# SOME RECENT EXTENSIONS OF JARNÍK'S WORK IN DIOPHANTINE APPROXIMATION

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The contribution Jarník made to mathematics has perhaps not been recognised fully and he has been rather overshadowed by better known mathematicians. This article begins with an outline of some of his remarkable and highly original work in number theory and then shows that the ideas he introduced are still flourishing.

Jarník was the first to use Hausdorff measure and dimension in the study of sets of measure zero which arise in the theory of Diophantine approximation and his paper "Zur metrischen Theorie der diophantischen Approximationen" [16] written around 1928 was rightly referred to by Rogers ([23], p. 135) as "pioneering". In it Jarník proved the interesting result that the Hausdorff dimension of the set ($M_\infty$ in his notation) of badly approximable numbers is 1. Badly approximable numbers are also referred to as numbers of constant type and are numbers $x$ for which there exists a positive constant $c$ depending only on $x$ such that

$$\|qx\| \geqslant c|q|^{-1}$$

for all non-negative integers $q$ ($\|t\| = \inf\{|t-k|:\ k \in \mathbb{Z}\}$ for all real $t$). Some 40 years later, this result was extended by W. M. Schmidt [24] to systems of linear forms and it is still attracting attention. S. J. Patterson has extended Jarník's theorem in considering Diophantine approximation in Fuchsian groups ([21], §10) and recently has extended it to quadratic forms ([22], Theorems 4 and 7). S. G. Dani ([9], [10]) has generalised these ideas still further to bounded orbits of flows on manifolds.

Jarník was also interested in the set $W(\tau)$ of real numbers $x$ for which the inequality

$$\|qx\| < |q|^{-\tau},$$

where $\tau > 1$, holds for infinitely many integers $q$. (Instead of (1), Jarník used the different but equivalent form

$$\left|x - \frac{p}{q}\right| < |q|^{-\alpha}$$

where $\alpha > 2$ and called the set $P_\alpha$). This set is a refinement of the complementary notion of well approximable numbers and in [17] Jarník showed that the Hausdorff dimension of $W(\tau)$ is $2/(\tau+1)$. His proof is based on arithmetic ideas and relies heavily on the properties of continued fractions and on a result from analysis, namely Cantor's intersection theorem. Shortly afterwards Jarník [18] extended this result to simultaneous Diophantine approximation by showing that rational approximations to $x_2, \ldots, x_n$ with common denominator $q$ can be found without affecting the approximation to $x_1$ significantly. He proved that the dimension of the set $W(\tau; n)$ (or $M(x^{-\tau}; n)$ in his notation) of points $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ for which the system of inequalities

$$(1) \qquad \|qx_i\| < |q|^{-\tau}, \quad i = 1, \ldots, n,$$

where $\tau > 1/n$, holds for infinitely many integers $q$, $p_i$, $i = 1, \ldots, n$, is $(n+1)/(\tau+1)$.

A few years later and apparently unaware of Jarník's work, Besicovitch [4] also determined $\dim W(\tau)$. Besicovitch used more geometric ideas than Jarník, as did Eggleston [14] who in a general paper on sets of number theoretic interest, also obtained the Hausdorff dimension in the $n$-dimensional case. Eggleston's more geometric approach enabled him to obtain more general, though less precise results than Jarník, in which the denominators $q$ lay in sequences, such as the primes, that were not too sparse.

Jarník's and Besicovitch's result was extended in a different direction by Baker and Schmidt [1] to sets defined more widely in terms of approximation by algebraic numbers of bounded degree. More recently, Kaufman [19] has shown that there is a positive measure $\mu$ supported on a compact subset of $W(\tau)$ with Fourier-Stieltjes transform $\hat{\mu}$ satisfying

$$\hat{\mu}(u) = o(|u|^{-1/(\tau+1)} \log |u|)$$

as $|u| \to \infty$.

