y \mid Y \in \mathscr{B}, \operatorname{card}(X \div Y) \leqq 1\}$, $\mathscr{N}_X = \mathsf{E}\{Y \mid Y \in \mathscr{B}, \operatorname{card}(X \div Y) \leqq 2\}$. It is easy to see that $\{\mathscr{M}_X\}$ star-refines $\{\mathscr{N}_X\}$ and even $\operatorname{St}\{\mathscr{M}_X\} = \mathscr{N}_X$. If $\mathscr{B}' \subset \mathscr{B}$ is arbitrary, then $\{\mathscr{B}' \cap \mathscr{M}_X\}$ star-refines $\{\mathscr{B}' \cap \mathscr{N}_X\}$, but the equality $\operatorname{St}\{\mathscr{B}' \cap \mathscr{M}_X\} = \{\mathscr{B}' \cap \mathscr{N}_X\}$ does not necessarily hold.

Remark. The quasi-order indicated above is neither distinguishing nor reflexive (although there exist covers $\mathscr{X}$ such that $\mathscr{X}$ star-refines $\mathscr{X}$).

12 A.14. *Let $\mathscr{X}$, $\mathscr{Y}$ be covers. Then $\mathscr{X}$ star-refines* $\operatorname{St}\mathscr{X}$. If *$\mathscr{X}$ star-refines $\mathscr{Y}$, then* $\operatorname{St}\mathscr{X}$ *star-refines* $\operatorname{St}\mathscr{Y}$, $\operatorname{St}(\mathscr{X},\mathscr{X})$ *star-refines* $\operatorname{St}(\mathscr{Y},\mathscr{Y})$.

12 A.15. *Let $\mathscr{X}$, $\mathscr{X}_1$, $\mathscr{Y}$, $\mathscr{Y}_1$ be covers. If $\mathscr{X}$ star-refines $\mathscr{Y}$, then every $\mathscr{X}_1$ refining $\mathscr{X}$ star-refines $\mathscr{Y}$, and $\mathscr{X}$ star-refines every $\mathscr{Y}_1$ refined by $\mathscr{Y}$. If $\mathscr{X}$ star-refines $\mathscr{Y}$ and $\mathscr{Y}_1$, then $\mathscr{X}$ star-refines every meet of $\mathscr{Y}$ and $\mathscr{Y}_1$. If $\mathscr{X}$ star-refines $\mathscr{Y}$ and $\mathscr{X}_1$ star-refines $\mathscr{Y}_1$, then $\mathscr{X} \times \mathscr{X}_1$ star-refines $\mathscr{Y} \times \mathscr{Y}_1$. If A is a set and $\mathscr{X}$ star-refines $\mathscr{Y}$, then $A \cap \mathscr{X}$ star-refines $A \cap \mathscr{Y}$.*

The proof of both propositions above is left to the reader.

Remark. Star-refinements are quite important for various questions of general topology. However, it is convenient to defer their consideration, apart from some almost trivial facts as above, to Section 24.

We conclude this subsection with definitions and some simple results concerning various kinds of "relative finiteness" of covers. An examination of these concepts is deferred to Section 30.

12 A.16. Definition. Let $\mathscr{X}$, $\mathscr{Y}$ be covers. If every fibre of $(\Sigma\mathscr{X})^{-1} \circ (\Sigma\mathscr{Y})$ is finite (if $\mathscr{X} = \{X_a\}$, $\mathscr{Y} = \{Y_b\}$, then this means that for any b there are only finitely many elements a such that X_a intersects Y_b) then we shall say that $\mathscr{X}$ is *finite relative to* $\mathscr{Y}$. If every fibre of $(\Sigma\mathscr{X})^{-1}$ is finite, then we shall say that $\mathscr{X}$ is *point-finite*. Finally, if $\mathscr{X}$ is finite relative to $\mathscr{X}$ (if $\mathscr{X} = \{X_a \mid a \in A\}$, then this means that, for any $a' \in A$, there is only a finite number of elements $a \in A$ such that X_a intersects $X_{a'}$), then we shall say that $\mathscr{X}$ is *star-finite*.

Remarks. 1) Clearly, a star-finite cover is point-finite, but not conversely. — 2) A cover $\mathscr{X}$ is point-finite if and only if it is finite relative to the cover $\{(x) \mid x \in \bigcup\mathscr{X}\}$. — 3) If $\mathscr{X}$ refines $\mathscr{Y}$, then it may happen that $\mathscr{X}$ is star-finite, etc., without $\mathscr{Y}$ being so, or conversely.

12 A.17. *Let $\mathscr{X} = \{X_a \mid a \in A\}$, $\mathscr{Y} = \{Y_b \mid b \in B\}$ be covers. If $\mathscr{X}, \mathscr{Y}$ are finite relative to a cover $\mathscr{Z}$, then $\{X_a \cap Y_b \mid a \in A, b \in A\}$ is finite relative to $\mathscr{Z}$; in particular, if $\mathscr{X}, \mathscr{Y}$ are point-finite, then so is $\{X_a \cap Y_b\}$. If $\mathscr{X}$, $\mathscr{Y}$ are star-finite, then $\{X_a \cap Y_b\}$ is also star-finite.*

12 A.18. *Let $\mathscr{X} = \{X_a \mid a \in A\}$ be a cover. Let $\{A_b \mid b \in B\}$ be a point-finite cover of A. Put $Y_b = \bigcup\{X_a \mid a \in A_b\}$, $\mathscr{Y} = \{Y_b \mid b \in B\}$. If $\mathscr{X}$ is finite relative to a cover $\mathscr{Z}$,* *then $\mathscr{Y}$ is also finite relative to $\mathscr{Z}$; in particular, if $\mathscr{X}$ is point-finite, then $\mathscr{Y}$ is point-finite.*

Proof. Let $\mathscr{Z} = \{Z_c\}$; for any c, let $A(c)$ denote the set of those elements a for which X_a intersects Z_c. Clearly, if $Y_b \cap Z_c \neq \emptyset$, then $A_b \cap A(c) \neq \emptyset$. Since $A(c)$ is finite and $\{A_b\}$ is point-finite, there is only a finite number of elements b such that $A_b \cap A(c) \neq \emptyset$. This proves that $\mathscr{Y}$ is finite relative to $\mathscr{Z}$.

B. FILTERS

Filters (see 10 E.4) of various kinds are of considerable importance in many topological and related questions. For instance, filters of covers (of a given set) play an important role (see Section 24). However, it seems that filters of non-empty subsets of a given set may be considered as the most important kind of filters.

12 B.1. *Let A be a set; let $\mathscr{F}$ be a non-empty collection of subsets of A. Then the following conditions are equivalent: (1) $\mathscr{F}$ is such that (a) $X \in \mathscr{F}$, $Y \in \mathscr{F} \Rightarrow X \cap Y \in \mathscr{F}$, (b) $X \in \mathscr{F}$, $X \subset Y \subset A \Rightarrow Y \in \mathscr{F}$. (2) $\mathscr{F}$ is a left filter (see* 10 E.4) *of the ordered set $\langle \exp A, \subset \rangle$; (3) $\mathscr{F}$ is an ideal (see* 8 D.4, 8 D.6) *of $\langle \exp A, \cap \rangle$. If these equivalent conditions are satisfied, then the following properties of $\mathscr{F}$ are equivalent: $\mathscr{F}$ does not contain the void set; $\mathscr{F}$ is a proper filter in $\langle \exp A, \subset \rangle$; $\mathscr{F}$ is a proper ideal in $\langle \exp A, \cap \rangle$.*

Proof. We are going to show that (1) is equivalent to both (2) and (3). Clearly, (1a) implies that $\mathscr{F}$ is left-directed, (1b) implies that $\mathscr{F}$ is right-saturated in $\langle \exp A, \subset \rangle$; conversely, if (2) is satisfied, then, clearly, (1) holds. The equivalence of (1) and (3) follows from the characterization of ideals of $\langle \exp A, \cap \rangle$ given in 8 D.6.

12 B.2. Definition. Let A be a set. A non-empty collection $\mathscr{F}$ of subsets of A is called a *filter* on A if (i) $X \in \mathscr{F}$, $Y \in \mathscr{F} \Rightarrow X \cap Y \in \mathscr{F}$, (ii) $X \in \mathscr{F}$, $X \subset Y \subset A \Rightarrow Y \in \mathscr{F}$. If, in addition, $\mathscr{F} \neq \exp A$, i.e. if $\emptyset \notin \mathscr{F}$, then $\mathscr{F}$ is called a *proper filter* on A.

Remark. This terminology deviates somewhat from the current one since we call "proper filters" what is usually called filters and, contrary to the usage, consider $\exp A$ as a filter. The terminology chosen seems to be in a better agreement with the terminology concerning ideals. In addition, it seems desirable that e.g. the set of all neighborhoods of the diagonal in a uniform space should always be a filter, the void space being no exception.

Examples. (A) If $\emptyset \neq X \subset A$, then the collection of all Y satisfying $X \subset Y \subset A$ is a proper filter on A. – (B) If A is an infinite set, card $A = a$, then the collection of all $X \subset A$ such that card $(A - X) < a$ is a proper filter. – (C) Let A be an infinite set; let $\{\xi_a \mid a \in A\}$ be a family of real numbers, $\xi_a \geqq 0$. For any $Y \subset A$

let $\sigma(Y)$ denote the set of all real numbers $\sum_{a\in K} \{\xi_a\}$ where $K \subset Y$, K is finite. Let $\mathscr{F}$ denote the collection of all $X \subset A$ such that $\sigma(A - X)$ is bounded (in R). Then either $\sigma(A)$ is bounded and $\mathscr{F} = \exp A$, or $\sigma(A)$ is not bounded and $\mathscr{F}$ is a proper filter on A.

12 B.3. *Let A be a set. The intersection of any non-empty family, as well as the union of every monotone family of filters (or proper filters) on A is a filter (proper filter, respectively) on A.*

12 B.4. *Let A be a non-empty set. Let the set $\mathbf{F}(A)$ of all proper filters on A be ordered by inclusion. Then the set (A) is the smallest element in $\mathbf{F}(A)$, and $\mathbf{F}(A)$ has no largest element unless A is a singleton. The ordered set $\mathbf{F}(A)$ is boundedly complete.*

Proof. Clearly, $A \in \mathscr{F}$ for any filter $\mathscr{F}$ on A, and (A) is a proper filter. If $a \in A$, $b \in A$, $a \neq b$, then $\mathbf{E}\{X \mid (a) \subset X \subset A\}$, $\mathbf{E}\{X \mid (b) \subset X \subset A\}$ are filters and there is no proper filter containing them both. By 12 B.3, every non-empty subset of $\mathbf{F}(A)$ has a meet; hence, by 10 G.3, $\mathbf{F}(A)$ is boundedly complete.

12 B.5. *Let $\mathscr{F}$ be a filter on a set A. If $B \subset A$, then the set, denoted by $B \cap [\mathscr{F}]$, which consists of all $B \cap X$, $X \in \mathscr{F}$, is a filter on B.*

We are now going to consider bases and sub-bases of filters of sets. It is convenient to consider a more general case, namely that of filters in a quasi-ordered class.

12 B.6. *Let $\langle B, \varrho \rangle$ be a quasi-ordered class; let $X \subset B$. Then X is left-directed (respectively, left-directed but not left-cofinal) if and only if the class of all y contained in some $⟦ x, \rightarrow ⟦$, $x \in X$, is a left filter (respectively, a proper left filter).*

Proof. Let $Y = \mathbf{E}\{y \mid y \in ⟦ x, \rightarrow ⟦ \text{ for some } x \in X\}$. Clearly, Y is right-saturated. Let X be left-directed; if y_1, y_2 belong to Y, choose x_1, x_2 with $y_i = x_i$ or $x_i \varrho y_i$, and choose $x \in X$ with $x_i \in ⟦ x, \rightarrow ⟦$. Hence $y_i \in ⟦ x, \rightarrow ⟦$. Conversely, if Y is a filter, let x_1, x_2 belong to X, choose $y \in Y$ with $y_i \in ⟦ y, \rightarrow ⟦$ and choose $x \in X$ with $y \in ⟦ x, \rightarrow ⟦$.

It is usual and convenient to give a special name to those left-directed collections of sets which "generate", in the way indicated above, a filter of sets on some set A.

12 B.7. Definition. Let $\mathscr{X}$ be a collection of sets. If A is a set, $\mathscr{X} \subset \exp A$, $\mathscr{F}$ is a filter on A, and $\mathscr{F}$ consists of those $Y \subset A$ for which there exists an $X \in \mathscr{X}$ with $X \subset Y$, then we shall say that $\mathscr{X}$ is a *base of the filter* $\mathscr{F}$ (on A). We shall say that $\mathscr{X}$ is a *filter base* (a *proper filter base*) *on* A, if there exists a filter (a proper filter) $\mathscr{F}$ on A of which $\mathscr{X}$ is a base (if there is such a filter $\mathscr{F}$, then it is, of course, uniquely determined). Finally, we shall say that $\mathscr{X}$ is a *filter base*, or, more explicitly, a *base of a filter of sets* if, for some set A, $\mathscr{X}$ is a base of a filter on A, and similarly for a *proper filter base*.

It follows at once from 12 B.6 that the following holds:

A collection of sets $\mathcal{X}$ is a filter base if and only if $(*)$ *for any $X \in \mathcal{X}$, $Y \in \mathcal{X}$, there exists a $Z \in \mathcal{X}$ with $Z \subset X \cap Y$; it is a proper filter base if and only if, in addition, $\emptyset$ does not belong to $\mathcal{X}$.*

If the above condition $(*)$ *is satisfied and A is a set, $A \supset \bigcup \mathcal{X}$, then there exists exactly one filter on A of which $\mathcal{X}$ is a base, namely the collection of all $Y \subset A$ such that $Y \supset X$ for some $X \in \mathcal{X}$.*

Examples. Consider the examples given in 12 B.2. In (A), the singleton (X) is a base of the filter in question. In (B), suppose that A can be well-ordered in such a way that (i) card $A_x < a$ for each segment A_x, (ii) if B is cofinal in A, then card $B = a$; then the collection of all $(A - A_x)$ is a base of the filter in question.

