27 Structure of the set of complete solutions of the differential equation (Qq)

This section is devoted to a study of the structure of the set of complete solutions of the differential equation (Qq); naturally, this structure depends on the character of the differential equations (q), (Q). In order to keep our study short, we shall first develop a theory applicable to the investigation of all cases, but then only study in full the case of general differential equations (q), (Q) of finite type (m), $m \geqslant 2$.

We consider two differential equations (q), (Q) in the intervals $j = (a, b)$, $J = (A, B)$ and assume that they are of the same character. From § 9.2 this means that (q), (Q) are both general, or are both special of the same finite type (m), $m \geqslant 1$, or are both oscillatory on one side, or finally are both oscillatory.

This is a necessary and sufficient condition for the existence of complete solutions of the differential equation (Qq) (§ 26.3).

27.1 Preliminary

We already have the following information:

1. If $t_0 \in j$, $X_0 \in J$ are arbitrary directly or indirectly associated numbers, then there always exist complete solutions of the differential equation (Qq) which take the value X_0 at the point t_0, i.e. $X(t_0) = X_0$.

Every such complete solution X is obtained from two directly or indirectly similar (first) normal phases α, $\mathbf{A}$ of the equations (q), (Q) with the zeros t_0, X_0, as a solution of the functional equation

$$\alpha(t) = \mathbf{A}X(t). \tag{27.1}$$

The function X increases or decreases according as the phases α, $\mathbf{A}$ are directly or indirectly similar.

We show further that:

2. Let X be a complete increasing or decreasing solution of the differential equation (Qq). Then every two numbers t, $X(t)$, with $t \in j$, $X(t) \in J$ are directly or indirectly associated respectively.

For, let $t_0 \in j$ be arbitrary. We select a normal phase α of (q) with zero t_0. Then there is a phase $\mathbf{A}$ of (Q) similar to α such that the relationship (1) holds in j. The phases α, $\mathbf{A}$ are directly or indirectly similar according as X is increasing or decreasing. From the fact that $\alpha(t_0) = 0$ we have $\mathbf{A}X(t_0) = 0$; consequently, $X(t_0)$ is the zero of $\mathbf{A}$, and on taking account of § 9.5, 2, we deduce that the numbers t_0, $X(t_0)$ are directly or indirectly associated respectively.

From the above results 1 and 2 it follows that:

3. For every number $t \in j$ the values $X(t)$ of all increasing or decreasing complete solutions of the differential equation (Qq) form respectively the set of numbers directly or indirectly associated with t.

27.2 Relations between complete solutions of the differential equation (Qq)

Let us choose, for definiteness, an increasing phase A of (Q). Then every complete solution X of the equation (Qq) is determined uniquely from a phase α of the differential equation (q) similar to A by a relation such as (1). We shall call the phase α the *generator* of X and say that X is generated by the phase α.

Obviously we have:

1. Two complete solutions X, $\bar{X}$ of the differential equation (Qq) coincide in the interval j if and only if their generators α, $\bar{\alpha}$ coincide.

The mean value theorem gives, for $t \in j$, the relationship

$$X(t) - \bar{X}(t) = \frac{1}{\dot{\mathrm{A}}(T)}[\alpha(t) - \bar{\alpha}(t)], \tag{27.2}$$

in which T is some number lying between $X(t)$ and $\bar{X}(t)$ when $X(t) \neq \bar{X}(t)$.

We see also that:

2. Two complete solutions X, $\bar{X}$ of the differential equation (Qq) are such that at every point $t \in j$ their difference has the same sign as the difference between their generators α, $\bar{\alpha}$; that is $\alpha(t) \geqslant \bar{\alpha}(t) \Rightarrow X(t) \geqslant \bar{X}(t)$ and $\alpha(t) < \bar{\alpha}(t) \Rightarrow X(t) < \bar{X}(t)$.

Moreover,

3. If in the interval J the function $\dot{\mathrm{A}}$ is always greater than some positive constant, then the difference $X(t) - \bar{X}(t)$ is bounded if $\alpha(t) - \bar{\alpha}(t)$ is bounded.

If for example the equation (Q) admits of two independent bounded integrals, then all its integrals are bounded; consequently the amplitudes of all bases of (Q) are bounded and then (by (5.14)) the function $\dot{\mathrm{A}}$ has the property described.

27.3 The structure of the set of complete solutions of the differential equation (Qq) in the case of differential equations (q), (Q) of finite type (m), $m \geqslant 2$

We assume that (q), (Q) are general differential equations of finite type (m), $m \geqslant 2$. Let M be the set of complete solutions of the differential equation (Qq). In what follows, we shall use the term *integral curve* of the differential equation (Qq) to mean the curve $[t, X(t)]$ determined by a complete solution $X \in M$ of the differential equation (Qq).

The set M obviously separates into classes M_1, M_{-1} where M_1 is formed from the increasing and M_{-1} from the decreasing functions. We shall for simplicity concern ourselves only with the set M_1, since the situation in the set M_{-1} is analogous.