I will discuss briefly Jarník's papers of 1929 and 1931 on the Hausdorff dimension of subsets of well-approximable numbers and then outline some recent extensions of his ideas. Jarník's starting point is the set of measure 0 which arises in the easy half of Khintchine's theorem ([6], [25]). Suppose $\psi(k)$, $k = 1, 2, \ldots$, is a sequence of positive numbers such that

$$\sum_{k=1}^{\infty} \psi(k)^n \tag{2}$$

converges. Then the system of inequalities

$$\|qx_i\| < \psi(|q|),\ i = 1, \ldots, n,$$

holds for infinitely many integers $q$ for almost no $x$ in $\mathbb{R}^n$. Thus the set of points which allow very good simultaneous rational approximations (in the sense that $\psi(k)$ decreases fast enough to ensure that the sum (2) is finite), is of measure zero. (The converse result that if the sum (2) diverges, almost no points $x$ satisfy the system of inequalities for only finitely many $q$ holds when $\psi(k)$ is monotonic.) In particular, when $\tau > 1/n$, the set $W(n;\tau)$ of points $\mathbf{x} \in \mathbb{R}^n$ which satisfy (1) for infinitely many integers $q$ has Lebesgue measure zero. Such sets can be considered thin or negligible and are sometimes referred to as exceptional. They can however be very different. For example the set of badly approximable numbers $M_\infty$ and the set $W(\tau)$ $(= W(1;\tau))$ when $\tau > 1$ both have Lebesgue measure 0 but have different Hausdorff dimension.

Hausdorff dimension is a generalisation of the familiar notion of dimension and the two dimensions coincide for sets such as the line or the plane or more generally for finite dimensional smooth manifolds such as the circle. Sets whose Hausdorff dimension exceeds their topological dimension are called fractals [20]. The fundamental difference between the two ideas is that any subset of finite dimensional Euclidean space can be assigned a Hausdorff dimension. The price of this generality is a somewhat complicated definition (see [15]).

Let $\Gamma$ be a finite or countable collection of open hypercubes $C$ in the $k$-dimensional Euclidean space $\mathbb{R}^k$ and let the length of the sides of a hypercube $C$ be denoted by $\ell(C)$. For each real number $s > 0$, the $s$-volume $V^s(\Gamma)$ of the collection $\Gamma$ is defined by

$$V^s(\Gamma) = \sum_{C \in \Gamma} \ell(C)^s.$$

Given any set $X \in \mathbb{R}^k$ and any positive number $\sigma$, let $m_\sigma(s, X) = \inf V^s(\Gamma_\sigma)$, where the infimum is taken over all covers $\Gamma_\sigma$ of $X$ with $\ell(C) < \sigma$ for each $C \in \Gamma_\sigma$. The $s$-dimensional outer Hausdorff measure $m(s, X)$ of $X$ is given by $m(s, X) = \sup\{m_\sigma(s, X) \colon \sigma > 0\}$. The Hausdorff dimension $\dim X$ of $X$ is defined by

$$\dim X = \inf\{s \in \mathbb{R} \colon m(s, X) = 0\}$$

and is the unique value of $s$ such that

$$m(s,X)=\begin{cases}\infty, & 0\leqslant s<\dim X,\\ 0, & \dim X<s<\infty.\end{cases}$$

The Lebesgue measure $m(k,X)$ of $X$ will be denoted by $|X|$.

It follows that if $X$ can be covered by a collection $\Gamma_\varrho$ with arbitrarily small $s$-volume $V^s(\Gamma_\varrho)$, then $\dim X\leqslant s$. On the other hand, if for each positive $\varepsilon$, there exists a positive number $\varrho=\varrho(\varepsilon)$ such that every cover $\Gamma_\varrho$ of $X$ with $\ell(C)\leqslant\varrho$ satisfies $V^s(\Gamma_\varrho)>\varepsilon$, then $\dim X\geqslant s$; roughly speaking if the $s$-volume of covers consisting of small hypercubes of $X$ is large, then $\dim X\geqslant s$.

Clearly a cover $\Gamma$ of $X$ will be a cover for any subset $X'$ of $X$ and it follows from the definition that if $X'\subset X\subset\mathbb{R}^n$, then

$$\dim X'\leqslant\dim X\leqslant n.$$

The determination of Hausdorff dimension can often be simplified by the observation that when