12 B.8. *Let $\mathcal{B} = \langle B, \varrho \rangle$ be a quasi-ordered class. Suppose that every non-empty finite left-bounded $K \subset B$ has a meet. Let $X \subset B$. Then the following conditions are equivalent:* (1) *every finite non-empty $K \subset X$ is left-bounded in $\mathcal{B}$,* (2) *there exists a left filter in $\mathcal{B}$ containing X;* (3) *there exists a smallest left filter in $\mathcal{B}$ containing X.*

Proof. If (1) holds, denote by Y the class of all y such that, for some finite non-empty $K \subset X$, the set $]\!] \leftarrow, y]\!]$ contains a meet of K. Clearly, Y is right-saturated, $X \subset Y$. Let $y_i \in Y$, $i = 1, 2$; let $K_i \subset X$ be finite, $K_i \neq \emptyset$ and let $]\!] \leftarrow, y_i]\!]$ contain a meet x_i of K_i. By the suppositions made, $K_1 \cup K_2$ is left-bounded, hence has a meet, say y. Clearly, $y \in Y$, y is a left bound of (y_1, y_2). This proves that Y is a left filter. Now, if Z is a left filter in $\mathcal{B}$, $X \subset Z$, then, for any finite non-empty $K \subset X$, there exists a $z \in Z$ which left-bounds K; since z left-bounds Inf K, we get Inf $K \subset Z$. From this, $Y \subset Z$ follows easily. Hence, Y is the smallest left filter containing X. Thus, we have proved $(1) \Rightarrow (3)$. Since $(3) \Rightarrow (2)$ is obvious, it remains to prove $(2) \Rightarrow (1)$. This is, however, clear; for if $Y \supset X$ is a left filter, then every finite non-empty $K \subset X$ has a left bound in Y.

Similarly as in the case considered in 12 B.6, 12 B.7, it is convenient to give a special name to the smallest filter of sets containing a given collection of sets.

12 B.9. Definition. Let $\mathcal{X}$ be a collection of sets. If A is a set, $\mathcal{X} \subset \exp A$, and $\mathcal{F}$ is a filter on A, then we shall say that $\mathcal{X}$ is a *sub-base* of the filter $\mathcal{X}$ (on A) provided the following holds: (i) $\mathcal{F} \subset \mathcal{X}$, (ii) if $\mathcal{F}_1$ is a filter on A, $\mathcal{F}_1 \supset \mathcal{X}$, then $\mathcal{F}_1 \supset \mathcal{F}$.

It can be easily proved that the following holds:

If $\mathcal{X}$ is a collection of sets and A is a set, $A \supset \bigcup \mathcal{X}$, then there exists exactly one filter on A of which $\mathcal{X}$ is a sub-base. This filter consists (i) *if $\mathcal{X} \neq \emptyset$, of all $Y \subset A$ such that $\bigcap \mathcal{Z} \subset Y$ for some finite non-empty $\mathcal{Z} \subset \mathcal{X}$,* (ii) *of a single element A if $\mathcal{X} = \emptyset$.*

12 B.10. Definition. Let $\mathcal{X}$ be a collection or a family of sets. If $\mathcal{X}$ is non-empty and $\bigcap \mathcal{Z} \neq \emptyset$ for every finite non-empty $\mathcal{Z} \subset \mathcal{X}$, then $\mathcal{X}$ is called *centred.* A centred $\mathcal{X}$ is also said to possess the "*finite intersection property*".

The proof of the following proposition is left to the reader.

A non-empty collection of sets is a sub-base of a proper filter if and only if it is centred.

12 B.11. Proposition and definition. *Let* $\{\mathscr{F}_b \mid b \in B\}$ *be a family of filters of sets; put* $A_b = \bigcup \mathscr{F}_b$, $A = \Pi\{A_b\}$. *The smallest filter* $\mathscr{F}$ *on* A *such that, for any* $b \in B, X \in \mathscr{F}_b$, *the set* $\mathrm{pr}_b^{-1}[X]$ *(consisting of all* $y \in A$ *such that* $\mathrm{pr}_b\, y \in X$*) belongs to* $\mathscr{F}$ *will be called the* cartesian product *(more explicitly, the* cartesian filter-product*) of* $\{\mathscr{F}_b\}$ *and will be denoted by* $\Pi_{\mathrm{filt}}\{\mathscr{F}_b \mid b \in B\}$ *or, provided there is no danger of misunderstanding, by* $\Pi\{\mathscr{F}_b \mid b \in B\}$ *or* $\Pi\{\mathscr{F}_b\}$ *or* $\prod_b \mathscr{F}_b$, *etc. If all* $\mathscr{F}_b$ *are proper, then* $\mathscr{F}$ *is also proper.*

Proof. By 12 B.9, it is sufficient to prove that the sets $\mathrm{pr}_b^{-1}[X]$, $X \in \mathscr{F}_b$, form a sub-base of a proper filter of sets. This is clear, however.

Remark. The reader is invited to formulate the definition of a filter-product (called also simply "product") $\mathscr{F}_1 \times_{\mathrm{filt}} \mathscr{F}_2$ or $\mathscr{F}_1 \times \mathscr{F}_2$ of two filters (this product is a filter on $(\bigcup \mathscr{F}_1) \times (\bigcup \mathscr{F}_2)$ containing all $X_1 \times X_2$ where $X_i \in \mathscr{F}_i$).

12 B.12. Proposition and definition. *Let* A, B *be sets, let* R *be a correspondence for* A *and* B. *Let* $\mathscr{F}$ *be a filter on* A. *Then the set of all* $R[X]$, $X \in \mathscr{F}$, *is a base of a filter on* B. *This filter (which is determined uniquely, see* 12 B.7*) will be called the* R-transform *of the filter* $\mathscr{F}$; *it is proper if and only if every* $X \in \mathscr{F}$ *intersects the actual domain* $\mathbf{D}\,\mathrm{gr}\, R$ *of* R.

Proof. If $X_1 \in \mathscr{F}$, $X_2 \in \mathscr{F}$, then $R[X_1 \cap X_2] \subset R[X_1] \cap R[X_2]$; hence the sets $R[X]$ form a base of a filter. If every $X \in \mathscr{F}$ intersects $\mathbf{D}$ gr R, then $R[X] \neq \emptyset$ for every $X \in \mathscr{F}$. This implies that $R[X]$ form a base of a proper filter.

Remark. In particular, if f is a mapping of a set A onto a set B, then the f-transform of every proper filter on A is a proper filter on B, and the f^{-1}-transform of every filter on B which "intersects" $f[A]$ (i.e. of a filter $\mathscr{Y}$ on B such that $\emptyset \notin f[A] \cap [\mathscr{Y}]$) is a proper filter on A.

Example. Let $\{A_b \mid b \in B\}$ be a family of sets, $A = \Pi\{A_b\}$. Let $\{\mathscr{F}_b \mid b \in B\}$ be a family, $\mathscr{F}_b$ being a filter on A_b; put $\mathscr{F} = \Pi\{\mathscr{F}_b\}$ (see 12 B.11). Put $p_b = \mathrm{pr}_b :$ $: A \to A_b$. Then, for any $b \in B$, $\mathscr{F}_b$ is the p_b-transform of $\mathscr{F}$.

We conclude this subsection by introducing the following concept.

12 B.13. Definition. A proper filter of sets $\mathscr{F}$ is called *fixed* if $\bigcap \mathscr{F} \neq \emptyset$, *free* if $\bigcap \mathscr{F} = \emptyset$.

Remarks. 1) If $\mathscr{F}$ is free, then every filter $B \cap \mathscr{F}$ is free (provided it is proper) as well as every proper filter $\mathscr{F}_1 \supset \mathscr{F}$; however, an R-transform (see 12 B.12) of $\mathscr{F}$ need not be free. If $\mathscr{F}_b$ are proper filters and one of them is free, then $\Pi\{\mathscr{F}_b\}$ is free. – 2) Every proper filter on a finite set A is fixed. If A is infinite, then there exist free proper filters on A; this can be proved e.g. by considering a well-order on A under which there is no largest element.

C. ULTRAFILTERS

12 C.1. Definition. Let A be a set. A maximal filter on A (i.e. a proper filter $\mathscr{F}$ on A such that if $\mathscr{F}_1$ is a proper filter on A, $\mathscr{F}_1 \supset \mathscr{F}$, then $\mathscr{F}_1 = \mathscr{F}$) will be also called an *ultrafilter* on A. The set of all ultrafilters on a set A will be denoted by ult A.

Clearly, a fixed filter $\mathscr{F}$ on A is an ultrafilter if and only if it has a base formed by a singleton, i.e. if there exists an $x \in A$ such that $\mathscr{F}$ consists of all Y with $(x) \subset Y \subset A$.

12 C.2. Theorem. *Let A be a set. If $\mathscr{F}$ is a free filter on A, then there exists an ultrafilter $\mathscr{F}^*$ on A such that $\mathscr{F}^* \supset \mathscr{F}$.*

This follows from 4 C.3 and 12 B.3.

We now intend to show that there exist "very many" ultrafilters on an infinite set, namely that there are exp exp a ultrafilters on a set A of an infinite cardinality a. To this end, some auxiliary propositions are necessary.

12 C.3. *Let $\mathscr{A}$ be a finite collection of sets. Then there exists a finite set $M \subset \bigcup\mathscr{A}$ such that $X \in \mathscr{A}$, $Y \in \mathscr{A}$, $X \neq Y \Rightarrow X \cap M \neq Y \cap M$.*

Proof. If $X \in \mathscr{A}$, $Y \in \mathscr{A}$, $X - Y \neq \emptyset$, choose an element $z_{X,Y} \in X - Y$. Let M be the set of all $z_{X,Y}$. If $X \in \mathscr{A}$, $Y \in \mathscr{A}$, $X \neq Y$, then either $X - Y \neq \emptyset$, $z_{X,Y} \in M \cap X$, $z_{X,Y} \notin M \cap Y$, or $Y - X \neq \emptyset$, $z_{Y,X} \notin M \cap X$, $z_{Y,X} \in M \cap Y$.

12 C.4. Definition. Let $\mathscr{Z}$ be a class of sets. We shall say that $\mathscr{Z}$ is *independent* (more explicitly, *inclusion-independent*) if the following holds: if $\mathscr{X} \subset \mathscr{Z}$, $\mathscr{Y} \subset \mathscr{Z}$ are finite and $\bigcap\mathscr{X} \subset \bigcup\mathscr{Y}$, then there exists a set M such that $M \in \mathscr{X}$, $M \in \mathscr{Y}$.

Example. Let A be a non-empty set. Let $\mathbf{M}$ consist of all collections $\mathscr{M}_t = \mathbf{E}\{X \mid t \in X \subset A\}$, where $t \in A$. Then $\mathbf{M}$ is independent.

12 C.5. *Let A be an infinite set; let* card $A = a$. *Then there exists an independent collection $\mathscr{M}$ of subsets of A such that* card $\mathscr{M} = \exp a$.

Proof. Denote by Φ the set of all $\langle K, \mathscr{Z}\rangle$ where $K \subset A$ is finite, $\mathscr{Z} \subset \exp K$. For any $X \in \exp A$ let $\Phi(X)$ denote the set of all $\langle K, \mathscr{Z}\rangle \in \Phi$ such that $K \cap X \in \mathscr{Z}$. It is easy to see that the cardinality of the set of all $\Phi(X)$ is equal to that of $\exp A$, i.e. to $\exp a$. We are going to prove that the collection $\mathbf{F}$ of all $\Phi(X)$ is an independent collection of sets.

Let $\mathscr{X} \subset \exp A$, $\mathscr{Y} \subset \exp A$ be finite, $\mathscr{X} \cap \mathscr{Y} = \emptyset$. By 12 C.3, there exists a finite $K_0 \subset A$ such that $S \in \mathscr{X} \cup \mathscr{Y}$, $T \in \mathscr{X} \cup \mathscr{Y}$, $S \neq T \Rightarrow K_0 \cap S \neq K_0 \cap T$. Denote by $\mathscr{Z}_0$ the set of all $K_0 \cap X$, $X \in \mathscr{X}$. Clearly, $X \in \mathscr{X} \Rightarrow \langle K_0, \mathscr{Z}_0\rangle \in \Phi(X)$. Since every $K_0 \cap Y$, $Y \in \mathscr{Y}$, is distinct from every $K_0 \cap X$, $X \in \mathscr{X}$, we obtain $Y \in \mathscr{Y} \Rightarrow \langle K_0, \mathscr{Z}_0\rangle \notin \Phi(Y)$. Thus, $\bigcap\{\Phi(X) \mid X \in \mathscr{X}\}$ is not contained in $\bigcup\{\Phi(Y) \mid Y \in \mathscr{Y}\}$. This proves that $\mathbf{F}$ is an independent collection of sets.

Clearly, card $\Phi = a$ and therefore there exists a bijective relation ψ on Φ onto A. Let $\mathscr{M}$ be the collection of all $\psi[\mathscr{F}]$, $\mathscr{F} \in \mathbf{F}$. Then, evidently, $\mathscr{M}$ is an independent collection of subsets of A, card $\mathscr{M} = \exp a$.

