

# Forgotten mathematician Henry Lowig (1904–1995)

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Löwig's works in algebra

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## LÖWIG'S WORK IN ALGEBRA

By the end of the thirties, Löwig's interest turned towards abstract algebra. The transition to this subject was probably facilitated by his study of papers dealing with Boolean rings and the rapidly developing lattice theory.<sup>1</sup> The authors of those papers were primarily Garrett Birkhoff (1911–1996), Edward Vermilye Huntington (1874–1952), Øystein Ore (1899–1968) and Marshall Harvey Stone (1903–1989).

Books dealing with lattice theory, its origin and problems include [Me], [Kö], [HK], [B11], [B12], [B13], [G4], [G5], [G7], [Ro], [ABH]. The following textbooks and monographs are recommended for those interested in the theory of lattices and Boolean algebras: [BD], [B10], [G1], [G2], [G3], [G6], [G8], [H], [Si], [Sz].

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In his papers on lattices and Boolean algebras, Löwig primarily follows the treatise of Birkhoff *On the structure of abstract algebras* [B5] published in 1935, which he referred to in his papers [L10], [L14], [L15], [L16], [L17], [L18], [L21] and [L23]. He also mentions Birkhoff's article *Rings of sets* [B8] published in 1937 in the References of his papers [L10], [L12], [L13] and [L14]. In addition, in [L10] he cites Birkhoff's papers *On the combination of topologies* [B7] published in 1936 and *Lattices and their applications* [B9] published in 1938. A reference to Birkhoff's monograph *Lattice Theory* (3rd edition) published in 1963 appears in his paper [L23].

In his article [L10], he also refers to van Dantzig's paper *Zur topologischen Algebra I. Kompletierungstheorie* [Da] published in 1933, to Fritz Klein's paper *Boole-Schrödersche Verbände* [K] published in 1936 and to the paper *Lineare halbgeordnete Räume* [Ka] by Kantorovič published in 1937.

In [L11], Löwig refers to Dedekind's treatise *Über die von drei Moduln erzeugte Dualgruppe* [De2] of 1900 and to Ore's work *On the foundation of abstract algebra I* [Or] published in 1935. Furthermore, in his paper [L12], he refers to Leopold Löwenheim's paper *Über die Auflösung von Gleichungen im logischen Gebietekalkül* [L] published in 1910, to Norbert Wiener's work *Certain formal invariances in Boolean algebras* [Wi] of 1917 and to the paper entitled *A generalization of the syllogism* [Be1] of B. A. Bernstein published in 1924. In [L13], he refers to Ore's work *On the foundation of abstract algebra I* [Or] of 1935.

In his paper [L19], Löwig makes a reference to George Neal Raney's paper *Completely distributive complete lattices* [Ra] published in 1952, in [L20]

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<sup>1</sup> Let us mention that in the papers reflecting the beginnings of functional analysis – the papers that Löwig followed – one can find also some considerations concerning the lattices of subspaces of Hilbert spaces.

and [L21] to Robert Kerkhoff's paper *Eine Konstruktion absolut freier Algebren* [Ke1] of 1965, and in [L10] and [L24] to the treatise *The theory of representations for Boolean algebras* [St1] of Marshall H. Stone published in 1936.

In several of his papers Löwig refers to the following monographs: in [L14] to *Mengenlehre* [Hau] of Felix Hausdorff (1868–1942) published in 1944, in [L16] to *Grundlagen der Analysis* [La] of Edmund Landau (1877–1938) published in 1930, in [L22] to *Lectures on Continuous Geometry* [No1] and to the monograph *Continuous Geometry* [No2] of John von Neumann (1903–1957) published in 1937 and 1960 respectively, as well as in [L24] to *Introduction to Lattice Theory* [Sz] of Gábor Szász published in 1963 and to *Distributive Lattices* [BD] of Raymond Balbes and Philip Dwinger published in 1974.

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In 1941, Löwig published an extensive study entitled *Intrinsic topology and completion of Boolean rings* [L10] in the American Annals of Mathematics; it concerned an extension of a Boolean ring to a  $\sigma$ -complete Boolean ring in which the original ring is dense. To this aim, he used a method based on the concept of an inner element that resembles the construction of the real numbers based on the concept of Cauchy sequences. In this study, Löwig generalized a number of results from the theory of Boolean  $\sigma$ -rings obtained earlier, by Holbrook Mann MacNeille (1907–1973) in his *Partially ordered sets* [MN], by John von Neumann in his *Lectures on Continuous Geometry* [No1] and by Stone in his treatise *Algebraic characterizations of special Boolean rings* [St2], to general Boolean rings.

Löwig's paper [L10] contains 151 theorems and is characterized by an extraordinary volume of detail. This is particularly evident by the painstaking care which Löwig demonstrates in the presentation of statements to the point where he could be viewed as being somewhat pedantic, a trait that is rather typical of all of Löwig's publications. For example, Theorem 108 contains 17 inequalities concerning the lower and upper limits of sequences. Such care necessarily results in a lengthy presentation that replicates the work already delivered in his paper *Komplexe euklidische Räume von beliebiger endlicher oder transfiniten Dimensionszahl* [L6] in comparison with the paper of Friedrich Riesz (1880–1956) entitled *Zur Theorie des Hilbertischen Raumes* that follows Löwig's article in *Acta Litterarum Scientiarum Szeged* in the year 1934.