$$X=\bigcup_{j=1}^{\infty}X_j,$$

then

$$\dim X=\sup\{\dim X_j:\ j=1,2,\ldots\}. \tag{3}$$

Determining the dimension $h$ say of a set $X$ is usually done in two separate stages. The upward inequality $\dim X\leqslant h$ is established by constructing for any positive $\varepsilon$ and any $s>h$, a cover $\Gamma$ with $s$-volume $V^s(\Gamma)<\varepsilon$. This is often straightforward (though not always; see for example [2], [3]). Unless there is some general lower bound for $\dim X$ that happens to coincide with $h$ (as for example in [6], [11], [26]), the complementary inequality $\dim X\geqslant h$ is usually much harder, and the cases considered here are no exception. It has to be shown that for any $s<h$, any cover $\Gamma$ consisting of arbitrarily small hypercubes of the set $X$ has $s$-volume $V^s(\Gamma)\geqslant\varepsilon$ for some positive $\varepsilon$. Jarník chose, as most workers have done since, to use the technically more convenient contrapositive form of the definition for the Hausdorff dimension: Let $s$ be any positive number with $s<h$. If any collection $\Gamma$ of arbitrarily small hypercubes with $V^s(\Gamma)<1$ cannot cover the set $X$, then $\dim X\geqslant h$.

To illustrate these ideas, we shall follow Jarník's proof [17] that

$$\dim W(\tau)\leqslant 2/(\tau+1).$$

First in view of (3), we need only consider numbers in the unit interval $I = (-\frac{1}{2}, \frac{1}{2}]$ instead of $\mathbb{R}$. The collection

$$\Gamma_N = \bigcup_{|q|>N} \bigcup_{p\in qI} \Big(\frac{p}{q} - |q|^{-\tau-1}, \frac{p}{q} + |q|^{-\tau-1}\Big)$$

is a cover by intervals for $W(\tau)$ for each $N = 1, 2, \dots$ since if $x \in W(\tau)$, then $|qx - p| < |q|^{-\tau}$ for infinitely many integers $p$, $q$ and we may choose $|q| > N$. Also the $s$-volume $V^s(\Gamma_N)$ (or more appropriately in this case, the $s$-length) of $\Gamma_N$ satisfies

$$\begin{aligned} V^s(\Gamma_N) &\ll \sum_{q>N} \sum_{|p|<q} 2q^{-(\tau+1)s} \\ &\ll \sum_{q>N} q^{1-(\tau+1)s} \ll N^{2-(\tau+1)s}. \end{aligned}$$

Thus when $s > 2/(\tau+1)$, $V^s(\Gamma_N) \to 0$ as $N \to \infty$ and for each $C \in \Gamma_N$, $\ell(C) \leqslant 2N^{-(\tau+1)}$. Hence by definition it follows that $\dim W(\tau) \leqslant 2/(\tau+1)$. Jarník [18] obtained the upper bound $(n+1)/(\tau+1)$, $\tau > 1/n$, for the Hausdorff dimension of $W(\tau; n)$ in a similar way.

Jarník's results can be extended to systems of $n$ linear forms

$$\sum_{i=1}^{m} \theta_i a_{ij}, \quad j = 1, \dots, n$$

in $m$ real variables, which we will write more concisely as

$$\theta A,$$

where $A$ is an $m \times n$ real matrix and $\theta = (\theta_1, \dots, \theta_m) \in \mathbb{R}^m$. If the sum

$$\sum_{k=1}^{\infty} k^{m-1} \psi(k)^n \tag{4}$$

converges, then by Groshev's extension of Khintchine's theorem ([25], pp. 33–34), the system of inequalities

$$\Big\| \sum_{i=1}^{m} q_i a_{ij} \Big\| < \psi(|\mathbf{q}|), \quad 1 \leqslant j \leqslant n, \tag{5}$$

where for each $\mathbf{x} \in \mathbb{R}^m$, $|\mathbf{x}| = \max\{x_1, \dots, x_m\}$, holds for infinitely many $\mathbf{q} \in \mathbb{Z}^m$ for almost no matrices $A \in \mathbb{R}^{mn}$. (The complementary result that when the sum

(4) diverges, almost all systems of linear forms satisfy (5) for infinitely many integer vectors $\mathbf{q}$ holds under natural monotonicity conditions.) Thus when $\tau > m/n$ and $\psi(k) = k^{-\tau}$, the set $W(m,n;\tau)$ of matrices (or systems of linear forms) satisfying (5) has Lebesgue measure 0 in $\mathbb{R}^{mn}$ (note that $W(n;\tau) = W(1,n;\tau)$).