12 C.6. *Let A be a set; let $\mathscr{M}$ be an independent non-empty collection of subsets of A. For any $\mathscr{X} \subset \mathscr{M}$, denote by $\tau\mathscr{X}$ the collection consisting of all $X \in \mathscr{X}$ and all $A - X, X \in \mathscr{M}, X \notin \mathscr{X}$, and denote by $\varphi\mathscr{X}$ the (uniquely determined) filter $\mathscr{F}$ on A such that $\tau\mathscr{X}$ is a sub-base of $\mathscr{F}$. Then (1) if $\mathscr{X} \subset \mathscr{M}$ and either $\mathscr{X}$ is infinite or $\mathscr{X} \neq \mathscr{M}$, then $\tau\mathscr{X}$ is centred, hence $\varphi\mathscr{X}$ is a proper filter; (2) if $\mathscr{X}_1 \subset \mathscr{M}$, $\mathscr{X}_2 \subset \mathscr{M}$, $\mathscr{X}_1 \neq \mathscr{X}_2$, then there is no proper filter of sets containing both $\varphi\mathscr{X}_1$ and $\varphi\mathscr{X}_2$.*

Proof. Let $\mathscr{X} \subset \mathscr{M}$, and suppose that either $\mathscr{X} \neq \mathscr{M}$ or $\mathscr{X}$ is infinite. Suppose that $\tau\mathscr{X}$ is not centred. Then there exist natural numbers $n \geqq 1$, $m \leqq n$, and distinct sets $X_i \in \mathscr{X}$, $i = 1, \ldots, n$, such that no elements x belong to both all X_i, $1 \leqq i \leqq m$, and all $A - X_i$, $m < i \leqq n$. If $m = 0$, we get $\bigcup\{X_i \mid i = 1, \ldots, n\} = A$; this implies that every $X \in \mathscr{M}$ is equal to some X_i; hence $\mathscr{X} = \mathscr{M}$ is finite which contradicts the supposition. If $m > 0$, then we get $\bigcap\{X_i \mid 1 \leqq i \leqq m\} \subset \bigcup\{X_i \mid m < i \leqq n\}$ which contradicts the fact that $\mathscr{M}$ is independent. We have proved that $\tau\mathscr{X}$ is centred; hence by 12 B.10, $\varphi\mathscr{X}$ is a proper filter. If $\mathscr{X}_1 \subset \mathscr{M}$, $\mathscr{X}_2 \subset \mathscr{M}$, $\mathscr{X}_1 \neq \mathscr{X}_2$, suppose e.g. that $\mathscr{X}_1 - \mathscr{X}_2 \neq \emptyset$ and choose $X \in \mathscr{X}_1$, $X \notin \mathscr{X}_2$. Then $X \in \tau\mathscr{X}_1$, $A - X \in \tau\mathscr{X}_2$, which proves the last assertion.

12 C.7. Theorem. *Let A be an infinite set,* card $A = a$. *Then the cardinality of the set* ult A *of all ultrafilters on A is equal to* exp exp a.

Proof. By 12 C.5, there exists an independent collection of subsets of A such that card $\mathscr{M}$ = exp a. Let $\varphi\mathscr{X}$, where $\mathscr{X} \subset \mathscr{M}$, have the same meaning as in 12 C.6. Then every $\varphi\mathscr{X}$ is a proper filter. Hence, by 12 C.2, for any $\mathscr{X} \subset \mathscr{M}$ there exists an ultrafilter $\mathscr{F} \supset \varphi\mathscr{X}$; by assertion (2) from 12 C.6, if $\mathscr{X}_1 \neq \mathscr{X}_2$, $\mathscr{F}_1 \supset \varphi\mathscr{X}_1$, $\mathscr{F}_2 \supset \varphi\mathscr{X}_2$ and $\mathscr{F}_1$, $\mathscr{F}_2$ are proper filters, then $\mathscr{F}_1 \neq \mathscr{F}_2$. From this it follows at once that the cardinality of ult A is not less than that of exp $\mathscr{M}$, that is exp exp a. On the other hand, since ult $A \subset$ exp exp A, we have card (ult A) $\leqq$ exp exp a. This proves the theorem.

Remark. It is easy to see that if A is finite, then ult A has as many elements as A.

12 C.8. Theorem. *A proper filter $\mathscr{F}$ on a set A is an ultrafilter if and only if for any $X \subset A$ either $X \in \mathscr{F}$ or $A - X \in \mathscr{F}$.*

Proof. If $\mathscr{F} \in$ ult A and $X \subset A$, consider the collection of all $X \cap F$, $F \in \mathscr{F}$. Clearly, this collection is a filter base. If it is a base of $\mathscr{F}$, then $X = X \cap A \in \mathscr{F}$. If not, then it is a base of exp A and therefore, for some $F_0 \in \mathscr{F}$, $X \cap F_0 = \emptyset$, $F_0 \subset A - X$, $A - X \in \mathscr{F}$. The proof of the inverse implication is quite easy.

12 C.9. Theorem. *Let $\mathscr{F}$ be an ultrafilter on a set A. If f is a mapping of A into a set B, then the f-transform of $\mathscr{F}$ is an ultrafilter. If g is an injective mapping of a set C into A, then the g^{-1}-transform of $\mathscr{F}$ is either an ultrafilter (on C) or equal to* exp C. *In particular, if $C \subset A$, then $C \cap [\mathscr{F}]$ is either an ultrafilter on C or equal to* exp C.

Proof. Consider $f = f : A \to B$. Let $\mathscr{F}^*$ denote the f-transform of $\mathscr{F}$. If $Y \subset B$, then either $f^{-1}(Y) \in \mathscr{F}$, hence $Y \in \mathscr{F}^*$ or $A - f^{-1}(Y) \in \mathscr{F}$, hence $B - Y \in \mathscr{F}^*$. Thus, by 12 C.8, $\mathscr{F}^*$ is an ultrafilter. Consider $g = g : C \to A$. Let $\mathscr{F}'$ denote the

g^{-1}-transform of $\mathscr{F}$. If $g[C] \in \mathscr{F}$, then $\mathscr{F}'$ is a g'-transform of $\mathscr{F}$ where $g' = g^{-1} : g[C] \to C$, hence $\mathscr{F}'$ is an ultrafilter. If not, then $A - g[C] \in \mathscr{F}$, hence $\mathscr{F}'$ contains the void set and therefore is equal to exp C.

12 C.10. Definition. Let f be a mapping of a set A into a set B. Then the mapping of ult A into ult B which assigns to every $\mathscr{F} \in$ ult A its f-transform (which is, by 12 C.9, an ultrafilter on B) will be denoted by ult f.

12 C.11. *Let f be a mapping of a set A into a set B. Then* ult f *is injective (respectively, surjective) if and only if f is injective (respectively, surjective). If g is a mapping of B into a set C, then* $\operatorname{ult}(g \circ f) = (\operatorname{ult} g) \circ (\operatorname{ult} f)$.

Proof. If $\mathscr{F}_1 \in$ ult A, $\mathscr{F}_2 \in$ ult A, $\mathscr{F}_1 \neq \mathscr{F}_2$, then there exist sets $X_i \in \mathscr{F}_i$ such that $X_1 \cap X_2 = \emptyset$ (this follows from 12 C.8). If f is injective, then $f[X_1] \cap f[X_2] = \emptyset$ and therefore f-transforms of $\mathscr{F}_1$ and of $\mathscr{F}_2$ are distinct. If f is not injective, then $fx_1 = fx_2$ for some distinct x_1, x_2. Denote, for any $a \in A$, by $\mathscr{F}(a)$ the ultrafilter consisting of all $X \subset A$ with $a \in X$. Clearly, f-transforms of $\mathscr{F}(x_1)$ and $\mathscr{F}(x_2)$ coincide. If f is surjective, $\mathscr{F} \in$ ult B, then let $\mathscr{F}^*$ be the f^{-1}-transform of $\mathscr{F}$. Since $f[A] = B$, we have $f^{-1}[Y] \neq 0$ for every $Y \in \mathscr{F}$. Therefore, by 12 C.2, there exists an ultrafilter $\mathscr{F}^*$ such that $Y \in \mathscr{F}$ implies $f^{-1}[Y] \in \mathscr{F}^*$. It is easy to show that $\mathscr{F}$ is the f-transform of $\mathscr{F}^*$. — The rest of the proof is left to the reader.

12 C.12. Clearly, there exist free ultrafilters $\mathscr{F}$ such that $\bigcap \mathscr{X} = \emptyset$ for some countable $\mathscr{X} \subset \mathscr{F}$. Indeed, if A is an infinite set, then there exist (see 9 B.9) sets A_n, $n = 0, 1, 2, \ldots$, such that $A_0 = A$, for every n, $A_n \supset A_{n+1}$, card A_n = card A, and $\bigcap \{A_n\} = \emptyset$. Clearly, the A_n form a base of a free filter, say $\mathscr{F}$, and by 12 C.2, there exists an ultrafilter $\mathscr{F}^* \supset \mathscr{F}$ on A; this ultrafilter has the property in question.

It seems that it is not possible to prove, on the basis of the system of axioms developed in this book (or of any of current systems of axioms) that there exist free ultrafilters $\mathscr{F}$ lacking the above property, i.e. such that $\bigcap \mathscr{X} \neq \emptyset$ for every countable $\mathscr{X} \subset \mathscr{F}$ (on the other hand, no means seem to be known to disprove the existence of such ultrafilters). However this may be, the case of an ultrafilter $\mathscr{F}$ such that $\bigcap \mathscr{X} = \emptyset$ for some countable $\mathscr{X} \subset \mathscr{F}$ may be considered as "normal".

Without going into this matter (connected with inaccessible cardinals, i.e. cardinals of non-accessible sets, see 4 ex. 12), we give two propositions concerning the filter-product (see 12 B.11) of ultrafilters; the first of these propositions refers to the "normal case"; the second shows that possibly there may occur quite different situations.

12 C.13. Theorem. *Let $\mathscr{F}_i$ be a free ultrafilter on a set A_i, $i = 1, 2$. Suppose that there exist countable $\mathscr{X}_i \subset \mathscr{F}_i$ such that $\bigcap \mathscr{X}_i = \emptyset$, $i = 1, 2$. Then $\mathscr{F}_1 \times_{\text{fil}} \mathscr{F}_2$ is not an ultrafilter.*

Proof. It follows at once from the suppositions that there exist infinite sequences of sets $\{B_n\}$, $\{C_n\}$ such that $B_n \in \mathscr{F}_1$, $C_n \in \mathscr{F}_2$, $B_0 = A_1$, $C_0 = A_2$, $B_n \supset B_{n+1}$ and $C_n \supset C_{n+1}$ for $n \in \mathrm{N}$, $\bigcap \{B_n\} = \emptyset$, $\bigcap \{C_n\} = \emptyset$. Let M denote the set of all $\langle x, y \rangle \in$

$\in A \times B$ such that, for some $n \in \mathrm{N}$, $x \notin B_n$, $y \in C_n$; then $(A \times B) - M$ consists of all $\langle x, y \rangle \in A \times B$ such that, for some $n \in \mathrm{N}$, $x \in B_n$, $y \notin C_{n+1}$.

Let $X \subset A$, $Y \subset B$ be sets such that $X \times Y \subset M$. Choose $y \in Y$ and choose k such that $y \notin C_k$. Then, clearly, for any $x \in X$ we have $x \notin B_k$; this implies $X \cap B_k = \emptyset$, $X \notin \mathscr{F}_1$. Quite similarly, it can be shown that $X \times Y \subset (A \times B) - M$ implies that $Y \notin \mathscr{F}_2$. We have shown that, for any $Z_i \in \mathscr{F}_i$, $i = 1, 2$, the set $Z_1 \times Z_2$ intersects both M and $(A \times B) - M$. This proves, according to 12 C.8, that the filter of all $Z \subset A \times B$ such that $Z \supset Z_1 \times Z_2$ for some $Z_i \in \mathscr{F}_i$, i.e. the filter $\mathscr{F}_1 \times_{\mathrm{filt}} \mathscr{F}_2$, is not an ultrafilter.

12 C.14. *Let $\mathscr{U}$ be an ultrafilter on a set A, and let $\mathscr{V}$ be an ultrafilter on a set B. If there exists a set $U \in \mathscr{U}$ such that $\bigcap \mathscr{Y} \neq \emptyset$ whenever $\mathscr{Y} \subset \mathscr{V}$, $\operatorname{card} \mathscr{Y} \leqq \operatorname{card} U$, then $\mathscr{U} \times_{\mathrm{filt}} \mathscr{V}$ is an ultrafilter.*

Proof. To prove that $\mathscr{U} \times_{\mathrm{filt}} \mathscr{V}$ is an ultrafilter, it is sufficient to show that, for any $M \subset A \times B$, either $M \supset X \times Y$ for some $X \in \mathscr{U}$, $Y \in \mathscr{V}$, or $M \cap (X \times Y) = \emptyset$ for some $X \in \mathscr{U}$, $Y \in \mathscr{V}$. For any $u \in M$, put $M_u = M[(u)]$. Denote by A' the set of those $u \in A$ for which $M_u \in \mathscr{V}$, any by A'' its complement in A. Since $\mathscr{V}$ is an ultrafilter, $B - M_u \in \mathscr{V}$ for every $u \in A''$. Two cases are possible: $A' \in \mathscr{U}$ or $A'' \in \mathscr{U}$. If $A' \in \mathscr{U}$, put $X = A' \cap U$, $Y = \bigcap\{M_u \mid u \in A' \cap U\}$. Then $X \in \mathscr{U}$, $Y \in \mathscr{V}$ (since $\bigcap \mathscr{Y} \neq \emptyset$ whenever $\mathscr{Y} \subset \mathscr{V}$, $\operatorname{card} \mathscr{Y} \leqq \operatorname{card} U$), and clearly $M \supset X \times Y$. If $A'' \in \mathscr{U}$, put $X = A''$, $Y = \bigcap\{B - M_u \mid u \in A'' \cap U\}$. Then $X \in \mathscr{U}$, $Y \in \mathscr{V}$, and clearly $M \cap (X \times Y) = \emptyset$.

13. CATEGORIES

The situation which we are going to describe has already been considered and will appear very often in the sequel. There are given certain objects and certain "transitions" from one object to another; if we perform successively a "transition" from a to b and a "transition" from b to c, then the result is a "transition" from a to c. An example: "objects" are sets, "transitions" are mappings. An example of a different kind (which has not yet been considered here): "objects" are arbitrary elements, "transitions" are pairs of elements, $\langle a, b\rangle$ being conceived as a "transition" from a to b.

It turns out that the situations indicated above can be usefully investigated in an abstract and general way, disregarding the special nature of the "objects" and "transitions" in question.

Such an investigation is the purpose of the theory of categories. The above remarks serve only to give a general idea of this concept and of its motivation. Its meaning and role will gradually become clear from the definition and subsequent examples.