In 1951, Löwig also published a short note dealing with Boolean algebras entitled *On transitive Boolean relations* [L12]. Here, he gave necessary and sufficient conditions in order that the relation defined by

$$xRy \stackrel{\text{def}}{=} axy + bx\bar{y} + c\bar{x}y + d\bar{x}\bar{y} = 0$$

be a transitive relation, or an equivalence, or a quasi-ordering, or a partial ordering of a Boolean algebra.

This publication [L12], as with some other Löwig's papers, demonstrates how closely the author followed the literature; it contains important and valuable references. Only here do we learn the motivation for the relation  $R$  that appeared in Löwenheim's 1910 paper *Über die Auflösung von Gleichungen im logischen Gebietekalkül* [L] and studied in 1917 by Wiener in *Certain formal invariances in Boolean algebras* [Wi] and by B. A. Bernstein in 1924 in his paper *A generalization of the syllogism* [Be1]. In connection with this subject, one should not omit to mention the contributions *On a relation between the logical theory of classes and the geometrical theory of points* [Kem] by Alfred Bray Kempe in 1890 and *Sets of independent postulates for the algebra of logic* [H1] by Huntington in 1904. The papers of these mathematicians preceded subsequent endeavours of Ore and Birkhoff who formulated the theory of algebraic structures, lattices in particular, that influenced a number of other algebraists, including Löwig.

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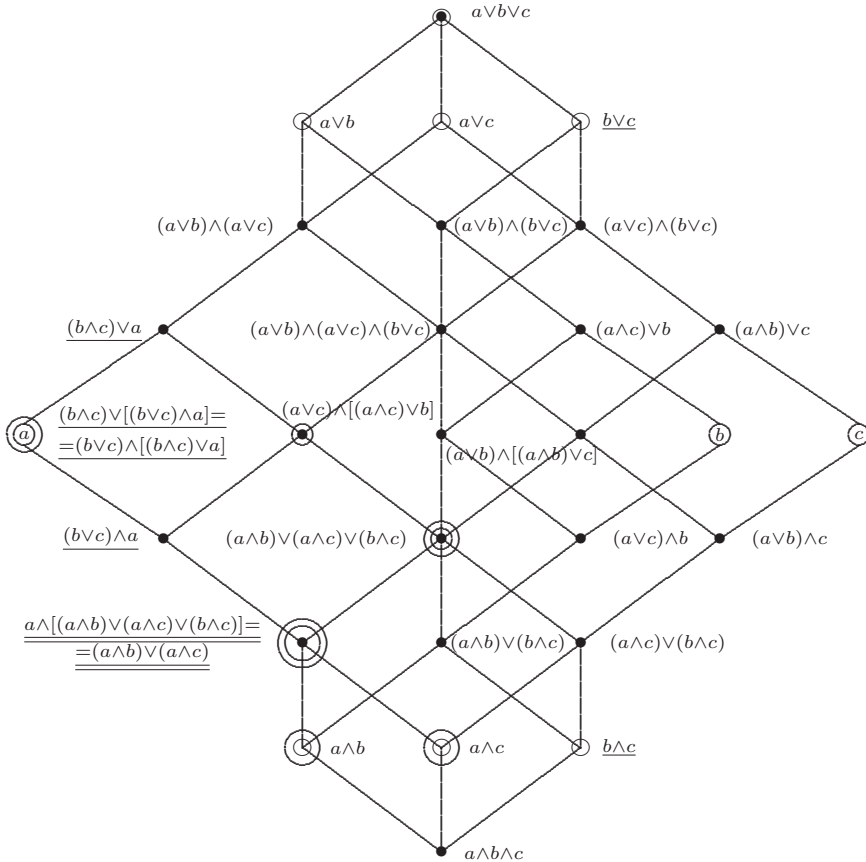


Figure 1. Free modular lattice generated by three elements.

Dedekind's paper, entitled *Über die von drei Moduln erzeugte Dualgruppe* [De2] and published in 1900, is of fundamental importance in the development of the theory of lattices.<sup>2</sup> In this paper, Dedekind abstracted the behaviour of normal subgroups and ideals that led to the concept of a modular lattice. He described the structure of a free modular lattice generated by three elements showing that it has 28 elements. Figure 1 provides a diagram of this free modular lattice. Let us mention in passing that a free modular lattice generated by four elements is infinite.

It is obvious that already at the very beginning of the theory of lattices, the importance of the concepts of modular and distributive lattices was well understood.

It is of interest to point out that Birkhoff's 1933 paper *On the combination of subalgebras* [B1], that deals with the theory of lattices, has no reference to Dedekind's paper [De2]. In this paper, Birkhoff defines the modular law and calls the modular lattices *B*-lattices. Even in the following papers *Applications of lattice algebra* [B2] and *On the lattice theory of ideals* [B3] in 1934 he uses the terminology of *B*-lattices. Moreover, here he defines also distributive lattices that he calls *C*-lattices, although he already briefly mentions Dedekind's paper [D2]. Only in the note entitled *Note on the paper "On the combination of algebras"* [B4] published in the same year did Birkhoff reveal that he was made aware of Dedekind's paper [De2] by Ore, and he then compared his results to those of Dedekind. In his papers *On the structures of abstract algebras* [B5] in 1935 and *Lattices and their applications* [B9] in 1938, Birkhoff was already using the contemporary terminology. In comparison, Ore refers to Dedekind's paper [De2] already at the very beginning of his paper *On the foundation of abstract algebra I* [Or] published in 1935 and uses the terminology of Dedekind lattices.