When $\tau \geqslant m/n$, the Hausdorff dimension of $W(m,n;\tau)$ is $(m-1)n + (n+1)/(\tau+1)$ [7]. The upper bound

$$\dim W(m,n;\tau) \leqslant (m-1)n + (n+1)/(\tau+1) \tag{6}$$

is obtained in a similar way to that of $W(\tau)$.

The proof that the Hausdorff dimension of $W(m,n;\tau)$ is at least $(m-1)n + (n+1)/(\tau+1)$ is much harder and has features in common with those of Jarník, Besicovitch and Eggleston. All these proofs rely on the zero sets of the associated system of Diophantine equations or "resonant" planes being regularly distributed in some way. In [7] and [13] this regularity is made explicit through a "second moment" argument which will now be sketched. This and the subsequent arguments are somewhat intricate, relying on a variance associated with the distribution of certain resonant planes not being too large and involving repeated subdivision. Some of the ideas resemble those arising in the study of iteration and self-similar sets, such as the Mandelbrot set, which reproduce themselves in some sense at certain scales and which are currently attracting considerable attention (see for example [5], [20]). When $m \geqslant 2$, more geometrical ideas can be used to obtain the fundamental "invariance of measure" and "independence" results (7) and (8) below ([13], (8.4) and (8.5)). The "independence" result is sharper than the corresponding one in [7], Lemma 3 and leads to the sharper second moment estimate (10) below. Incidentally, since the "invariance of measure" and "independence" results (7) and (8) correspond to Lemmas 8 and 9 respectively in [25], they provide a different proof of the version of Groshev's theorem given above. In view of this and of the results of Jarník [18] and Eggleston [14] which deal with the case $m = 1$, we shall take $m \geqslant 2$.

Some notation and definitions are needed. Let $I = (-\frac{1}{2}, \frac{1}{2}]$ and for each $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, let

$$\mathbf{k}_{\mathbf{x}}$$

be the unique integer vector (symmetrised fractional part) such that $\mathbf{x} - \mathbf{k}_{\mathbf{x}} \in I^n$. Write

$$\langle \mathbf{x} \rangle = \mathbf{x} - \mathbf{k}_{\mathbf{x}}.$$

The map $\langle . \rangle \colon \mathbb{R}^n \to I^n \colon \mathbf{x} \mapsto \langle \mathbf{x} \rangle$ projects each $\mathbf{x} \in \mathbb{R}^n$ to the cube $I^n$; note too that $|\langle u \rangle| = \|u\|$ for $u \in \mathbb{R}$. For each $\mathbf{q} \in \mathbb{Z}^m$, define the function $\Phi_{\mathbf{q}} \colon I^{mn} \to I^n$

(where we can regard $I^{mn} \cong \mathbb{R}^{mn}/\mathbb{Z}^{mn} \cong \mathbb{T}^{mn}$, the $n$-dimensional torus) by

$$\Phi_{\mathbf{q}}(A) = \langle \mathbf{q}A \rangle.$$

Let $\chi$ be the characteristic function of the $n$-dimensional cube $B(\varrho) = \{\mathbf{x} \in \mathbb{R}^n\colon |\mathbf{x}| < \varrho\}$. Then

$$\chi(\mathbf{q}A - \mathbf{p}) = \sum_{\mathbf{p}} \chi_U(A),$$

where the sum is over all vectors $\mathbf{p} \in \mathbb{Z}^n$ with $\mathbf{p} \in |\mathbf{q}|I^n$ and where the set $U = \Phi_{\mathbf{q}}^{-1}(B(\varrho))$. When $n \geqslant 2$, the measure of the inverse image under $\Phi_q$ of any hypercube $C$ in $I^n$ is preserved, i.e.,

$$|\Phi_q^{-1}(C)| = |C| \tag{7}$$

and the sets $\Phi_q^{-1}(C)$ and $\Phi_{q'}^{-1}(C')$ are "independent" in the sense that for any linearly independent $\mathbf{q}$, $\mathbf{q}'$,

$$|\Phi_{\mathbf{q}}^{-1}(C) \cap \Phi_{\mathbf{q}'}^{-1}(C')| = |\Phi_{\mathbf{q}}^{-1}(C)| \cdot |\Phi_{\mathbf{q}'}^{-1}(C')| = |C||C'|. \tag{8}$$

These equations follow from the geometry of the torus $\mathbb{R}^{mn}/\mathbb{Z}^{mn}$ and require only translation invariance (see [13], (8.4) and (8.5)). The proof given there is for the case $m \geqslant 2$, $n = 1$ but it can be extended to general $n$. Sprindžuk ([25], Chapter I, Lemmas 8 and 9) proves a more general result, although linearity is needed.