We shall restrict ourselves to the basic concepts and results here, since we do not intend to give an exposition of the theory of categories to an extent which would permit applications in general topology, but merely to facilitate an approach to various topological concepts and theorems from a general and unifying point of view.

A. CATEGOROIDS

13 A.1. Definition. We shall say that σ is a *partial composition* in a class X if σ is a single-valued relation, $\mathbf{D}\sigma \subset X \times X$, $\mathbf{E}\sigma \subset X$.

Observe that a composition on X is a partial composition in any class $Y \supset X$.

Examples. (A) Let Φ be a quasi-order (i.e. a transitive relation); let σ consist of all $\langle\langle\xi, \eta\rangle, \zeta\rangle$ where ξ, η, ζ belong to Φ and, for some elements a, b, c, we have $\xi = \langle b, c\rangle$, $\eta = \langle a, b\rangle$, $\zeta = \langle a, c\rangle$. Then σ is a partial composition in Φ; $\mathbf{D}\sigma$ consists of all $\langle\langle b, c\rangle, \langle a, b\rangle\rangle$ such that $\langle b, c\rangle \in \Phi$, $\langle a, b\rangle \in \Phi$. Clearly, σ is not a composition, unless $\Phi = \emptyset$ or $\Phi = (\langle a, a\rangle)$. – (B) Let $\mathscr{R}$ be a class of comprisable relations. Then $\{\langle\varrho, \sigma\rangle \to \varrho \circ \sigma \mid \varrho, \sigma, \varrho \circ \sigma \text{ belong to } \mathscr{R}\}$ is a partial composition in $\mathscr{R}$. — (C) Consider a class A and the set $\mathscr{F}$ of all mappings $f = \langle\varphi, X, Y\rangle$ where $X \subset A$, $Y \subset A$ are sets. Let σ assign to $\langle f, g\rangle \in \mathscr{F} \times \mathscr{F}$ the element $f \circ g$ provided it exists (see Section 7). Then σ is a partial composition in $\mathscr{F}$. If $S \subset A$ is a set and $\mathscr{S}$ denotes the set of all $f = \langle\varphi, S, S\rangle \in \mathscr{F}$, then the "restriction of σ to $\mathscr{S}$", i.e. the relation $\sigma \cap ((\mathscr{S} \times \mathscr{S}) \times \mathscr{S})$, is a composition on $\mathscr{S}$ and even a semi-group structure on $\mathscr{S}$. — (D) Let Φ be the class of all homomorphisms $\langle\varphi, \mathscr{G}, \mathscr{H}\rangle$ where $\mathscr{G}$, $\mathscr{H}$ are groups, and let σ assign to $\langle f, g\rangle \in \Phi \times \Phi$ the homomorphism $f \circ g$ provided it exists. Then σ is a partial composition in Φ. — (E) Let A, B be given classes and let B contain more than one element. Consider the class $\mathscr{F}$ of all comprisable single-valued relations f such that $\mathbf{D}f \subset A$, $\mathbf{E}f \subset B$. Let σ consist of all $\langle\langle f, g\rangle, h\rangle$ where f, g, h belong to $\mathscr{F}$, $h = f \cup g$. Then σ is a partial composition in $\mathscr{F}$; σ is not a composition unless A is void. Clearly $\mathbf{D}\sigma$ consists of all $\langle f, g\rangle \in \mathscr{F} \times \mathscr{F}$ such that $fx = gx$ whenever $x \in \mathbf{D}f \cap \mathbf{D}g$.

Convention. If σ is a partial composition, and $\langle x, y\rangle \in \mathbf{D}\sigma$, we shall denote by $x\sigma y$ the unique element z such that $\langle\langle x, y\rangle, z\rangle \in \sigma$. — This symbol must be used with care, for if $\langle x, y\rangle \notin \mathbf{D}\sigma$, then "$x\sigma y$" has no meaning; in such a case, we shall also say, for convenience, that $x\sigma y$ is not defined or that "$x\sigma y$ does not exist".

We shall use freely, with appropriate changes, various conventions introduced in Section 6 for compositions. In particular, if σ is a partial composition in X and $Y \subset X$, then σ_Y will denote the relation $\sigma \cap ((Y \times Y) \times Y)$.

13 A.2. Definition. A partial composition σ is called *associative* if it possesses the following property:

If $\langle x, y\rangle \in \mathbf{D}\sigma$ and $\langle x\sigma y, z\rangle \in \mathbf{D}\sigma$, then $\langle y, z\rangle \in \mathbf{D}\sigma$ and $\langle x, y\sigma z\rangle \in \mathbf{D}\sigma$, and conversely; if these conditions are satisfied, then $(x\sigma y)\,\sigma z = x\sigma(y\sigma z)$.

A partial composition σ is called *strongly associative* if it is associative and $\langle x, y\rangle \in \mathbf{D}\sigma$, $\langle y, z\rangle \in \mathbf{D}\sigma$ implies $\langle x, y\sigma z\rangle \in \mathbf{D}\sigma$, $\langle x\sigma y, z\rangle \in \mathbf{D}\sigma$.

Clearly, if σ is a composition (that is, if there is a class X such that $\mathbf{D}\sigma = X \times X$, $\mathbf{E}\sigma \subset X$), then the associativity and strong associativity described above as well as the associativity in the sense of Section 6 are equivalent properties.

Observe that the partial compositions in the above examples (A), (C), (D) are strongly associative. The partial composition from example (B) may be strongly associative or associative but not strongly associative or not associative according to the properties of $\mathscr{R}$. The partial composition σ in example (E) is not strongly associative (provided A is not void) which is shown as follows: choose $a \in A$, $b_1 \in B$, $b_2 \in B$, $b_1 \neq b_2$, and put $f_i = (\langle a, b_i\rangle)$; then $\langle f_1, \emptyset\rangle \in \mathbf{D}\sigma$, $\langle\emptyset, f_2\rangle \in \mathbf{D}\sigma$, $\emptyset\sigma f_2 = f_2$ $\langle f_1, f_2\rangle \notin \mathbf{D}\sigma$.

13 A.3. Definition. Let σ be a partial composition. We shall say that a is *neutral* or that a is a *unit element* (or simply a *unit*) *under* σ if (1) there exist x, y such that $\langle a, x\rangle \in \mathbf{D}\sigma$, $\langle y, a\rangle \in \mathbf{D}\sigma$, (2) $a\sigma x = x$ whenever $\langle a, x\rangle \in \mathbf{D}\sigma$, and $y\sigma a = y$ whenever $\langle y, a\rangle \in \mathbf{D}\sigma$.

Examples. Consider the examples given in 13 A.1. In example (A), the units are precisely all elements $\langle a, a\rangle$ belonging to Φ. In (B), if $\mathscr{R} = \exp(A \times A)$, A being a class, then there is exactly one unit J_A if A is a set and there is no unit if A is non-comprisable. In example (C), the units are mappings of the form $\mathsf{J} : X \to X$. In example (D), $\emptyset$ is the only unit.

13 A.4. Definition. Let X be a non-void class, σ a partial composition in X. If σ is strongly associative, and for every $x \in X$ there exist unit elements (under σ) e' and e'' such that $e'\sigma x = x$, $x\sigma e'' = x$, then $\langle X, \sigma\rangle$ will be called a *categoroid*, and σ will be called a *categoroid structure on X*.

Remarks. 1) Clearly, a categoroid is a struct, and therefore all definitions and conventions introduced for structs apply to categoroids. – 2) Clearly, if $\langle X, \sigma\rangle$ is a categoroid and e is a unit under σ, then $\langle e, e\rangle \in \mathbf{D}\sigma$, $e\sigma e = e$.

Consider examples from 13 A.1. – In example (A), σ is a categoroid structure on Φ if and only if Φ is reflexive. In example (C), σ is a categoroid structure on $\mathscr{F}$; it is comprisable if and only if A is a set. In example (D), $\langle \Phi, \sigma\rangle$ is a categoroid. In example (E), σ is not strongly associative, hence σ is not a categoroid structure.

Convention. If $\mathscr{K}^* = \langle \Phi, \sigma\rangle$ is a categoroid, then the elements of Φ will sometimes be called *morphisms of* (or *from*) $\mathscr{K}^*$ or $\mathscr{K}^*$*-morphisms*; if x, y are morphisms of $\mathscr{K}^*$, we shall sometimes write $x \,.\, y$ (or simply xy) instead of $x\sigma y$.

13 A.5. *If $\langle X, \sigma\rangle$ is a categoroid, then for every $x \in X$ there exists exactly one unit element $e' \in X$ such that $e'\sigma x = x$ and exactly one unit element $e'' \in X$ such that $x\sigma e'' = x$.*

Proof. Elements e', e'' such that $e'\sigma x = x$, $x\sigma e'' = x$ exist by the definition 13 A.4. If $\bar{e}$ is a unit element, $\bar{e}\sigma x = x$, then $e'\sigma(\bar{e}\sigma x)$ "is defined" (that is $\langle e', \bar{e}\sigma x\rangle \in \mathbf{D}\sigma$) and therefore, by the associativity of σ, $\langle e', \bar{e}\rangle \in \mathbf{D}\sigma$; hence, e' and $\bar{e}$ being units, $e'\sigma\bar{e} = e'$, $e'\sigma\bar{e} = \bar{e}$, $\bar{e} = e'$.

Definition. If σ is a categoroid structure on X and $x \in X$, then the elements e', e'' described above will be called the *left unit* and the *right unit* for x (under σ). The left unit will also be called the *range unit*, and the right unit will also be called the *domain unit*.

Consider the examples from 13 A.1. In example (A) with a reflexive Φ, the left (right) unit for $\langle a, b\rangle$ is $\langle b, b\rangle$ (respectively, $\langle a, a\rangle$). In example (C), the left (right) unit for $\langle \varphi, X, Y\rangle$ is $\mathsf{J} : Y \to Y$ (respectively, $\mathsf{J} : X \to X$).

13 A.6. *Let σ be a categoroid structure on X. Then $\langle x, y\rangle \in \mathbf{D}\sigma$ if and only if the right (domain) unit for x coincides with the left (range) unit for y.*

Proof. Let e_x, e_y be the units in question. If $\langle x, y\rangle \in \mathbf{D}\sigma$, then $x\sigma(e_y\sigma y)$ "is defined" (i.e. $\langle x, e_y\sigma y\rangle \in \mathbf{D}\sigma$), hence also $x\sigma e_y$ "is defined" and therefore, by 13 A.5, $e_y = e_x$. If $e_x = e_y = e$, then $\langle x, e\rangle \in \mathbf{D}\sigma$, $\langle e, y\rangle \in \mathbf{D}\sigma$; by strong associativity, this implies $\langle x, y\rangle = \langle x, e\sigma y\rangle \in \mathbf{D}\sigma$.

13 A.7. Consider a class X and a partial composition σ in X. If $Y \subset X$, put $\sigma_Y = \sigma \cap ((Y \times Y) \times Y)$. It seems natural to consider $\langle Y, \sigma_Y \rangle$ as a substruct of $\langle X, \sigma \rangle$ and, in particular, to say that $\langle Y, \mu \rangle$ is a subcategoroid of a categoroid $\langle X, \sigma \rangle$ if $\langle Y, \mu \rangle$ is a categoroid and $\mu = \sigma_Y$. However, it turns out in connection with the investigation of categories, that it is more expedient to define subcategoroids in a different and somewhat more restrictive way.

Definition. Let $\mathscr{A} = \langle A, \sigma \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be categoroids. We shall say that $\mathscr{B} = \langle B, \mu \rangle$ is a *subcategoroid of* $\mathscr{A}$ (or that $\mathscr{B}$ is *identically embedded in* $\mathscr{A}$) if (i) $\mu \subset \sigma$, $B \subset A$, (ii) every unit under μ is a unit under σ.

Convention. If $\langle B, \mu \rangle$ is a subcategoroid of $\mathscr{A}$, then we shall also say that the class B is a *subcategoroid of* $\mathscr{A}$.

Examples. (A) Consider example (A) from 13 A.1; suppose that Φ is reflexive. Every $\langle X, \sigma_X \rangle$ where $X \subset \Phi$ is an arbitrary non-void reflexive transitive relation is a subcategoroid, and every subcategoroid is of this form. – (B) Let $T = (a, b)$, $a \neq b$. The set $F = \mathbf{F}(T, T)$ consists of four mappings: the identity mapping ε, two constant mappings α, β and a mapping γ with $\gamma \neq \varepsilon$, $\gamma \circ \gamma = \varepsilon$. Clearly, F endowed with the usual composition is a categoroid possessing exactly one unit ε; it has six subcategoroids: F itself, (ε), (ε, α), (ε, β), (ε, γ), $(\varepsilon, \alpha, \beta)$. The set (α) as well as (β) satisfy condition (i) above, but do not satisfy (ii), and therefore are not subcategoroids, although they are sub-semi-groups of F.

13 A.8. *Let $\langle A, \sigma \rangle$ be a categoroid, let $\emptyset \neq B \subset A$. If (1) $x \in B$, $y \in B$, $\langle x, y \rangle \in \mathbf{D}\sigma \Rightarrow x\sigma y \in B$, (2) for every $x \in B$, the left and right unit for x (under σ) belong to B, then there exists exactly one partial composition μ in B such that $\langle B, \mu \rangle$ is a subcategoroid of $\langle A, \sigma \rangle$. This partial composition μ is equal to $\sigma \cap ((B \times B) \times B)$. Conversely, if there exists a μ such that $\langle B, \mu \rangle$ is a subcategoroid of $\langle A, \sigma \rangle$, then B satisfies conditions (1) and (2).*

The proof is left to the reader.

Remark. Let $\langle X, \sigma \rangle$ be a categoroid possessing exactly one unit. Then, by 13 A.6, $\mathbf{D}\sigma = X \times X$ and therefore σ is a semi-group structure, and X contains a neutral element under σ. Clearly, $Y \subset X$ is a subcategoroid if and only if Y is stable under σ and contains the neutral element of X. Conversely, if σ is a semi-group structure on X and X contains a neutral element, then $\langle X, \sigma \rangle$ is a categoroid with exactly one unit.