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The concept of modularity was the motivation for the first of Löwig's works concerning the theory of lattices. For every three elements  $a, b, c$  of an arbitrary lattice, there holds the following relation

$$(a \wedge b) \vee (a \wedge c) \leq a \wedge [(a \wedge b) \vee (a \wedge c) \vee (b \wedge c)]$$

and dually

$$(a \vee b) \wedge (a \vee c) \leq a \vee [(a \vee b) \wedge (a \vee c) \wedge (b \vee c)].$$

In his paper [Or], Ore pointed out that for three arbitrary elements  $a, b, c$  of a modular lattice, there holds the identity

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<sup>2</sup> Some ideas that relate to the contemporary lattice theory can also be found in other Dedekind's papers, such as *Über Zerlegung von Zahlen durch ihre größten gemeinsamen Teiler* [De1] in 1897.

$$(a \wedge b) \vee (a \wedge c) = a \wedge [(a \wedge b) \vee (a \wedge c) \vee (b \wedge c)] \quad (1)$$

and dually

$$(a \vee b) \wedge (a \vee c) = a \vee [(a \vee b) \wedge (a \vee c) \wedge (b \vee c)]. \quad (2)$$

At the same time, he added: *It should be observed, and it is easily verified, that each of the relations (1) to (5) may be considered as a restatement of the axiom of Dedekind. In a structure in which the Dedekind axiom does not hold these identities must be replaced by inequalities.* ([Or], p. 413)

In the lattice whose diagram is presented in Figure 2 (and commonly denoted by the symbol  $N_5$  and called a pentagon), there is

$$[a \wedge (b \vee c)] \vee (b \wedge c) \neq [a \vee (b \wedge c)] \wedge (b \vee c),$$

that is, the lattice  $N_5$  is not modular. It provides a characterization of non-modular lattices in the sense that every lattice that is not modular contains a sublattice that is isomorphic with  $N_5$ . At the same time, it is easy to see that the elements of the lattice  $N_5$  satisfy (1) and (2).

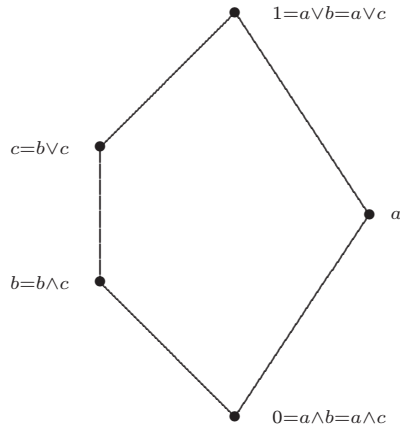


Figure 2. Lattice  $N_5$ :  $b = [a \wedge (b \vee c)] \vee (b \wedge c) \neq [a \vee (b \wedge c)] \wedge (b \vee c) = c$

A detailed explanation of this phenomenon is given by Löwig in his paper *On the importance of the relation  $[(A, B), (A, C)] = (A, [(B, C), (C, A), (A, B)])$  between three elements of a structure* [L11]. He has constructed a (non-modular) lattice  $L_9$  (see Figure 3) whose elements  $a, b, c$  do not satisfy the equality (1) and has proved the following statement that is parallel to the previously mentioned characteristic of modular lattices, that is, lattices satisfying the identity

$$[a \wedge (b \vee c)] \vee (b \wedge c) = [a \vee (b \wedge c)] \wedge (b \vee c).$$

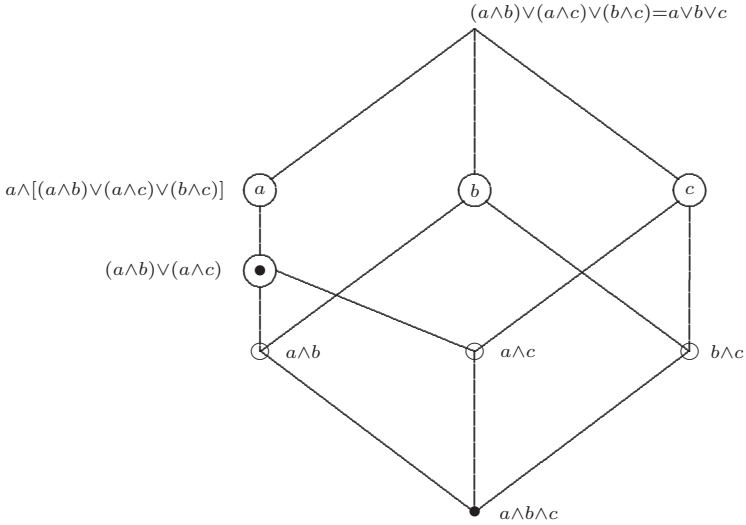


Figure 3. Lattice  $L_9$ :  $a \wedge [(a \wedge b) \vee (a \wedge c) \vee (b \wedge c)] \neq (a \wedge b) \vee (a \wedge c)$