The "resonant" planes

$$\Pi(\mathbf{q}, \mathbf{p}) = \{A \in I^{mn}\colon\ \mathbf{q}A = \mathbf{p}\} \subset W(m, n; \tau),$$

where $\mathbf{q} \in \mathbb{Z}^m$ and $\mathbf{p} \in \mathbb{Z}^n$, play a fundamental role and form a kind of "skeleton" for $W(m, n; \tau)$. For technical reasons the integer vectors $\mathbf{q} \in \mathbb{Z}^m$ are taken to be *primitive*, i.e., the components $q_1, \ldots, q_m$ have no common factors. Two distinct primitive (integer) vectors cannot be collinear with the origin and so are linearly independent. Let

$$S_N = \{\Pi(\mathbf{q}, \mathbf{p})\colon\ \mathbf{p} \in \mathbf{q}I^{mn},\ \mathbf{q} \text{ primitive}, N < |\mathbf{q}| < 2N\}.$$

From now on $\mathbf{q}$ will be a primitive vector in $\mathbb{Z}^m$ with $N < |\mathbf{q}| < 2N$ and $\mathbf{p}$ will be a vector in $\mathbb{Z}^n$ satisfying $\mathbf{p} \in \mathbf{q}I^{mn}$. The function $\nu_N\colon\ I^{mn} \to \mathbb{Z}$ given by

$$\nu_N(A) = \sum_{\mathbf{q}} \sum_{\mathbf{p}} \chi(\mathbf{q}A - \mathbf{p}) = \sum_{\mathbf{q}} \chi_U(A)$$

is the number of $\mathbf{p}, \mathbf{q}$ such that $|\mathbf{q}A - \mathbf{p}| < \varrho$ or, roughly speaking, the number of planes $\Pi(\mathbf{q}, \mathbf{p}) \in S_N$ within about $\varrho/N$ of $A$. The mean $\mu_N$ of $\nu_N$ is by definition

$$\mu_N = \int_{I^{mn}} \nu_N(A)\,\mathrm{d}A = \int_{I^{mn}} \sum \chi_U(A)\,\mathrm{d}A \\ = \sum_{\mathbf{q}} |\Phi_q^{-1}(B(\varrho))| = 2^n \varrho^n \sum_{\mathbf{q}} 1 \tag{9}$$

by (7). The sum $\sum_{\mathbf{q}} 1$ can be estimated when $m > 2$ is $\sim N^{m-1}$ and when $m = 2$ is $\sim N\varphi(N)$ where $\varphi(N)$ is the Euler divisor function ($a \sim b$ means that $a \ll b$ and $b \ll a$). Next we estimate the second moment of $\nu_N$:

$$\int_{I^{mn}} \nu_N^2(A)\,\mathrm{d}A = \int_{I^{mn}} \sum_{\mathbf{q}} \sum_{\mathbf{q}'} \chi_U(A)\chi_{U'}(A)\,\mathrm{d}A,$$

where $U' = \Phi_{\mathbf{q}'}^{-1} B(\varrho)$. Hence

$$\begin{aligned} \int_{I^{mn}} \nu_N^2(A)\,\mathrm{d}A &= \sum_{\mathbf{q}} \int_{I^{mn}} \chi_U(A)\,\mathrm{d}A + \sum_{\mathbf{q}\neq\mathbf{q}'} \int_{I^{mn}} \chi_U(A)\chi_{U'}(A)\,\mathrm{d}A \\ &= \mu_N + \sum_{\mathbf{q}\neq\mathbf{q}'} |\Phi_{\mathbf{q}}^{-1}(-\varrho,\varrho) \cap \Phi_{\mathbf{q}'}^{-1}(-\varrho,\varrho)| \\ &= \mu_N + \sum_{\mathbf{q}\neq\mathbf{q}'} |\Phi_{\mathbf{q}}^{-1}(-\varrho,\varrho)||\Phi_{\mathbf{q}'}^{-1}(-\varrho,\varrho)| \end{aligned}$$