13 A.9. We are going to consider those mappings of categoroids which will play a role analogous to that of homomorphisms in the theory of algebraic structures. If $\langle X, \sigma \rangle$, $\langle X', \sigma' \rangle$ are categoroids, it would seem appropriate to define a single-valued "homomorphism-relation under σ and σ'" as a relation f on X into X' such that $(fx)\,\sigma\,(fy) = f(x\sigma y)$ whenever $\langle x, y \rangle \in \mathbf{D}\sigma$. However, we shall consider a different though closely related concept.

Definition. Let $\langle X, \sigma\rangle$, $\langle Y, \mu\rangle$ be categoroids. A single-valued relation φ will be called *covariant under* σ and μ (or (σ, μ)-*covariant*) provided (1) if $x \in \mathbf{D}\varphi$, $y \in \mathbf{D}\varphi$, $\langle x, y\rangle \in \mathbf{D}\sigma$, then $x\sigma y \in \mathbf{D}\varphi$, $\langle \varphi x, \varphi y\rangle \in \mathbf{D}\mu$, $(\varphi x)\,\mu\,(\varphi y) = \varphi(x\sigma y)$, (2) if e is a unit under σ, $e \in \mathbf{D}\varphi$, then φe is a unit under μ.

Remark. The above definition overlaps that of 6 E.1. Namely, they are both meaningful if φ is single-valued, $\mathbf{D}\varphi \subset X$, $\mathbf{E}\varphi \subset Y$ and semi-group structures σ on X and μ on Y possess neutral elements. It is easy to see that in such a case both properties considered ("covariant" and "stable") are equivalent.

Examples. (A) Let $\mathscr{X} = \langle X, \varrho\rangle$ be an arbitrary categoroid. Let $\Phi = V \times V$ be the class of all pairs of elements; consider the categoroid $\langle \Phi, \sigma\rangle = \langle V \times V, \sigma\rangle$ described in 13 A.1, example (A). For any $x \in X$, put $fx = \langle e', e''\rangle$ where e', e'' are the domain unit and the range unit for x under ϱ. It is easy to see that the relation f is covariant under ϱ and σ. – (B) Denote by $\mathscr{F}(A)$ the categoroid described in 13 A.1, example (C). If $B \subset A$, then for every $F \in \mathscr{F}(A)$, $F = \langle f, X, Y\rangle$, put $F^* = F : X \cap B \to Y \cap B$. It is easy to see that the relation $\{F \to F^*\}$ is covariant. – (C) Consider $\mathscr{F}(V)$, V being the universal class. If $F \in \mathscr{F}(V)$, $F = \langle f, X, Y\rangle$, let $\hat{F}$ denote the mapping $\langle\{Z \to f[Z]\}, \exp X, \exp Y\rangle$. It is easy to see that $\{F \to \hat{F}\}$ is a covariant relation.

13 A.10. Definition. Let $\mathscr{X} = \langle X, \sigma\rangle$, $\mathscr{Y} = \langle Y, \mu\rangle$ be categoroids. A mapping $\langle \varphi, \mathscr{X}, \mathscr{Y}\rangle$ will be called a *covariant functor on* $\mathscr{X}$ *into* $\mathscr{Y}$ (*onto* $\mathscr{Y}$ if $\mathbf{E}\varphi = Y$) if φ is covariant under σ and μ.

Clearly, in the above examples we have covariant functors on $\mathscr{X}$ into $\langle V \times V, \sigma\rangle$, on $\mathscr{F}(A)$ onto $\mathscr{F}(B)$, on $\mathscr{F}(V)$ into $\mathscr{F}(V)$.

13 A.11. *If* $\mathscr{X} = \langle X, \sigma\rangle$, $\mathscr{Y} = \langle Y, \mu\rangle$ *are categoroids and* f *is an injective covariant functor on* $\mathscr{X}$ *onto* $\mathscr{Y}$, *then* f^{-1} *is a covariant functor.*

Proof. Let $\langle \eta_1, \eta_2\rangle \in \mathbf{D}\mu$ and let $\eta_i = f\xi_i$. Let e_1 be the right unit for ξ_1 and let e_2 be the left unit for ξ_2. Suppose $\langle \xi_1, \xi_2\rangle \notin \mathbf{D}\sigma$; then, by 13 A.6, $e_1 \neq e_2$, hence $fe_1 \neq fe_2$ (for f is injective). Clearly $\eta_1\mu(fe_1) = \eta_1$, $(fe_2)\,\mu\eta_2 = \eta_2$; by 13 A.6, this contradicts $\langle \eta_1, \eta_2\rangle \in \mathbf{D}\mu$. Thus $\langle \xi_1, \xi_2\rangle \in \mathbf{D}\sigma$ and therefore $\eta_1\mu\eta_2 = f(\xi_1\sigma\xi_2)$, $\xi_1\sigma\xi_2 = f^{-1}(\eta_1\mu\eta_2)$. – The rest of the proof is left to the reader.

13 A.12. *If* $\mathscr{A}$, $\mathscr{B}$ *are categoroids, then* $\mathscr{B}$ *is a subcategoroid of* $\mathscr{A}$ *if and only if* $\mathsf{J} : \mathscr{B} \to \mathscr{A}$ *is a covariant functor on* $\mathscr{B}$ *into* $\mathscr{A}$.

Proof. Let $\mathscr{A} = \langle A, \sigma\rangle$, $\mathscr{B} = \langle B, \mu\rangle$. If $\mathscr{B}$ is a subcategoroid, then $B \subset A$, $\mu \subset \sigma$, and every unit under μ is a unit under σ. We have to prove that the relation J_B is covariant under σ and μ; but this is immediate since conditions (1), (2) in 13 A.8 are clearly satisfied. On the other hand, if $\mathsf{J} : \mathscr{B} \to \mathscr{A}$ is a covariant functor, then J_B is a covariant relation on B into A, from which it follows that $B \subset A$, $\mu \subset \sigma$ and every unit under μ is a unit under σ.

13 A.13. Definition. Let $\mathscr{A} = \langle A, \sigma\rangle$, $\mathscr{B} = \langle B, \mu\rangle$ be categoroids. If φ is a bijective relation on A onto B and $\langle x, y\rangle \in \mathbf{D}\sigma$, $z = x\sigma y \Leftrightarrow \langle \varphi x, \varphi y\rangle \in \mathbf{D}\mu$, $\varphi z =$

$= (\varphi x)\,\mu\,(\varphi y)$, then φ is called an *isomorphism-relation* (*under* σ *and* μ) and $\langle \varphi, \mathscr{A}, \mathscr{B} \rangle$ is called an *isomorphism functor* or simply an *isomorphism* (*on* $\mathscr{A}$ *onto* $\mathscr{B}$). Two categoroids $\mathscr{A}$ and $\mathscr{B}$ are said to be *isomorphic* if there exists an isomorphism of $\mathscr{A}$ onto $\mathscr{B}$.

Example. Let Φ be a reflexive quasi-order. Consider the categoroid $\langle \Phi, \sigma \rangle$ described in 13 A.1, example (A). Let $\mathscr{R}$ consist of all singletons $(\langle a, b \rangle) \subset \Phi$. Then $\langle \Phi, \sigma \rangle$ and $\langle \mathscr{R}, \mu \rangle$ (where μ is the composition of relations restricted to $\mathscr{R}$) are isomorphic.

Let $\mathscr{A}$, $\mathscr{B}$ *be categoroids. A bijective mapping F of* $\mathscr{A}$ *onto* $\mathscr{B}$ *is an isomorphism if and only if F or* F^{-1} *is covariant.*

This follows from 13 A.11.

13 A.14. Definition. Let $\mathscr{X} = \langle X, \sigma \rangle$, $\mathscr{Y} = \langle Y, \mu \rangle$ be categoroids. A single-valued relation φ will be called *contravariant under* σ *and* μ or (σ, μ)*-contravariant* if (1) $x \in \mathbf{D}\varphi$, $y \in \mathbf{D}\varphi$, $\langle x, y \rangle \in \mathbf{D}\sigma$ imply $x\sigma y \in \mathbf{D}\varphi$, $\langle \varphi y, \varphi x \rangle \in \mathbf{D}\mu$, $(\varphi y)\,\mu(\varphi x) = \varphi(x\sigma y)$, (2) if e is a unit under σ, $e \in \mathbf{D}\varphi$, then φe is a unit under μ. A mapping $\langle \varphi, \mathscr{X}, \mathscr{Y} \rangle$ will be called a *contravariant functor on* $\mathscr{X}$ *into* $\mathscr{Y}$ if its graph φ is a contravariant relation under σ and μ.

Examples. (A) Consider the categoroid $\mathscr{F}(V)$ (see 13 A.9, example (B)), V being the universal class. If $F \in \mathscr{F}(V)$, $F = \langle f, X, Y \rangle$, let $\tilde{F}$ denote the mapping $\{Z \to f^{-1}[Z] \mid Z \subset Y\} : \exp Y \to \exp X$. It is easy to prove that $\{F \to \tilde{F}\} : \mathscr{F}(V) \to \mathscr{F}(V)$ is a contravariant relation on $\mathscr{F}(V)$ into $\mathscr{F}(V)$. — (B) Let A be a fixed group. For any group G denote by $h(G)$ the set of all homomorphisms of G into A. Consider the categoroid $\mathscr{X}$ equal to the class X of all homomorphisms of groups endowed with the usual composition of mappings. For any $f \in X$, $f : G_1 \to G_2$, let f^* denote the mapping of $h(G_2)$ into $h(G_1)$ which assigns $\varphi \circ f$ to $\varphi \in h(G_2)$. It is easy to see that $\{f \to f^*\}$ is a $(\circ, \circ)$-contravariant relation which maps X into the class of all mappings.

13 A.15. Definition. Let $\mathscr{A} = \langle A, \sigma \rangle$, $\mathscr{B} = \langle B, \mu \rangle$ be categoroids. If φ is a bijective relation on A onto B and $\langle\langle x, y \rangle, z \rangle \in \sigma \Leftrightarrow \langle\langle \varphi y, \varphi x \rangle, \varphi z \rangle \in \mu$, then φ is called an *anti-isomorphism relation* (*under* σ *and* μ) and $\langle \varphi, \mathscr{A}, \mathscr{B} \rangle$ is called an *anti-isomorphism functor* (or simply an *anti-isomorphism* provided there is no danger of misunderstanding). If $\mathscr{X}$, $\mathscr{Y}$ are categoroids and there exists an anti-isomorphism of $\mathscr{X}$ onto $\mathscr{Y}$ we shall say that $\mathscr{X}$ and $\mathscr{Y}$ are *anti-isomorphic.*

Clearly, if $\mathscr{X}$, $\mathscr{Y}$ are categoroids, then $f = \langle \varphi, \mathscr{X}, \mathscr{Y} \rangle$ is an anti-isomorphism if and only if both f and f^{-1} are contravariant functors.

13 A.16. Definition. Let σ be a partial composition (in a class X). Then the partial composition consisting of all $\langle\langle x, y \rangle, z \rangle$ such that $\langle\langle y, x \rangle, z \rangle \in \sigma$ is called *contragredient* (or *opposite*) to σ.

If $\bar{\sigma}$ is contragredient to σ, then clearly σ is contragredient to $\bar{\sigma}$, and $\mathbf{D}\bar{\sigma}$ consists of all $\langle x, y \rangle$ such that $\langle y, x \rangle \in \mathbf{D}\sigma$.

13 A.17. *Let $\mathscr{X} = \langle X, \sigma \rangle$ be a categoroid. Let $\bar{\sigma}$ be contragredient to σ. Then $\bar{\mathscr{X}} = \langle X, \bar{\sigma} \rangle$ is a categoroid and $\mathsf{J} : \mathscr{X} \to \bar{\mathscr{X}}$ as well as $\mathsf{J} : \bar{\mathscr{X}} \to \mathscr{X}$ are anti-isomorphisms. Every unit under σ is a unit under $\bar{\sigma}$ and conversely; if $x \in X$ and e is the left (right) unit for x under σ, then e is the right (left) unit for x under $\bar{\sigma}$.*

The proof consists in a straightforward verification of conditions indicated in the corresponding definitions.

Definition. We shall say that the categoroid $\bar{\mathscr{X}}$ described above is *contragredient* (or *opposite*) to $\mathscr{X}$.

13 A.18. *Let $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ be categoroids. Let F, G be covariant or contravariant functors, F on $\mathscr{X}$ into $\mathscr{Y}$, G on $\mathscr{Y}$ into $\mathscr{Z}$. Then $G \circ F$ is either covariant or contravariant: covariant if either both F and G are covariant or both F and G are contravariant; contravariant if F is contravariant and G is covariant or vice versa.*

13 A.19. Proposition and definition. *For $i = 1, 2$, let σ_i be a partial composition in a class X_i. Then the relation consisting of all $\langle\langle x, y\rangle, z\rangle$ such that $x = \langle x_1, x_2\rangle$, $y = \langle y_1, y_2\rangle$, $z = \langle z_1, z_2\rangle$, $z_i = x_i\sigma_i y_i$, is a partial composition in $X_1 \times X_2$. It will be called the* compositional product *of σ_1 and σ_2 and will be denoted by $\sigma_1 \times_{\text{comp}} \sigma_2$ (or by $\sigma_1 \times \sigma_2$ if there is no danger of misunderstanding).*

Remark. The compositional product of two compositions has been defined in 6 E.9. Clearly, the present definition coincides with that given in 6 E.9 if σ_1 and σ_2 are compositions.

13 A.20. Proposition and definition. *For $i = 1, 2$, let $\mathscr{X}_i = \langle X_i, \sigma_i \rangle$ be a categoroid. Then $\mathscr{X} = \langle X_1 \times X_2, \sigma_1 \times_{\text{comp}} \sigma_2 \rangle$ is also a categoroid. It will be called the* product of categoroids *$\mathscr{X}_1$ and $\mathscr{X}_2$.*

The proof is left to the reader (observe that units of $\mathscr{X}$ are precisely those $\langle x_1, x_2 \rangle \in$ $\in X_1 \times X_2$ such that x_i is a unit of $\mathscr{X}_i$).

We omit an investigation of the product of categoroids here. Some facts are given in the exercises.