Theorem. Every lattice that does not satisfy the equality (1) for any choice of elements  $a, b, c$ , contains a sublattice isomorphic with the lattice  $L_9$ .<sup>3</sup>

Löwig explained, by this theorem, why every lattice that has at most eight elements always satisfies the identity (1) for every choice of the elements  $a, b, c$ . Moreover, he drew attention to the fact that the lattice  $L_9$  satisfies the (dual) identity (2).<sup>4</sup>

The set of all elements satisfying for a given prime quotient  $a/b$  of a distributive lattice the equality  $(b \wedge x) \vee a = b$  forms a prime ideal  $P_{a/b}$ . In his note *Bemerkung zu den Primquotienten eines distributiven Verbandes* [L13] Löwig showed that  $P_{a/b} = P_{c/d}$  if and only if the prime quotients  $a/b$  and  $c/d$  are similar. The papers *Note on the self-duality of the unrestricted distributive law in complete lattice* [L19] and *On the completion of relatively complemented distributive lattices* [L24] deal also with distributive lattices. In the first one, the author re-proved the duality of the distributive laws for complete lattices formulated in 1952 by Raney in his paper *Completely distributive complete lattices* [Ra]. In the second paper, Löwig extended the inclusion of a relative

<sup>3</sup> THEOREM 8. Any structure in which the equation (2) is not generally valid contains at least one sub-structure of the ninth order having the same property.

THEOREM 9. Any structure of the ninth order in which the equation (2) is not generally valid is isomorphic with the structure  $\Lambda$ . ([L11], p. 578)

<sup>4</sup> This fact is pointed out explicitly by Padmanabhan and Rudeanu in their monograph *Axioms for Lattices and Boolean Algebras* [PR] in 2008.

complementary distributive lattice with zero into a complete Boolean algebra (formed by its normal ideals) to the case that the relative complementary lattice does not have zero. Such a minimal completion is attained by the lattice of the normal filters in the lattice of the normal ideals of the original relative complementary distributive lattice.

A motivation for *Note on the theory of independence in continuous geometries* [L22] is the formulation of Theorem 2.7 in the monograph *Lectures on Continuous Geometry* by John von Neumann in 1937. Theorem 2.7 provides a characterization of the concept of independence in a complete complementary modular lattice  $L$ .<sup>5</sup> Von Neumann defined an independent subset  $A \subseteq L$  by requiring

$$\left( \bigvee_{a \in X} a \right) \wedge \left( \bigvee_{a \in Y} a \right) = 0,$$

for every (disjoint) decomposition  $A = X \cup Y$ , and pointed out that  $A$  is independent if and only if

$$B = \left\{ \bigvee_{a \in X} a \mid X \subseteq A \right\}$$

is isomorphic to the Boolean algebra of all subsets of the set  $A$  (where the joins and meets correspond to the unions and intersections). Löwig showed that this statement does not hold in general, and proved the following fact: *A is independent if and only if the correspondence mapping every subset X of A the corresponding element  $\bigvee_{a \in X} a \in B$  is an isomorphism.* In addition, he pointed out that von Neumann's formulation is valid in the case that  $A$  is a finite subset.

In his typically rigorous way, Löwig reformulates in the paper *On the properties of freely generated algebras* [L14] Birkhoff's concept of freely generated algebra in the case that there are no restrictions on the number of operations and on their arities. Using a strictly formal notation, he derives fifty theorems describing the basic properties of free algebras, their homomorphisms, free bases and the independence of their elements. This paper has become the basis for an extensive series of subsequent publications [L15], [L16], [L17], [L18], [L20], [L21] and [L23] that follow each other and are mentioned in Grätzer's monograph *Universal Algebra* [G1]. All of them show Löwig's extraordinary pedantic care that leads to an exclusive abstract notation. The aim of Löwig's papers is to remove some of the logical difficulties that are implicitly connected with some of the definitions that Birkhoff introduced in the paper *On the structures of algebras* [B5] in 1935. They concern mainly the problems related to transfinite constructions required in the proofs of the existence of free algebras. Thurston underlined this extraordinary care in his review of Löwig's paper *Gesetzrelationen über frei erzeugten Algebren* [L15] by pointing

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<sup>5</sup> See also the papers [B6], [Wh], [H2].



out that Löwig's formulations allow to decide if an algebra over a set  $A$  with no operations whatsoever is or is not the same as an algebra over the set  $A$  with operations that are functions with an empty domain (of definition).<sup>6</sup> The mere fact that these papers contain 240 theorems can give one a great deal of reassurance.

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An evaluation of Löwig's contributions to algebra would not be complete without mentioning his meticulous reviewing activities. Here, one can see how closely he followed contemporary literature and how much care he put into preparing his reviews. Let us illustrate this with a few examples.