by (8). Therefore by (7),

$$\int_{I^{mn}} \nu_N^2(A)\,\mathrm{d}A = \mu_N + \sum_{\mathbf{p}\neq\mathbf{p}'} (2\varrho)^{2n} \leqslant \mu_N + \mu_N^2$$

by (9). It follows immediately that

$$\sigma_N^2 \leqslant \mu_N. \tag{10}$$

Thus in a sense the resonant hyperplanes in $S_N$ are regularly distributed and we will show that they are well enough distributed to construct a "sampling" set $T(N)$ which "measures" the volume of a set and which tends to a subset of $W(m, n; \tau)$ as $N \to \infty$. Volume considerations can then be used to show that $\Gamma$ cannot cover $W(m, n; \tau)$ (in fact $\Gamma$ fails to cover one point in a particular subset of $W(m, n; \tau)$).

By Tchebycheff's inequality, the Lebesgue measure of the set $\{A \in I^{mn}\colon |\nu_N(A) - \mu_N| \geqslant \mu_N\}$ satisfies

$$|\{A \in I^{mn}\colon\ |\nu_N(A) - \mu_N| \geqslant \sigma_N^2/\mu_N\}| \leqslant 1/\mu_N. \tag{11}$$

Now choose $\varrho = N^{-m/n+\eta}$, where $0 < \eta < \min\{m/n, \tau - m/n, (1+\tau)\delta/n\}$. Then $\varrho^n \to 0$ and $\mu_N \to \infty$ as $N \to \infty$. It follows from (11) that the measure of the set of points (matrices) $A$ for which $\nu_N(A) = 0$, i.e., which are not within $\varrho/N$ of a resonant plane $\Pi(\mathbf{q}, \mathbf{p}) \in S_N$ (more precisely the measure of the set of points $A \in I^{mn}$ not satisfying $|\mathbf{q}A - \mathbf{p}| < \varrho$ for some $\mathbf{p}$, $\mathbf{q}$) tends to 0 as $N \to \infty$.

We are now in a position to construct the "sampling" set $T(N)$ by selecting well distributed resonant planes in $S_N$ and then thickening them slightly. In addition, as $N \to \infty$, $T(N)$ tends to a subset of $W(m, n; \tau) \cap I^{mn}$.

Dissect $I^{mn}$ into $[N/(16\varrho)]^{mn} \sim N^{m(n-m+n\eta)}$ congruent hypercubes $H$ with length of side $\ell(H) = [N/(16\varrho)]^{-1} \sim N^{1-(m/n)+\eta}$. By volume considerations and (11), a plane $\Pi(\mathbf{q}, \mathbf{p})$ passes through $\gg \ell(H)^{-mn} \sim N^{m(n-m+n\eta)}$ of these hypercubes $H$ within $\varrho/N$ of its centre. Choose one such "slice" $S$ say from each such hypercube $H$ and let

$$V = V(S) = \Big\{A \in \mathrm{cl}\Big(\frac{1}{2}H\Big) \colon\ |A - R| \leqslant m^{-1}(2N)^{-\tau-1} \text{ for some } R \in S\Big\}$$

($\frac{1}{2}H$ is the hypercube of half the length of $H$ and the same centre) so that $V = V(S)$ is a closed neighbourhood of a contracted slice $S$. Let $T(N)$ be the collection of such $V$'s. The $mn$-dimensional volume $|T(N)|$ of $T(N)$ satisfies

$$|T(N)| \sim \sum_{V \in T(N)} |V| \sim N^{m+n-(\tau+1)n-n\eta},$$

where the implied constants do not depend on $N$. It follows from volume considerations and a counting argument that the set $T(N)$ is sufficiently regular and numerous to "measure" in a rough sense the volume of a set (see [7], Lemmas 5 and 6; [13], Lemmas 8.4 and 8.5). Indeed for any set $X \in I^{mn}$ with boundary of measure 0, the measure of the intersection of $X$ and $T(N)$ satisfies

$$|X \cap T(N)| \sim |X||T(N)|. \tag{12}$$

When the set $X$ depends on $N$, i.e., when $X = X(N)$, the result breaks down but a similar counting argument gives the estimate

$$|C \cap T(N)| \ll |C||T(N)| + \ell(C)^{(m-1)n} N^{-(\tau+1)n} \tag{13}$$

for hypercubes $C$ with $\ell(C) \geqslant N^{-\tau-1}$.