B. CATEGORIES

Categoroids satisfying certain comprisability conditions and categories, to be introduced below, are, in fact, equivalent concepts, in a sense which will be made clear somewhat later in this subsection.

Before defining categories, we give some important examples describing them in a somewhat informal manner.

13 B.1. (A) The category of all sets. – Consider all sets, calling every set an object of the category in question. Every mapping of a set X into a set Y will be called a morphism; the composition of mappings (see 7 C.1) is taken as the partial composition, in fact a categoroid structure, on the class of all morphisms. To every morphism $f = \langle \varphi, X, Y \rangle$, two objects are assigned, namely the sets $X = \mathbf{D}^*f$, $Y =$ $= \mathbf{E}^*f$; in other words, there is a relation assigning, to every pair $\langle X, Y \rangle$ of objects, those morphisms that transform X into Y. It may happen, in general, that, for certain X, Y, there are no morphisms from X to Y; in the case considered, this happens if and only if $Y = \emptyset$, $X \neq \emptyset$. – (B) The category of all groups. – Every group is an object, every group homomorphism is a morphism of the category in question. The partial composition of morphisms is defined as the composition of mappings. To every pair $\langle G_1, G_2 \rangle$ of groups there are assigned homomorphisms with domain G_1 and ranging in G_2. – (C) The category of all modules over a given ring $\mathscr{A}$. – Every $\mathscr{A}$-module is an object, homomorphisms of $\mathscr{A}$-modules are morphisms; the partial composition of morphisms is defined in the usual way. To every pair $\langle M_1, M_2 \rangle$ of $\mathscr{A}$-modules there correspond homomorphisms with domain-carrier M_1 and range-carrier M_2. – (D) The category of all comprisable structs and their arbitrary mappings. – Every comprisable struct (i.e. every set X and every pair $\langle Y, \alpha \rangle$, Y being a set and α an arbitrary element) is an object of the category. Every mapping $\langle \varphi, \mathscr{X}, \mathscr{Y} \rangle$, where $\mathscr{X}$ and $\mathscr{Y}$ are comprisable structs, is a morphism; the partial composition is defined as the usual composition of mappings. It is clear that the categories described in (A), (B), (C) are subcategories (see 13 B.10) of this category. – (E) Let A be a non-void class and let Φ be a reflexive quasi-order on A. Consider the following category: objects are elements of A; morphisms are elements of Φ (more precisely, $\langle a, b \rangle \in \Phi$ is a morphism from a into b); the partial composition is as described in 13 A.1, example (A).

Summing up, we can say that a category is determined by the following data: a class of "objects", a class of "morphisms" (which are, in most cases, mappings of one "object" into another), a partial composition in the class of all "morphisms", and a relation which determines, for any given "objects" a, b, which "morphisms" are considered as "transforming a into b".

13 B.2. According to the above consideration as well as to the current approach, we should define a category as a pair of classes (the class of "objects" and the class of "morphisms") endowed with two relations: (a) the partial composition of morphisms and (b) a relation assigning to every morphism its "domain object" and "range object", or conversely, relating pairs $\langle x, y \rangle$ of "objects" with those "morphisms" which "transform x into y". A disadvantage of this approach, from the point of view of the present book, consists in the fact that a category conceived in this manner has, so to say, "two underlying classes", the class of "objects" and that of "morphisms". This makes it difficult to include categories, covariant functors for categories, etc., into the general pattern of structs and their correspondences described in Section 7.

Therefore we adopt a formally different definition in which it is unambiguously stressed that a category can and will be considered as a struct.

13 B.3. Definition. A *category* is a quadruple $\langle \Phi, \mu, A, \kappa \rangle$ such that Φ is a non-void class, μ is a relation, A is a class, κ is a relation, and the following conditions are satisfied:

(1) μ is a strongly associative partial composition in Φ;

(2) for every $\varphi \in \Phi$ there exist unit elements (under μ) ε and ε' such that $\langle \varphi, \varepsilon \rangle \in \mathbf{D}\mu$, $\langle \varepsilon', \varphi \rangle \in \mathbf{D}\mu$;

(3) κ is a relation such that $\mathsf{J}_A \subset \mathbf{D}\kappa \subset A \times A$, $\mathbf{E}\kappa = \Phi$; if a, b, c, d belong to A and $\langle\langle a, b\rangle, \varphi\rangle \in \kappa$, $\langle\langle c, d\rangle, \psi\rangle \in \kappa$, then (i) if $b = c$, then $\langle \psi, \varphi \rangle \in \mathbf{D}\mu$, $\langle\langle a, d\rangle, \psi \circ \varphi\rangle \in \kappa$; (ii) if $b \neq c$, then $\langle \psi, \varphi \rangle \notin \mathbf{D}\mu$;

(4) κ is a fibering relation, i.e. $\langle\langle a, b\rangle, \varphi\rangle \in \kappa$, $\langle\langle c, d\rangle, \varphi\rangle \in \kappa$ implies $\langle a, b\rangle = \langle c, d\rangle$;

(5) every fibre $\kappa[\langle x, y\rangle]$ of κ is comprisable.

13 B.4. Definition. Let $\mathscr{K} = \langle \Phi, \mu, A, \kappa \rangle$ be a category. Then the underlying class Φ of the struct $\mathscr{K}$ will be called the *class of morphisms of* $\mathscr{K}$ and every $\varphi \in \Phi$ will be called a *morphism of* $\mathscr{K}$. The relation μ will be called the *partial composition* (*of morphisms*) *of* $\mathscr{K}$ or the *categoroid structure of* $\mathscr{K}$; if $\varphi \in \Phi$, $\psi \in \Phi$, then $\varphi\mu\psi$ will be called the *composite of* φ *and* ψ (*in* $\mathscr{K}$). The pair $\langle \Phi, \mu \rangle$ (which is a categoroid, by conditions (1) and (2)) will be called the *underlying categoroid of* $\mathscr{K}$. The class A will be called the *class of objects of* $\mathscr{K}$, and every $x \in A$ will be called an *object of the category* $\mathscr{K}$. Finally, the relation κ will be called the *assignment relation of* $\mathscr{K}$.

Examples. Consider example (A) from 13 B.1. The category in question, called the category of all sets, will often be denoted by $\mathscr{M}$. Its class of morphisms consists of all comprisable abstract mappings, i.e. mappings of a set into a set. Its partial composition consists of all $\langle\langle f, g\rangle, h\rangle$ where f, g, h are mappings of sets into sets, $h = f \circ g$. The class of objects is the class of all sets and κ consists of all $\langle\langle X, Y\rangle, \langle \varphi, X, Y\rangle\rangle$ where X, Y are sets, φ is a single-valued relation, $\mathbf{D}\varphi = X$, $\mathbf{E}\varphi \subset Y$. – Consider example (E) from 13 B.1, let A be the universal class and let $\Phi = A \times A$. The category in question will sometimes be denoted by $\mathscr{P}$. We have $\mathscr{P} = \langle A \times A, \mu, A, \mathsf{J}_{A \times A}\rangle$ where A is the universal class, μ consists of all elements of the form $\langle\langle\langle b, c\rangle, \langle a, b\rangle\rangle, \langle a, c\rangle\rangle$.

13 B.5. Definition. Let $\mathscr{K} = \langle \Phi, \mu, A, \kappa \rangle$ be a category. If $a \in A$, $b \in A$, then the set $\kappa[\langle a, b\rangle]$ will be denoted by $\mathrm{Hom}_{\mathscr{K}} \langle a, b\rangle$ (the subscript $\mathscr{K}$ will often be omitted) and called the *set of morphisms from* a *to* b. If $\varphi \in \Phi$, $\langle a, b\rangle = \kappa^{-1}\varphi$, then a will be denoted by $\mathrm{D}_{\mathscr{K}}\varphi$ and called the *domain-object of* φ (*in* $\mathscr{K}$); b will be denoted by $\mathrm{E}_{\mathscr{K}}\varphi$ and called the *range-object of* φ (*in* $\mathscr{K}$); the subscript $\mathscr{K}$ will often be omitted. If a morphism $\varphi \in \Phi$ is a unit under μ (see 13 A.3), then we shall say that φ is a *unit morphism* (or simply a *unit*) *of* $\mathscr{K}$; if φ is a unit, $\mathrm{D}\varphi = \mathrm{E}\varphi = a$, we shall say that φ is a *unit at* a.

Examples. If f is a morphism of $\mathscr{M}$, then $\mathrm{D}f = \mathbf{D}^*f$, $\mathrm{E}f = \mathbf{E}^*f$; if A, B are objects of $\mathscr{M}$ (i.e. sets), then $\mathrm{Hom}_{\mathscr{M}} \langle A, B\rangle = \mathbf{F}(A, B)$. – If φ is a morphism of $\mathscr{P}$, then $\varphi = \langle \mathrm{D}\varphi, \mathrm{E}\varphi\rangle$. – If a, b are elements, then $\mathrm{Hom}_{\mathscr{P}} \langle a, b\rangle = (\langle a, b\rangle)$.

13 B.6. *Let* $\mathscr{K} = \langle \Phi, \mu, A, \kappa \rangle$ *be a category. If* φ *is a morphism of* $\mathscr{K}$, e' *is the left* (*i.e. range*) *unit for* φ, *and* e'' *is the right* (*i.e. domain*) *unit for* φ, *then*

$\mathrm{D}\varphi = \mathrm{D}e'' = \mathrm{E}e''$, $\mathrm{E}\varphi = \mathrm{D}e' = \mathrm{E}e'$. *If* φ *and* ψ *are morphisms of* $\mathscr{K}$, *then* $\langle \varphi, \psi \rangle \in \mathrm{D}\mu$ *if and only if* $\mathrm{D}\varphi = \mathrm{E}\psi$. *If* a *is an object of* $\mathscr{K}$, *then there exists exactly one unit at* a; *consequently*, $\{x \to \mathrm{D}x\}$ *is a bijective relation on the class of all units of* $\mathscr{K}$ *onto the class of all objects of* $\mathscr{K}$.

13 B.7. Proposition and definition. *Let* $\mathscr{K} = \langle \Phi, \mu, A, \kappa \rangle$ and $\mathscr{K}' = \langle \Phi', \mu', A', \kappa' \rangle$ *be categories. If* τ *is a* (μ, μ')*-covariant single-valued relation with* $\mathbf{D}\tau \supset \Phi$, *then there exists exactly one relation* $\bar{\tau}$ *on* A *into* A' *such that* $\varphi \in \mathrm{Hom}_{\mathscr{K}} \langle a, b \rangle$ *if and only if* $\tau\varphi \in \mathrm{Hom}_{\mathscr{K}'} \langle \bar{\tau}a, \bar{\tau}b \rangle$. *This relation* $\bar{\tau}$ *is single-valued. It will be called the* associated (with τ) relation for objects *of* $\mathscr{K}$ *and* $\mathscr{K}'$.

Proof. Let φ, ψ be morphisms of $\mathscr{K}$, $\mathrm{D}\varphi = \mathrm{D}\psi$. Then the right unit e for φ coincides with the right unit for ψ (see 13 B.6), $\mathrm{D}e = \mathrm{D}\varphi = \mathrm{D}\psi$. Clearly, τe is the right unit (in $\mathscr{K}'$) for $\tau\varphi$ as well as for $\tau\psi$ and therefore $\mathrm{D}(\tau\varphi) = \mathrm{D}(\tau e) = \mathrm{D}(\tau\psi)$. Now, we put $\bar{\tau}a = \mathrm{D}\varphi$ where φ is a morphism with $\mathrm{D}\varphi = a$ (e.g. φ may be the unit at a). The rest of the proof is left to the reader.

13 B.8. Definition. Let $\mathscr{K} = \langle \Phi, \mu, A, \varkappa \rangle$ and $\mathscr{K}' = \langle \Phi', \mu', A', \varkappa' \rangle$ be categories. A mapping $\langle \tau, \mathscr{K}, \mathscr{K}' \rangle$ will be called *covariant* or a *covariant functor* if $\tau : \langle \Phi, \mu \rangle \to \langle \Phi', \mu' \rangle$ is covariant (see 13 A.10); *contravariant* or a *contravariant functor* if $\tau : \langle \Phi, \mu \rangle \to \langle \Phi', \mu' \rangle$ is contravariant (see 13 A.14); an *isomorphism functor* (or simply an *isomorphism*) if $\tau : \langle \Phi, \mu \rangle \to \langle \Phi', \mu' \rangle$ is an isomorphism functor (see 13 A.13); an *anti-isomorphism functor* (or simply an *anti-isomorphism*) if $\tau : \langle \Phi, \mu \rangle \to \langle \Phi', \mu' \rangle$ is an anti-isomorphism functor (see 13 A.15). Two categories $\mathscr{K}$, $\mathscr{K}'$ will be called *isomorphic* (respectively, *anti-isomorphic*) if there exists an isomorphism (an anti-isomorphism) of $\mathscr{K}$ onto $\mathscr{K}'$.

Remarks. 1) If $\mathscr{K}_1$, $\mathscr{K}_2$, $\mathscr{K}_3$ are categories, $f_1 = \langle \tau_1, \mathscr{K}_1, \mathscr{K}_2 \rangle$, $f_2 = \langle \tau_2, \mathscr{K}_2, \mathscr{K}_3 \rangle$ are covariant functors, then $f_3 = f_2 \circ f_1$ is a covariant functor. Denote by $\bar{\tau}_i$ the relation associated with τ_i in the sense of 13 B.7; it is easy to see that $\bar{\tau}_3 = \bar{\tau}_2 \circ \bar{\tau}_1$. — 2) Clearly, a mapping f of a category $\mathscr{K}$ into a category $\mathscr{K}'$ is an isomorphism if and only if both f and f^{-1} are covariant functors. — 3) It is to be pointed out that if $\mathscr{K} = \langle \Phi, \mu, A, \kappa \rangle$, $\mathscr{K}' = \langle \Phi, \mu, A', \kappa' \rangle$ are categories, then the relation for objects of $\mathscr{K}$ and $\mathscr{K}'$ associated with J is bijective for A and A' but is not, of course, an identical relation in general.