During the Second World War, Löwig learned about the lectures *Lectures of Continuous Geometry* [No1] that John von Neumann read at the Institute for Advanced Studies in Princeton during the years 1935–1937.<sup>7</sup> He studied them thoroughly and found a few shortcomings. He wrote about these findings to von Neumann in 1946. It was only in 1957 that Halperin, who was preparing a publication of von Neumann's lectures *Continuous Geometry* [No2], replied.<sup>8</sup> He included Löwig's corrections into the book and mentioned their relevance in the appendix (see [No2], p. 291). Halperin's edition of the lectures of von Neumann was reviewed by Fumitomo Maeda;<sup>9</sup> he devoted nearly a third of his review to clarifying Löwig's formulations.<sup>10</sup>

In 1946, Wilcox and Smiley published a correction of their article *Metric lattices* [WS] that appeared in 1939; in it they made the following comment:

*Henry Loewig, in a letter dated September 23, 1940, which reached us March 28, 1946, pointed out and corrected a flaw in the paper cited.*<sup>11</sup>

In his 1963 paper *Two theorems about relations* [Ken] Kenyon thanked Löwig for pointing out a mistake in an earlier version of the paper.

Löwig also pointed out a mistake in Sakiho Ôhashi's 1968 paper *On definitions of Boolean rings and distributive lattices* [Ôh], which the author subsequently corrected in a short joint note co-authored in 1970 with Kiyoshi Iséki in *Axiom systems of distributive lattice* [IÔ]:<sup>12</sup>

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<sup>6</sup> Löwig's precise formulations enable one to decide, for example, whether or not the algebra over  $A$  with no operations is the same as the algebra over  $A$  all of whose operations are functions on no elements. (MR0067853 (16,786e))

<sup>7</sup> The details are mentioned on pp. 40–41 of this book.

<sup>8</sup> John von Neumann died in 1957.

<sup>9</sup> Mathematical Reviews MR0120174 (22#10931).

<sup>10</sup> The relevant part of Maeda's review is presented in this book on pp. 40–41. Let us point out that Maeda published a German translation of his monograph of 1950 under the title *Kontinuierliche Geometrien* [M] in 1958. A very detailed review of this book has been published in the Bulletin of the American Mathematical Society 64 (1958), pp. 386–390, by Halperin, where he mentioned Löwig's improvements of von Neumann's formulations.

<sup>11</sup> Let us point out that the discrepancy of the dates is a result of the war events.

<sup>12</sup> This fact is mentioned also in the monograph [PH] on pp. 62 and 201.

In a letter of Dr. H. F. J. Löwig to Ôhashi, he noted that Theorem 2 in [1], is true under an additional condition ... ([IÔ], p. 409)

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Löwig's paper [L10] was rather successful. In 1951, it is mentioned by Basil C. Rennie (1920–1996) in an article *Lattices* [Re] and in the same year, it is listed by Leonid Vital'evič Kantorovič (1912–1986), Boris Zacharovič Vulich (1913–1978) a Aron Grigor'evič Pinsker (1905–1985) in the references of their extensive joint work *Poluuporjadočennye grupy i linejnye poluuporjadočennye prostranstva* [KVP].

David O. Ellis and H. D. Sprinkle mentioned the paper [L10] both in their little note *Topology in B-metrized spaces* [ES1] in 1952 and in *Topology of B-metric spaces* [ES2] in 1956. In 1967, in their second paper they wrote:

*The work was suggested mainly by the interesting comparison between the distance geometries of ordinary metric spaces and the autometrized Boolean algebras studied by one of us (10, 11) and, more recently, by L. M. Blumenthal and others (5).<sup>13</sup> There is a hint of the program, however, in a paper of Löwig around 1936 (22).* ([ES2], p. 250)

The paper [L10] is also mentioned by Fredos Papangelou in the following papers: *Some considerations on convergence in abelian lattice-groups* [Pa] in 1965, by Jerold Chase Mathews and R. F. Anderson in *A comparison of two modes of order convergence* [MA] in 1967, by Jörg Stephan in *Varieties generated by lattices of breadth two* [Ste] in 1993, and by George Georgescu and Andrei Popescu in *Similarity convergence in residuated structures* [GP] in 2005.

The work [L11] was cited by R. V. Petropavlovskaja in her 1950 paper *O zakonach v strukturach* [Pe], as well as by T. Tamura and F. M. Yaqub in their 1965 article *Examples related to attainability of identities on lattices and rings* [TY]. The contribution of Löwig was also acknowledged by Hans-Jürgen Hoehnke (born 1925), the reviewer of [TY] in *Mathematical Reviews*.<sup>14</sup>

Omarov cited the work [L11] in his 1988 paper *O mnogoobrazii rešetok, opredelennom toždestvom Ikbalunnisy* [Om1], and in the 1990 paper *O peresečenii necharakterizuemych mnogoobrazij rešetok* [Om2].

Birkhoff cited the works [L10] and [L11] in the second edition of his monograph *Lattice Theory* [B10] from 1948.<sup>15</sup> In the third edition of *Lattice Theory*

<sup>13</sup> The reference is to of Leonard M. Blumenthal's *Boolean geometry I*, National Bureau of Standards, 1952. See also *Rendiconti del Circolo matematico di Palermo* 1 (1952), pp. 343–360.

<sup>14</sup> MR0191852(33#79).