To establish the lower bound for the Hausdorff dimension, let $\delta$ be any positive number and put $s = (m-1)n + (n+1)/(\tau+1) - \delta$. Suppose that the countable collection $\Gamma$ of sufficiently small hypercubes $C$ satisfies

$$V^s(\Gamma) = \sum_{C \in \Gamma} \ell(C)^s < \varepsilon \tag{14}$$

for some positive $\varepsilon$. The bound (14) for the $s$-volume $V^s(\Gamma)$ restricts the Lebesgue measure $V^{mn}(\Gamma)$ of $\Gamma$ and the extent to which it can cover $T(N)$. It turns out that no such collection $\Gamma$ can cover $W(m,n;\tau)$ and hence that

$$\dim W(m,n;\tau) \geqslant (m-1)n + (n+1)/(\tau+1).$$

The argument uses repeated subdivision. Dissect $I^{mn}$ into $[N_1/16\varrho_1]^{mn} \sim N_1^{m(n-m+n\eta)}$ congruent hypercubes as above. The volume estimates (12), (13) and (14) imply that the hypercubes $C$ in $\Gamma$ with $N_1^{-\tau-1} \leqslant \ell(C) < N_0^{-\tau-1}$, where $N_0, N_1$ are suitably large (in particular so that $N_0^{-\tau-1}$ is sufficiently small), cannot cover $T(N_1)$. Indeed the closed set

$$G_1 = T(N_1) \setminus \{C \in \Gamma\colon\ N_1^{-\tau-1} \leqslant \ell(C) < N_0^{-\tau-1}\}$$

has positive measure. Dissect $I^{mn}$ into $[N_2/16\varrho_2]^{mn}$ congruent hypercubes where $N_2$ is sufficently large and construct $T(N_2)$. The set

$$G_2 = T(N_2) \setminus \{C \in \Gamma\colon\ N_2^{-\tau-1} \leqslant \ell(C) < N_1^{-\tau-1}\} \subset G_1$$

also has positive measure. Repeated application with a suitably rapid increasing sequence $N_r\colon\ r = 1, 2, \ldots$ yields a decreasing sequence of non-empty closed bounded sets $G_r$, $r = 1, 2, \ldots$, with $G_{r+1} \supset G_r$ (for a detailed proof see [7], Lemma 8). Thus by Cantor's finite intersection theorem,

$$G_\infty = \bigcap_{r=1}^{\infty} G_r \neq \emptyset.$$

Now each $C \in \Gamma$ does not meet $G_\infty$, since every $C$ in $\Gamma$ is in some range $N_r^{-\tau-1} \leqslant \ell(C) < N_{r-1}^{-\tau-1}, r \geqslant 1$, and hence cannot be in

$$G_r = T(N_r) \setminus \{C \in \Gamma\colon\ N_r^{-\tau-1} \leqslant \ell(C) < N_{r-1}^{-\tau-1}\} \supset G_\infty.$$

Hence the collection $\Gamma$ cannot cover $G_\infty$. But $G_\infty \subset W(m,n;\tau)$. For suppose $A \in G_\infty$. Then $A \in G_r$ for $r = 1, 2, \ldots$, and hence $A \in T(N_r)$ for $r = 1, 2, \ldots$ Thus for each $r = 1, 2, \ldots$, there is a point $R$ in some $\Pi(\mathbf{q}, \mathbf{p}) \in S_{N_r}$ with

$$|A - R| \leqslant n^{-1} \cdot (2N_r)^{-(\tau+1)},$$

and therefore since $R \in \Pi(\mathbf{q}, \mathbf{p})$ and $|\mathbf{q}| < 2N_r$,

$$|\mathbf{q}A - \mathbf{p}| = |\mathbf{q}(A - R)| \leqslant m|\mathbf{q}||A - R| \leqslant |\mathbf{q}|(2N_r)^{-(\tau+1)} < |\mathbf{q}|^{-\tau}.$$