Conventions. If $\mathscr{K} = \langle \Phi, \mu, A, \kappa \rangle$, $\mathscr{K}' = \langle \Phi', \mu', A', \kappa' \rangle$ are categories, then a (μ, μ')-covariant relation τ will also be called $(\mathscr{K}, \mathscr{K}')$*-covariant*; if $\mathbf{D}\tau \supset \Phi$, then the relation $\bar{\tau}$ described in 13 B.7 will also be called $(\mathscr{K}, \mathscr{K}')$*-associated with* τ or *with* $f = \tau : \mathscr{K} \to \mathscr{K}'$.

13 B.9. Theorem. *Let* $\mathscr{K}^* = \langle \Phi, \mu \rangle$ *be a categoroid. Then* $\mathscr{K}^*$ *is an underlying categoroid of a category if and only if, for any* $\alpha \in \Phi$, $\beta \in \Phi$ *the class of elements* $\varphi \in \Phi$ *such that* $\langle \alpha, \varphi \rangle \in \mathbf{D}\mu$, $\langle \varphi, \beta \rangle \in \mathbf{D}\mu$, *is comprisable. If* $\mathscr{K}_1$, $\mathscr{K}_2$ *are categories, and* $\mathscr{K}_1^*$, $\mathscr{K}_2^*$ *are the underlying categoroids of* $\mathscr{K}_1$, $\mathscr{K}_2$, *then a mapping*

$F : \mathscr{K}_1 \to \mathscr{K}_2$ is an isomorphism if and only if $F : \mathscr{K}_1^* \to \mathscr{K}_2^*$ is an isomorphism. In particular, if $\mathscr{K}_1^*$, $\mathscr{K}_2^*$ coincide, then $\mathscr{K}_1$, $\mathscr{K}_2$ are isomorphic.

Proof. I. Let $\mathscr{K}^* = \langle \Phi, \mu \rangle$ be a categoroid. Let A denote the class of all units of $\mathscr{K}^*$. Let κ consist of all $\langle\langle a, b\rangle, \varphi\rangle$ such that $a \in A$, $b \in A$, $\varphi \in \Phi$, a is the domain unit and b is the range unit for φ. It is easy to see that conditions (1)–(4) from 13 B.3 are satisfied and that condition (5) is fulfilled if and only if the condition of the present theorem is satisfied. The rest of the proof is left to the reader.

Remark. In view of the theorem just proved, terms and symbols introduced for categoroids (for categories) will often be tacitly considered as also defined for categories (respectively, for categoroids).

13 B.10. Definition. Let $\mathscr{K} = \langle \Phi, \mu, A, \kappa \rangle$, $\mathscr{K}' = \langle \Phi', \mu', A', \kappa' \rangle$ be categories. If $\langle \Phi', \mu' \rangle$ is a subcategoroid of $\langle \Phi, \mu \rangle$, $A' \subset A$ and $\kappa' \subset \kappa$, then we shall say that $\mathscr{K}'$ is a *subcategory of* $\mathscr{K}$.

For example, as mentioned in 13 B.1, example (D), the category of all sets, as well as the category of all groups and of all $\mathscr{A}$-modules (see 13 B.1), is a subcategory of the category of all structs.

13 B.11. *Let $\mathscr{K}$ be a category; let Φ by the class of morphisms and let A be the class of objects of the category $\mathscr{K}$. Let $\Phi' \subset \Phi$, $A' \subset A$. Then the following conditions are necessary and sufficient for the existence of a subcategory $\mathscr{K}'$ of $\mathscr{K}$ such that Φ' is the class of morphisms and A' is the class of objects of $\mathscr{K}'$:*

(1) *Φ' is a subcategoroid of Φ, which means that:*

(1a) $\Phi' \neq \emptyset$; (1b) *if $\varphi_1 \in \Phi'$, $\varphi_2 \in \Phi'$, $\langle \varphi_1, \varphi_2 \rangle \in \mathbf{D}\sigma$, then the composite of φ_1 and φ_2 belongs to Φ'*; (1c) *if $\varphi \in \Phi'$, then the left as well as the right unit for φ (in $\mathscr{K}$) belongs to Φ.*

(2) *A' consists of all $a \in A$ such that the unit at a (in $\mathscr{K}$) belongs to Φ'.*

If these conditions are satisfied then there exists exactly one subcategory of $\mathscr{K}$ for which Φ' is the class of morphisms and A' is the class of objects.

Proof. Let (1) and (2) be satisfied. Put $\mu' = ((\Phi' \times \Phi') \times \Phi') \cap \mu$, and let κ' be the range-restriction of κ to Φ'. It is easy to show that $\mathscr{K}' = \langle \Phi', \mu', A', \kappa' \rangle$ is a subcategory of $\mathscr{K}$. The rest of the proof is left to the reader.

Definition. The category $\mathscr{K}'$ described above will be called the *subcategory of $\mathscr{K}$ generated by Φ' and A'*.

13 B.12. Definition. Let $\mathscr{K} = \langle \Phi, \mu, A, \kappa \rangle$ be a category. A subcategory $\mathscr{K}' = \langle \Phi', \mu', A', \kappa' \rangle$ of $\mathscr{K}$ will be called a *full* (more explicitly, *morphism-full*) subcategory if $a \in A'$, $b \in A'$, $\langle a, b \rangle = \kappa^{-1}\varphi$ implies $\varphi \in \Phi'$; it will be called an *object-full* subcategory if $A' = A$.

If $\mathscr{K} = \langle \Phi, \mu, A, \kappa \rangle$ is a category and $\emptyset \neq A' \subset A$, then there exists exactly one full subcategory $\mathscr{K}'$ of $\mathscr{K}$ such that the class of objects of $\mathscr{K}'$ is equal to A'.

Proof. It is sufficient to put $\Phi' = \kappa[A' \times A']$ and to use proposition 13 B.11.

Convention. Let $\mathscr{K} = \langle \Phi, \mu, A, \kappa \rangle$ be a given category. If $\Phi' \subset \Phi$ is a subcategoroid, that is if conditions (1a), (1b), (1c) from 13 B.11 are satisfied, then the subcategory generated by Φ' and A will be called the (*object-full*) *subcategory generated by* Φ' or also, not quite correctly, "the category of all $\varphi \in \Phi'$" (e.g. we speak of the "category of all injective homomorphisms of abelian groups"). If $A' \subset A$, then the subcategory generated by A' and $\Phi' = \kappa[A' \times A']$ will be called the (*morphism-full*) *subcategory generated by* A' or also, not quite correctly, "the category of all $a \in A'$" (e.g. we speak of the "category of all finite groups" provided the category of all structs described in 13 B.1, example (D) is considered as given).

Examples. Consider the category of all sets (see 13 B.1, example (A)). The category of all subsets of a given set A is comprisable. The category of all singletons is isomorphic with the category $\mathscr{P}$ (see 13 B.4, examples). The category of all mappings of the form $\mathrm{J} : X \to Y$ is isomorphic with a category, the underlying categoroid of which is of the form $\langle \Phi, \sigma \rangle$ described in 13 A.1, example (A), where $\Phi = \{X \to Y \mid X \subset Y\}$.

13 B.13. To conclude this subsection, we are going to carry over to categories some definitions introduced for categoroids in 13 A.17, and 13 A.20.

Proposition and definition. *Let* $\mathscr{K} = \langle \Phi, \mu, A, \kappa \rangle$ *be a category; let* $\bar{\mu}$ *be contragredient to* μ *and let* $\bar{\kappa}$ *consist of all* $\langle \langle a, b \rangle, \varphi \rangle$ *such that* $\langle \langle b, a \rangle, \varphi \rangle \in \kappa$. *Then* $\langle \Phi, \bar{\mu}, A, \bar{\kappa} \rangle$ *is a category; it will be called* contragredient *or* opposite to $\mathscr{K}$.

13 B.14. Proposition and definition. *For* $i = 1, 2$, *let* $\mathscr{K}_i = \langle \Phi_i, \mu_i, A_i, \kappa_i \rangle$ *be a category. Then* $\langle \Phi_1 \times \Phi_2, \mu_1 \times_{\mathrm{comp}} \mu_2, A_1 \times A_2, \kappa \rangle$, *where* κ *consists of all* $\langle \langle \langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \rangle, \langle \varphi_1, \varphi_2 \rangle \rangle$ *such that* $\langle a_i, b_i \rangle \kappa_i \varphi_i$, $i = 1, 2$, *is a category. It will be called the* product of categories $\mathscr{K}_1$ *and* $\mathscr{K}_2$, *and will be denoted by* $\mathscr{K}_1 \times \times \mathscr{K}_2$.

C. PROPERTIES OF MORPHISMS

We shall now consider some properties of morphisms (of a given category) which are related to the property (of elements of a class endowed with a composition) of being invertible (see 6 B.10) or virtually invertible (see 6 B.11), and which correspond to the properties (of mappings) of being injective or surjective.

13 C.1. Definition. Let $\mathscr{K}^* = \langle \Phi, \mu \rangle$ be a categoroid. If $\langle \xi, \eta \rangle \in \mathrm{D}\mu$ and $\xi \, . \, \eta$ is a unit (in $\mathscr{K}^*$), then we shall say that ξ is a *left inverse of* η (*in* $\mathscr{K}^*$) and that η is a *right inverse of* ξ (*in* $\mathscr{K}^*$). If $\mathscr{K} = \langle \Phi, \mu, A, \kappa \rangle$ is a category, then we shall say that α is a *left* (*right*) *inverse of* β *in* $\mathscr{K}$ if α is a left (right) inverse of β in the categoroid $\langle \Phi, \mu \rangle$.

If $\xi \in \Phi$ and there exists an η such that $\xi \, . \, \eta$ is a unit (i.e. if ξ is a left inverse of some η), then we shall say that ξ is a *left inverse* (*in* $\mathscr{K}^*$ or *in* $\mathscr{K}$) or that ξ *possesses a right inverse* (*in* $\mathscr{K}^*$ or *in* $\mathscr{K}$). Similarly, if $\xi \in \Phi$ and there exists a ζ such that $\zeta \, . \, \xi$ is a unit, we shall say that ξ *possesses a left inverse* or that ξ *is a right inverse in* $\mathscr{K}$.

We avoid the use of terms "left-invertible", "right-invertible" which may be confusing.

13 C.2. A morphism may possess none, one, or many left (right) inverses. However, the following proposition holds:

If $\mathscr{K}^*$ is a categoroid, α, ξ, η are morphisms of $\mathscr{K}^*$, ξ is a left inverse and η is a right inverse of α, then $\xi = \eta$. Consequently, if α has a left as well as a right inverse, then it possesses exactly one left and exactly one right inverse, and these inverses coincide.

Indeed, $\xi = \xi . (\alpha . \eta) = (\xi . \alpha) . \eta = \eta$.

13 C.3. Examples. (A) Consider the category $\mathscr{M}$ of all sets (see 13 B.1, example (A)). If f, g are mappings (of sets into sets) and $f \circ g$ is a unit (in $\mathscr{M}$), then $f(gx) = x$ for every $x \in \mathbf{D}g$ and therefore g is injective, f is surjective. It is easy to see that, in $\mathscr{M}$, a mapping f is (1) a left inverse (in other words, possesses a right inverse) if and only if it is surjective, (2) a right inverse if and only if it is injective. – (B) Consider the category $\mathscr{G}$ of all groups (see 13 B.1, example (B)). If φ, ψ are morphisms of $\mathscr{G}$ and $\varphi \circ \psi$ is a unit morphism, then it is easy to see that ψ is injective, φ is surjective. Let $\psi = \psi : G \to H$, $\varphi = \varphi : H \to G$; put $A = \psi[G]$, $B = \varphi^{-1}[0]$. It is easy to prove that A, B are subgroups of H, that B is invariant (see e.g. 8 D.8) and that every $x \in H$ may be uniquely expressed as a composite of u and v, $u \in A$, $v \in B$. Thus an injective homomorphism need not be a right inverse in $\mathscr{G}$; if e.g. H has no non-trivial invariant subgroups, then a morphism $\psi : G \to H$ is a right inverse in $\mathscr{G}$ if and only if either G contains exactly one element or ψ is an isomorphism.

13 C.4. Definition. Let $\mathscr{K}$ be a category. A morphism φ of $\mathscr{K}$ will be called an *isomorphism* (or an *invertible morphism*) *of* $\mathscr{K}$ (or of the underlying categoroid $\mathscr{K}^*$) if there exists a morphism ψ such that $\varphi . \psi$, $\psi . \varphi$ are unit morphisms. If this is the case, then the morphism ψ (which is unique, by 13 C.2) will be called the *inverse of* φ and will be denoted by φ^{-1}.

Remark. The word "isomorphism" is used in a twofold sense here: it means either a morphism satisfying certain condition or a certain mapping of a category onto another one. If a misunderstanding seems possible, we shall use terms "isomorphism-functor" and "invertible morphism".

Convention. A left (right) inverse in a category $\mathscr{K}$ will also be called a *left (right) semi-isomorphism of* $\mathscr{K}$.

The proof of the following propositions is left to the reader.

(a) *A morphism is an isomorphism if and only if it is a left as well as a right semi-isomorphism.*

(b) *If $\mathscr{K}$ and $\overline{\mathscr{K}}$ are contragredient categories, then every left (right) semi-isomorphism of $\mathscr{K}$ is a right (left) semi-isomorphism of $\overline{\mathscr{K}}$.*

(c) *If $\overline{\mathscr{K}}$ is a subcategory of $\mathscr{K}$, then every left (right) semi-isomorphism of $\overline{\mathscr{K}}$ is a left (right) semi-isomorphism of $\mathscr{K}$.*

The converse assertion does not hold, however. Example: the category of all injective mappings of sets.

13 C.5. *Let* φ and ψ *be morphisms of a category and let* $\varphi \,.\, \psi$ *exist. If* φ *and* ψ *are left (respectively, right) semi-isomorphisms, then* $\varphi \,.\, \psi$ *is a left (respectively, right) semi-isomorphism; if* $\varphi \,.\, \psi$ *is a left (respectively, right) semi-isomorphism, then* ψ *is a left semi-isomorphism (respectively,* φ *is a right semi-isomorphism).*

The proof is left to the reader.