<sup>15</sup> Löwig's results are mentioned on pages 66, 109, and 166, but his works are missing in the final list of references on pp. 272–274. In the Russian translation *Teorija struktur* published in 1952, Löwig's works are cited on pages 104, 160, and 233, as well as in the list of references on p. 384.

from 1967, Löwig's name appears on p. 142 in connection with the works of Schmidt and Słomiński,<sup>16</sup> but the works [L10] and [L11] are not included in the final bibliography (pp. 411–413).

George Grätzer (born 1936) cited the work [L11] in the first chapter of his book *Lattice Theory. First Concepts and Distributive Lattices* [G2] published in 1971,<sup>17</sup> as well as in his 1978 book *General Lattice Theory* [G3]<sup>18</sup> and its subsequent editions (1998, 2003, 2007, Russian translation 1982) and in his monograph *Lattice Theory. Foundations* [G8] published in 2010.

The results from [L11] were mentioned by Ranganathan Padmanabhan and Sergiu Rudeanu in their book *Axioms for Lattices and Boolean Algebras* [PR].

Roman Sikorski (1920–1983) cited Löwig's work [L12] in his book *Boolean Algebras* [Si] published in 1960. The second and third editions, which appeared in 1964 and 1969, include the works [L10] and [L12] among the references.<sup>19</sup>

Löwig's works [L14], [L15], and [L16] have received a fairly large number of citations.

In 1959, A. Nerode (born 1932) cited the work [L15] in his article *Composita, equations, and freely generated algebras* [Ne]. In 1965, Walter Felscher (1931–2000) made a reference to the works [L14] and [L16] in his paper *Zur Algebra unendlich langer Zeichenreihen* [F]. Among other things he wrote:

*In der mathematischen Logik hat man in den letzten Jahren viele und berechtigende Gründe gefunden, sich mit solchen formalen Sprachen zu befassen, denen unendlich lange Zeichenreihen zugrunde liegen ... Die Term- und Formelalgebren solcher Sprachen sind dann absolut freie Algebren mit infinitären Operationen, und eine algebraische Theorie solcher Algebren ist von LOWIG [1], [2], SŁOMIŃSKI [1], DIENER [1] und KERKHOFF [1] dargestellt worden.*<sup>20</sup> ([F], p. 5)

Numerous references to Löwig's works can be found in the publications of Jürgen Schmidt.

In 1960, he cited the work [L14] in the article *Peano-Bäume* [S1].<sup>21</sup> His subsequent paper *Algebraic operations and algebraic independence in algebras*

<sup>16</sup> See also p. 188 of the Russian translation *Teorija rešetok*.

<sup>17</sup> *The examples of lattice identities we have seen so far seem to suggest that all such identities are self-dual. The identity  $(a \wedge b) \vee (a \wedge c) = a$ , where  $a = x \wedge ((x \wedge y) \vee (y \wedge z) \vee (z \wedge x))$ ,  $b = (x \wedge y) \vee (y \wedge z)$ ,  $c = (x \wedge z) \vee (y \wedge z)$ , is an example of a nonself-dual identity. (H. F. J. Löwig [1943]).* ([G2], p. 63)

<sup>18</sup> *This identity holds in a lattice  $L$  iff  $L$  does not have a sublattice isomorphic to the dual of the lattice of Figure V.2.7; see H. F. J. Löwig [1943].* ([G3], p. 53)

<sup>19</sup> The second edition of the monograph [Si] published in 1964 appeared in a Russian translation in 1969, including the references to [L10] and [L12].

<sup>20</sup> The references correspond to Słomiński [S1], Diener [Di0], Kerkhoff [Ke1].

<sup>21</sup> The article was dedicated to the 60th birthday of Karl Dörge (1899–1975).

with infinitary operations [S2] cites the works [L14] and [L15]; the following sentence appears in the introduction:

*Since in more recent times, especially following the needs of metamathematics, a deeper interest in infinitary operations has arisen (e.g. cf. DÖRGE [4], LÖWIG [6], [7], SOMINSKI [11]), we want to contribute to this development by extending MARCZEWSKI's theory to infinitary operations.*<sup>22</sup> ([S2], p. 77)

Further in the text we find the following footnote:

*In fact, LOWIG seems to be the only author on general algebra who profits by this convenient general form of induction (containing ordinary complete induction on natural numbers as an extremely special case), whereas other authors, e.e. SOMINSKI [11], make a sometimes more than exhaustive use of transfinite induction on rank numbers. Still, algebraic induction is frequently used in metamathematics, even by authors of the intuitionistic schools.*<sup>23</sup> ([S2], p. 84)

Löwig's articles [L15] and [L16] are cited several times in Schmidt's work *Die Charakteristik einer allgemeinen Algebra. I.* [S3]<sup>24</sup> published in 1962. In 1964, he cited Löwig's work [L14] in the paper *Some properties of algebraically independent sets in algebras with infinitary operations* [S4]. He wrote:

*... the existence of the element basis for all elements  $x$  in the algebraically independent generating set  $M$  is secured in the special cases of finitary algebras and (reproducing a result of Löwig [1]) of absolutely free algebras ...* ([S4], p. 123)

Löwig's work [L14] is cited in many places of Schmidt's subsequent paper *Über die Dimension einer partiellen Algebra mit endlichen oder unendlichen Operationen* [S5] from 1965. The paper *Die überinvarianten und verwandte Kongruenzrelationen einer allgemeinen Algebra* [S6], published in the same year, contains numerous references to the works [L14] and [L15].