And since $|\mathbf{q}| > N_r$, for each $A \in G_\infty$ there are infinitely many $(\mathbf{q}, \mathbf{p}) \in \mathbb{Z}^{m+n}$ such that $|\mathbf{q}A - \mathbf{p}| < |\mathbf{q}|^{-\tau}$, i.e., $|\langle \mathbf{q}A \rangle| < |\mathbf{q}|^{-\tau}$ holds for infinitely many $\mathbf{q}$ in $\mathbb{Z}^m$. Therefore if $A \in G_\infty$, then $A \in W(m,n;\tau)$, i.e., $G_\infty \subset W(m,n;\tau)$. Because it is not a cover of $G_\infty$, $\Gamma$ certainly cannot cover $W(m,n;\tau)$ and it follows from the definition of dimension that

$$\dim W(m,n;\tau) \geqslant (m-1)n + \frac{n+1}{\tau+1}, \quad \tau > \frac{m}{n},$$

which together with the complementary inequality (6) proves that when $\tau > m/n$,

$$\dim W(m,n;\tau) = (m-1)n + \frac{n+1}{\tau+1}.$$

By Groshev's extension of Khintchine's theorem [25], when $\tau \geqslant m/n$, almost all matrices $A$ in $I^{mn}$ are in $W(m,n;\tau)$ because then the sum $\sum |\mathbf{q}|^{-\tau}$, where the summation is over all non-zero $\mathbf{q}$ in $\mathbb{Z}^n$, diverges.

Just as the geometric arguments of Eggleston [14] permitted the extension to integers $q$ lying in sequences which were not too sparse, the fairly general geometric and statistical character of the above arguments allows the vectors $\mathbf{q}$ to lie in subsets of $\mathbb{Z}^n$ which are not too irregular or sparse. For instance each coordinate of $\mathbf{q}$ can be taken to be a prime in arithmetic progression ([7], p. 353).

So far we have looked at metric Diophantine approximation and Hausdorff dimension for points in Euclidean space. The extension to points lying on manifolds $M$ such as curves or surfaces embedded in Euclidean space and which are extremal (i.e. almost all points in the induced measure on $M$ are not well approximable) is a natural but difficult question. Resonant planes which are tangent or near-tangent to $M$ are hard to handle but R. C. Baker [2] has shown that the Hausdorff dimension of the set of points $x$ on a $C^3$ (three times differentiable) curve in $\mathbb{R}^2$ with non-zero curvature everywhere except on a set of zero Hausdorff dimension such that

$$\|\mathbf{q} \cdot \mathbf{x}\| < |\mathbf{q}|^{-\tau} \tag{15}$$

for infinitely many $\mathbf{q}$ in $\mathbb{Z}^2$ has Hausdorff dimension $3/(\tau+1)$ when $\tau > 2$. In a recent joint work with Bryan Rynne and James Vickers of Southampton University, this result has been extended to smooth manifolds of dimension $m$ and codimension $n$ in $\mathbb{R}^{m+n}$ which satisfy a certain curvature condition [12]. For such manifolds the Hausdorff dimension of the set of points $\mathbf{x}$ in $M$ for which (15) holds for infinitely many $\mathbf{q} \in \mathbb{Z}^m$ is $m - 1 + (m+n+1)/(\tau+1)$ when $\tau > m+n$. The methods used differ in some important respects from those used in the Euclidean case.

In particular they are less statistical and rely on the geometry of numbers to supply the well-distributed sets $T(N)$. Other applications of these ideas include determining the Hausdorff dimension of the set of points $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2n}$ satisfying the pair of Diophantine inequalities

$$|\mathbf{q} \cdot \mathbf{x}| < |\mathbf{q}|^{-\tau}, \|\mathbf{q} \cdot \mathbf{y}\| < |\mathbf{q}|^{-\tau}$$

for infinitely many $\mathbf{q} \in \mathbb{Z}^n$. These inequalities arise in connection with Schröder's functional equation and with normal forms for periodic holomorphic vector fields and indeed the number theoretic sets discussed here are closely related to exceptional sets associated with a variety of results in analysis and dynamical systems, e.g. the Kolmogorov-Arnol'd-Moser theorem (see [13]).

Thus the ideas and research begun by Jarník nearly 60 years ago still flourish and promise to yield more interesting developments and results in the future.

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