13 C.6. Definition. Let μ be a partial composition in a class X. An element $x \in X$ is called *idempotent under* μ (or *in* $\langle X, \mu \rangle$) if $\langle x, x \rangle \in \mathbf{D}\mu$, $x\mu x = x$.

If φ *and* ψ *are morphisms of a category* $\mathscr{K}$ *and* $\varphi \,.\, \psi$ *is a unit, then* $\psi \,.\, \varphi$ *is idempotent. A semi-isomorphism (left or right) is idempotent if and only if it is a unit morphism.*

Proof. If $\varphi \,.\, \psi$ is a unit, then $(\psi \,.\, \varphi) \,.\, (\psi \,.\, \varphi) = \psi \,.\, (\varphi \,.\, \psi) \,.\, \varphi = \psi \,.\, \varphi$. Let φ be a left semi-isomorphism, $\varphi \,.\, \varphi = \varphi$; choose ψ such that $\varphi \,.\, \psi$ is a unit. Then $\varphi \,.\, (\varphi \,.\, \psi) = \varphi$, $(\varphi \,.\, \varphi) \,.\, \psi = \varphi \,.\, \psi$, hence $\varphi = \varphi \,.\, \psi$.

13 C.7. Definition. Two objects a and b of a category $\mathscr{K}$ are called *isomorphic in* $\mathscr{K}$ if there exists an isomorphism φ with $\mathrm{D}\varphi = a$, $\mathrm{E}\varphi = b$.

Examples. In the category of all sets, two sets X, Y are isomorphic if and only if $\operatorname{card} X = \operatorname{card} Y$. In this category, the following assertion holds: if there exists a left (right) semi-isomorphism from X to Y as well as a left (right) semi-isomorphism from Y to X, then X and Y are isomorphic. In the category of all groups, this assertion is not valid.

13 C.8. Now we introduce, in a somewhat informal manner, the category of quasi-ordered sets and some of its subcategories which, besides being quite important in themselves, yield various useful examples.

The category of all quasi-ordered sets is a subcategory of the category of all structs; its objects are quasi-ordered sets, and the morphisms from a quasi-ordered set A to a quasi-ordered set B are mappings $\langle \varphi, A, B \rangle$ such that if $x\varrho y$, then $(\varphi x)\, \sigma\, (\varphi y)$ or $\varphi x = \varphi y$, where ϱ is the quasi-order of A, σ is the quasi-order of B. Thus every order-preserving mapping of A into B is a morphism, but a morphism is not necessarily an order-preserving mapping.

The following useful subcategories of the category of all quasi-ordered sets may be mentioned: the category of all order-preserving mappings, the category of all lattice-preserving mappings, the category of all completely lattice-preserving mappings, the category of all ordered sets, the category of all monotone ordered sets, the category of all well-ordered sets.

It is easy to prove that, in the category of all monotone ordered sets, a morphism is a right semi-isomorphism if and only if it is a surjective mapping, and an injective order-preserving mapping f of X into a well-ordered set Y is a left semi-isomorphism if and only if $f[X]$ either possesses a least element or is cofinal in Y.

13 C.9. Definition. A morphism φ of a category $\mathscr{K}$ will be called a *monomorphism* (*of* $\mathscr{K}$) if $\xi = \eta$ whenever ξ and η are morphisms of $\mathscr{K}$ and $\varphi . \xi = \varphi . \eta$; it will be called an *epimorphism* (*of* $\mathscr{K}$) if $\xi = \eta$ whenever ξ and η are morphisms of $\mathscr{K}$ and $\xi . \varphi = \eta . \varphi$.

Examples. (A) In the category of all sets, monomorphisms coincide with injective mappings, epimorphisms with surjective mappings. – (B) In any category of structs (this means, in every subcategory of the category of all structs as defined in 13 B.1, example (D)) every injective mapping is a monomorphism and every surjective mapping is an epimorphism. Indeed, supposing $\varphi . \xi = \varphi . \eta$, $\xi \neq \eta$, we get $\mathbf{D}^*\xi = \mathbf{D}^*\eta$, $\xi a \neq \eta a$ for some $a \in |\mathbf{D}^*\xi|$ and therefore $\varphi(\xi a) \neq \varphi(\eta a)$, which is a contradiction. – (C) There are important categories of structs for which the converse holds, that is (i) monomorphisms are injective, (ii) epimorphisms are surjective mappings. As a matter of fact, (i) seems to hold in all categories of algebraic structs occurring "in practice". However, (ii) does not hold even for semi-groups. Indeed, let $\varphi = \mathrm{J} : \langle \mathrm{N}, + \rangle \to \langle \mathrm{Z}, + \rangle$; let ξ, η be homomorphisms of $\langle \mathrm{Z}, + \rangle$ onto a semi-group $\mathscr{H}$, and suppose $\xi . \varphi = \eta . \varphi$. Then (denoting by $+$ the composition of $\mathscr{H}$) we get $\xi(-1) = \xi(-1) + \eta 1 + \eta(-1) = (\xi(-1) + \xi 1) + \eta(-1) = \eta(-1)$, hence $\xi(-n) = \eta(-n)$ for any $n \in \mathrm{N}$; thus, $\xi = \eta$. – (D) Consider the category of all ordered sets. Then a morphism φ is a monomorphism (an epimorphism) if and only if it is an injective (surjective) mapping. Indeed, let, for instance, $\varphi : \langle A, \varrho \rangle \to \langle B, \sigma \rangle$ be an epimorphism and suppose that φ is not surjective. Choose $b \in B - \varphi[A]$; let B be embedded in $\langle B^*, \sigma^* \rangle$ where $B^* = B \cup (b^*)$, b^* non $\in B$, and let σ^* consist of (i) all $\langle x, y \rangle \in \sigma$, (ii) all $\langle b^*, y \rangle$ where $b \sigma y$, $y \neq b$, (iii) all $\langle x, b^* \rangle$ where $x \sigma b$, $x \neq b$, (iv) the element $\langle b^*, b^* \rangle$. Let ξ, η be mappings of $\langle B, \sigma \rangle$ into $\langle B, \sigma^* \rangle$, $\xi z = \eta z = z$ for $z \neq b$, $\xi b = b$, $\eta b = b^*$. Then ξ, η are morphisms, $\xi . \varphi = \eta . \varphi$, $\xi \neq \eta$.

Remark. The concept of a right semi-isomorphism is too narrow to cover, in "practically" important cases, all mappings currently considered as embeddings ("isomorphic mappings into"); cf. 13 C.3, example (B). On the other hand, the broader concept (see 13 C.12) of a monomorphism seems too wide (e.g. in the category of ordered sets, $\mathrm{J} : \langle A, \mathrm{J}_A \rangle \to \langle A, \sigma \rangle$ is a monomorphism and an epimorphism, for any set A and any order σ on A). An analogous remark applies for left semi-isomorphisms and epimorphisms.

13 C.10. *Let φ and ψ be morphisms of a category $\mathscr{K}$ and let $\varphi . \psi$ exist. If φ and ψ are monomorphisms (respectively, epimorphisms), then $\varphi . \psi$ is a monomorphism (respectively, an epimorphism). If $\varphi . \psi$ is a monomorphism (respectively, an epimorphism), then ψ is a monomorphism (respectively φ is an epimorphism).*

Proof. If φ, ψ are monomorphisms, let $(\varphi . \psi) . \xi = (\varphi . \psi) . \eta$; then $\varphi . (\psi . \xi) = \varphi . (\psi . \eta)$, hence $\psi . \xi = \psi . \eta$ and therefore $\xi = \eta$. If $\varphi . \psi$ is a monomorphism, let $\psi . \xi = \psi . \eta$; then $\varphi . \psi . \xi = \varphi . \psi . \eta$, hence $\xi = \eta$.

13 C.11. *Let $\mathscr{K}$ and $\overline{\mathscr{K}}$ be contragredient categories. Then every monomorphism (epimorphism) of $\mathscr{K}$ is an epimorphism (a monomorphism) of $\mathscr{K}$.*

This proposition (the proof of which is immediate) makes it possible to obtain propositions on epimorphisms from propositions on monomorphisms and vice versa, as well as to perform proofs, in many cases, for monomorphisms or for epimorphisms only (such a "duality" also holds for left and right semi-isomorphisms, as indicated in 13 C.4, and for strong monomorphisms and strong epimorphisms, see below, 13 C.16).

13 C.12. *Let $\mathscr{K}$ be a category. Every left (right) semi-isomorphism φ of $\mathscr{K}$ is an epimorphism (a monomorphism) of $\mathscr{K}$. If, in addition, φ is a monomorphism (an epimorphism), then it is an isomorphism.*

Proof. Let φ be a left semi-isomorphism; let φ, ψ be a unit. If $\xi . \varphi = \eta . \varphi$, then $\xi . \varphi . \psi = \eta . \varphi . \psi$, hence $\xi = \eta$. If, in addition, φ is a monomorphism, let ε be the right unit for φ. Then $\varphi . \psi . \varphi = \varphi = \varphi . \varepsilon$, hence, φ being a monomorphism, we have $\psi . \varphi = \varepsilon$.

13 C.13. *If $\mathscr{K}$ is a category, $\mathscr{K}'$ is a subcategory of $\mathscr{K}$ and $\varphi \in \mathscr{K}'$ is a monomorphism (an epimorphism) of $\mathscr{K}$, then φ is a monomorphism (an epimorphism) of $\mathscr{K}'$.*

This is clear. – Observe that the converse does not hold, in general.

13 C.14. Definition. If φ is a monomorphism as well as an epimorphism of a category $\mathscr{K}$, then φ is called a *bimorphism of* $\mathscr{K}$.

Examples. (A) In the category of all ordered sets, bimorphisms coincide with bijective order-preserving mappings (see 13 C.9, example (D)), but a bimorphism need not be an isomorphism (see 10 C.2, remark). – (B) It can be proved that, in the category of all groups, bimorphisms and isomorphisms coincide. – (C) In the category of all semi-groups, $J : \langle N, + \rangle \to \langle Z, + \rangle$ is a bimorphism.

Remark. Clearly, if φ is a bimorphism of $\mathscr{K}$, then it is also a bimorphism of the contragredient category $\overline{\mathscr{K}}$,

13 C.15. *Let φ and ψ be morphisms of category $\mathscr{K}$ and let $\varphi . \psi$ exist. If φ, ψ are bimorphisms, then $\varphi . \psi$ is also a bimorphism. If $\varphi . \psi$ is a bimorphism, then φ is an epimorphism and ψ is a monomorphism; if, in addition, $\psi . \varphi$ is also a bimorphism, then φ and ψ are bimorphisms.*

13 C.16. Definition. A morphism φ of a category $\mathscr{K}$ will be called a *strong monomorphism of* $\mathscr{K}$ *if* (1) φ is a monomorphism, (2) if ξ, η are monomorphisms $\varphi = \eta . \xi$ and ξ is an epimorphism, then ξ is an isomorphism. A morphism ψ of a category $\mathscr{K}$ will be called a *strong epimorphism of* $\mathscr{K}$ if (1) ψ is an epimorphism, (2) if ξ, η are epimorphisms, $\psi = \eta . \xi$ and η is a monomorphism, then η is an isomorphism.

Remarks. 1) Every right semi-isomorphism is a strong monomorphism (see below, 13 C.17) and, evidently, every strong monomorphism is a monomorphism, and similarly for epimorphisms. – 2) In some important cases (see the example below), strong monomorphisms (epimorphisms) correspond to the concept of embedding (respectively of mapping onto a "quotient", cf. e.g. 8 C.12), whereas the concept of a monomorphism or epimorphism is, as pointed out, too wide, that of a semi-isomorphism too restrictive. – 3) It is easy to see that a strong monomorphism of $\mathscr{K}$ is a strong epimorphism of the contragredient category $\overline{\mathscr{K}}$ and vice versa.

Example. Consider the category of all semi-groups. It is clear that monomorphisms, strong monomorphisms and injective homomorphisms coincide. As we know (13 C.9, example (C)) an epimorphism need not be a surjective mapping. Let $f = f : \mathscr{G} \to \mathscr{H}$ be a strong epimorphism. Put $\mathscr{H}' = f[\mathscr{G}]$; $\mathscr{H}'$ is clearly a semi-group. Then $f_1 = f : \mathscr{G} \to f[\mathscr{G}]$ is an epimorphism (since it is a surjective mapping), $f_2 = \mathrm{J} : f[\mathscr{G}] \to \mathscr{H}$ is clearly an epimorphism, and $f = f_2 \circ f_1$. Hence f_1 is an isomorphism and therefore a surjective mapping. We have shown that, for the category of all semi-groups, strong epimorphisms and surjective homomorphisms coincide.

13 C.17. *Every left (right) semi-isomorphism is a strong epimorphism (strong monomorphism).*

Proof. Let ψ be a right semi-isomorphism, let $\psi . \varphi$ be a unit. Let $\varphi = \eta . \xi$, where ξ, η are monomorphisms and, in addition, ξ is an epimorphism. Then $\psi . \eta . \xi$ is a unit, $\xi . \psi . \eta . \xi = \xi$, hence, ξ being an epimorphism, $\xi . \psi . \eta$ is a unit and therefore $\psi . \eta$ is an inverse of ξ, ξ is an isomorphism. The rest of the proof is omitted in view of 13 C.16, remark 3.

Remark. We observe that, in fact, we have proved more than was asserted; namely we have shown that every right semi-isomorphism φ possesses the following property:

(*) if $\varphi = \eta . \xi$ where ξ is an epimorphism, then ξ is an isomorphism.

Clearly, every monomorphism satisfying condition (*) is a strong monomorphism. An analogous remark applies, of course, for epimorphisms.

13 C.18. *Let α, β, γ be morphisms of a category $\mathscr{K}$, let $\alpha = \beta . \gamma$ and let α be a strong monomorphism. If β is a monomorphism, then γ is a strong monomorphism; if γ is an epimorphism, then β is a strong monomorphism. An analogous assertion holds if α is supposed to be a strong epimorphism.*

The proof is left to the reader.