*LOWIG, SŁOMIŃSKI und B. H. NEUMANN haben es unternommen*<sup>25</sup>, *die Birkhoffsche Gleichungstheorie rein innermathematisch nachzubauen. An die Stelle der der Metamathematik angehörenden formalen Sprache der algebraischen Gleichungen treten als genaue innermathematische Gegenstücke die absolut freien Algebren*<sup>26</sup> ...<sup>27</sup> ([S6], p. 131)

<sup>22</sup> The references correspond to [Dö], [Mar1], and [Mar2].

<sup>23</sup> In the whole article the surname of Słomiński is misspelled as Sominski.

<sup>24</sup> The work is dedicated to the 60th birthday of Reinhold Baer (1902–1979).

<sup>25</sup> *LOWIG [5], [6] und SŁOMIŃSKI [13] für beliebige endlich- oder unendlichstellige Operationen, B. H. NEUMANN [9] unter Beschränkung auf endlichstellige Operationen.*

<sup>26</sup> *LOWIG [5]: freely generated algebras; SŁOMIŃSKI [13] p. 21: absolutely free algebras; B. H. NEUMANN [9] p. 50: free anarchic algebras; BIRKHOFF [3] p. 162: primitive algebras.*

<sup>27</sup> The references correspond to Słomiński [S1], Neumann [Neu], Birkhoff [B10].

In 1966, Peter Burmeister and Jürgen Schmidt cited the work [L15] in their paper *Über die Dimension einer partiellen Algebra mit endlichen oder unendlichen Operationen II* [BS]. In 1968, Słomiński cited Löwig's articles [L14], [L15] in his work *Peano-algebras and quasi-algebras* [Sł2]. Schmidt also cited the works [L14], [L15] and [L16] in the paper *Clones and semiclones of operations* [S7] from 1977 (published in 1982); Löwig's work [L14] was mentioned by Hoehnke in the review of Schmidt's work [S7].<sup>28</sup>

E. G. Manes cited the work [L16] in his article *Free algebraic theories* [Ma] written in 1977 and published in [CFS] as late as 1982.

Robert Kerkhoff cited the articles [L14] and [L16] in his works *Eine Konstruktion absolut freier Algebren* [Ke1] from 1965 and *Über verallgemeinerte Peano-Algebren* [Ke2] from 1969.<sup>29</sup>

The introduction of the paper [Ke1] contains the following definition of an absolutely free algebra:

*Die Algebra  $A$  heißt absolut frei über einer Menge  $B$ , wenn eine Abbildung  $d$  von  $B$  in  $A$  existiert, so daß zu jeder Abbildung  $h$  von  $B$  in die Grundmenge einer Algebra  $D$  vom Typ  $\Delta$  genau ein Homomorphismus  $\varphi$  von  $A$  in  $D$  existiert mit  $\varphi \circ d = h$ . Bekanntlich (DIENER [2], §2; LOWIG [5], [6]; SŁOMIŃSKI [7], Chap. III, §1) ist eine absolut freie Algebra  $A$  über der Menge  $B$  durch folgende drei Eigenschaften (verallgemeinerte Peano-Axiome) charakterisiert:*

- (P<sub>1</sub>) *Es existiert eine Injektion  $d$  von  $B$  in  $A$ , so daß  $A$  von  $B^* = d(B)$  erzeugt wird und die Elemente aus  $B^*$  nicht als Bilder unter Operationen auftreten;*
- (P<sub>2</sub>) *verschieden indizierte Operationen haben disjunkte Bilder;*
- (P<sub>3</sub>) *die Operationen sind umkehrbar. ([Ke1], p. 109)*

In 1966, Egbert Harzheim cited the work [L16] in his paper *Über die Grundlagen der universellen Algebra* [Ha].<sup>30</sup>

Karl-Heinz Diener, a pupil of Jürgen Schmidt, made a reference to the works [L14] and [L15] in his article *Order in absolutely free and related algebras* [Di1] published in 1966; the following passage can be found in the introduction:

<sup>28</sup> MR660904 (83m:08005).

<sup>29</sup> Let us quote from Diener's review of the book *Consequences of the Axiom of Choice* [HR] by Paul Howard and Jean Estelle Rubin (1967–2002); the review was published in Zentralblatt für Mathematik (Zbl 0947.03001): *The proof of the existence of Peano algebras is a different matter. In 1965, Kerkhoff [Math. Ann. 158, 109–112 (1965; Zbl. 0192.09402)] published a surprisingly simple construction which does not make use of any form of the axiom of choice! Several other proofs had been published before (e.g., by Birkhoff, Löwig, Dörge, Harzheim, Peirce, Słomiński etc.); they all use consequences of the axiom of choice by assuming that the "arities" have well-ordered cardinalities, and that there exists a regular cardinal above them. The book [HR] cites Löwig's work [L7].*

<sup>30</sup> This paper was dedicated to the 65th birthday of Karl Dörge.

The class of all well-formed formulas of a formal language together with the operations corresponding to the logical connectives forms an absolutely free algebra; moreover, the recent study of formal languages with infinitely long expressions requires the admission of infinitary operations. On the other hand, absolutely free algebras, being special cases of free algebras, have been investigated in *General Algebra* both for their own interest and as a tool to define such notions as “equation” etc. without using metamathematical concepts (see e.g. Löwig [6], [7], Stomiński [11]). ([Di1], p. 63)

Diener also cited the works [L14] and [L16] in his contribution *On constructing infinitary languages  $L_{\alpha\beta}$  without the axiom of choice* [Di2] from 1983, and in his paper *On the transitive hull of a  $\kappa$ -narrow relation* [D3] from 1992. His articles *On the natural order relation in Peano algebras with finitary or infinitary operations* [Di4] and *On the predecessor relation in abstract algebras* [Di5], published in 1993, cite Löwig’s work [L14].

George Grätzer (born 1936) listed Löwig’s works [L14], [L15], [L16], [L17], and [L18] among the references in his monograph *Universal Algebra* [G1] published in 1968.<sup>31</sup> In the subsequent edition published in 2008, he added references to the works [L20] and [L21] (see *Additional Bibliography*).

The work [L19] is mentioned in the cumulative bibliography (*The Continuous Lattices Bibliography*) published in the proceedings *Continuous Lattices and their Applications* [HH] in 1985, as well as in the article *Teorija struktura* [GSF] written jointly by Gluchov, Stelleckij and Fofanova in 1970.

Artamonov included Löwig’s papers [L20] and [L21] in his extensive survey article *Universal’nye algebrы* [Ar] published in 1976 in *Itogi nauki i tehniki. Algebra. Topologija. Geometrija*.

The website of Jaroslav Ježek (1945–2011)

<http://www.karlin.mff.cuni.cz/~jezek/bibua.pdf>

includes, in the bibliography of papers on universal algebras and related topics (in April 2008, 321 pages), Löwig’s papers [L11], [L14], [L15], [L16], [L17], [L18], [L20] and [L21].

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In Prague, in the forties and fifties of the last century, the lattice theory was a subject of research not only of Löwig, but also of Vladimír Kořínek (1899–1981)<sup>32</sup> and his students Ludvík Janoš, Čestmír Vitner (1925–2010) and Václav Vilhelm (1925–2011). In 1941, in the year when Löwig’s paper [L10] appeared, Kořínek published his paper *Der Schreiersche Satz und das Zassenhaussche*

<sup>31</sup> See p. 346 (*Bibliography*). In the introduction (on page ix), he acknowledged the help and suggestions of several colleagues, including Löwig.

<sup>32</sup> More information about Kořínek can be found in the monograph [KB] by Kohoutová and Bečvář.

*Verfahren in Verbänden* [Ko1]. A closer contact between these two algebraists that continued till the eighties seemed to have started in 1943.<sup>33</sup> Kořínek published his second paper on the lattice theory entitled *Lattices in which the theorem of Jordan-Hölder is generally true* [Ko2] in 1949; at that time, Löwig was already in Hobart.

The doctoral theses *Vlastnosti Zassenhausova zjemnění* (1949/50) of Janoš, *Podmínky semimodularity ve svazech* (1951/52) by Vitner, and *Jordan-Hölderova věta ve svazech bez konečnosti řetězců* (1951/52) by Vilhelm, all published in the Czechoslovak Mathematical Journal in 1953 and 1954 (see [J], [Vit], [Vil]), were supervised by Kořínek. It seems that there was no contact between these algebraists and Löwig. No information can be gleaned about this from available sources.

It is obvious that Löwig tried to keep close contact with the Czech environment. As was already mentioned, there is a long history of correspondence between him and Kořínek. Moreover, he published some of his papers in the Czechoslovak Mathematical Journal (see [L12], [L20] and [L22]). He carefully followed the work on the lattice theory and universal algebra of some of the Czech and Slovak authors, and reviewed them for Mathematical Reviews (the reviews include papers of E. Gedeonová, V. Slavík, J. Ježek, J. Havrda, V. Vilhelm, D. Jakubíková-Studenovská, V. Rödl).

The work of the Czech and Slovak algebraists on the lattice theory is summarized in the doctoral thesis of Štěpánka Bilová entitled *Lattice theory in Czech and Slovak Mathematics until 1963* [Bi] written in 2004.

Štěpánka Bilová did not mention the work of Löwig in her dissertation paper, however the influence of Löwig's contributions can be seen in the work of Ján Jakubík (born 1923) and Milan Kolibiar (1922–1994). In their joint paper *O nekotorych svojstvach par struktur* [JK], they refer to his paper [L12]. Moreover, Kolibiar refers to Löwig's paper [L19] in his article *Distributive sublattices of a lattice* [Kol] in 1972 and Ján Jakubík makes a reference to [L10] in his paper *Sequential convergences in Boolean algebras* [J1]. Ten years later, in 1998, he refers to it again in the article *Disjoint sequences in Boolean algebras* [J2], and in 2002 again in his *Sequential convergences on generalized Boolean algebras* [J3].

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<sup>33</sup> Excerpts from their correspondence are presented in the first chapter of this book. See also [KB], pp. 218–220.



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