1. The region covered by the integral curves of the differential equation (Qq). We set

$$j_\mu = (a_\mu, b_{-m+\mu+1}), \qquad j'_\nu = (b_{-m+\nu+1}, a_{\nu+1}),$$
$$J_\mu = (A_\mu, B_{-m+\mu+1}), \qquad J'_\nu = (B_{-m+\nu+1}, A_{\nu+1});$$
$$\lambda = 1, \ldots, m-1; \qquad \mu = 0, \ldots, m-1; \qquad \nu = 0, \ldots, m-2;$$
$$a_0 = a, \quad b_0 = b; \quad A_0 = A, \quad B_0 = B.$$

Theorem. All integral curves $[t, X(t)]$, $X \in M_1$ *pass through the* $2(m-1)$ *points* $P(a_\lambda, A_\lambda)$, $P(b_{-\lambda}, B_{-\lambda})$ *and their union covers simply and completely the region* D_1 *formed by the union of the open rectangular regions* $j_\mu \times J_\mu, j'_\nu \times J'_\nu$.

All integral curves $[t, X(t)]$, $X \in M_{-1}$ *pass through the* $2(m-1)$ *points* $P(a_\lambda, B_{-\lambda})$, $P(b_{-\lambda}, A_\lambda)$ *and cover simply and completely the region* D_{-1} *formed by the union of the rectangular open regions* $j_\mu \times J_{m-\mu-1}, j'_\nu \times J'_{m-\nu-2}$.

Proof. We restrict ourselves to the proof of the first part of this theorem.

Let $X \in M_1$. From § 27.1, 2, $X(a_\lambda)$ is a number directly associated with a_λ, that is $X(a_\lambda) = A_\lambda$. Let $P(t_0, X_0) \in D_1$, so that $t_0 \in j_\mu$, $X_0 \in J_\mu$, say. Then the numbers t_0, X_0 are directly associated and not singular. Consequently (from § 26.4) there exists precisely one complete solution of the differential equation (Qq) whose value at the point t_0 is precisely X_0. This completes the proof.

2. Normalization of the generators.

Let (u, v) $(uv' - u'v < 0)$ be a principal basis of the differential equation (q) and (U, V) $(UV' - U'V < 0)$ a principal basis of (Q). We assume that u and v respectively are left and right 1-fundamental integrals of (q) and that U and V are respectively such integrals of (Q).

We choose a number r $(= 1, \ldots, m-1)$ and further choose a normal phase **A** of the basis (U, V) with the zero A_r and another normal phase $\bar{\mathbf{A}}$ with the zero B_{-r}. The boundary characteristic of **A** is $(A_r; -r\pi, (m - r - \frac{1}{2})\pi)$ and that of $\bar{\mathbf{A}}$ is $(B_{-r}, -(m - r - \frac{1}{2})\pi, r\pi)$.

Let $P(a_r)$ be the phase bunch (§ 7.10) formed by those normal phases which vanish at the point a_r of the 1-parameter basis system $(\rho u, v)$ of (q) with $\rho \neq 0$.

For every number ρ $(\neq 0)$ we shall denote by α_ρ the normal phase of the basis $(\rho u, v)$ which is included in the phase bunch $P(a_r)$. We know that $P(a_r)$ breaks up into two sub-bunches, one of which, $P_1(a_r)$, consists of increasing phases and the other, $P_{-1}(a_r)$, of decreasing phases.

Every normal phase $\alpha_\rho \in P_\varepsilon(a_r)$ has the boundary characteristic $(a_r; -r\pi\varepsilon, (m - r - \frac{1}{2})\pi\varepsilon)$ $(\varepsilon = \pm 1)$; consequently the phase α_ρ is directly similar to **A** in the case $\varepsilon = 1$ and indirectly similar to $\bar{\mathbf{A}}$ in the case $\varepsilon = -1$. Conversely, every phase of the differential equation (q) which is directly similar to **A** or indirectly similar to $\bar{\mathbf{A}}$ has the above boundary characteristic; we deduce that it is included in the sub-bunch $P_1(a_r)$ or $P_{-1}(a_r)$ respectively.

Hence the phases of the differential equation (q) which are directly similar to **A** are precisely the elements of the sub-bunch $P_1(a_r)$; the phases of (q) which are indirectly similar to $\bar{\mathbf{A}}$ are precisely the elements of the sub-bunch $P_{-1}(a_r)$.

It follows that the increasing complete solutions of the differential equation (Qq) are (for the above choice of the phase **A**) generated by the elements in the sub-bunch

$P_1(a_r)$ while the decreasing complete solutions of (Qq) are (for the above choice of the phase $\bar{\mathbf{A}}$) generated by the elements of the sub-bunch $P_{-1}(a_r)$.

3. Properties of the structure of the set M_ε.

Let I_1 and I_{-1} be the intervals comprising all positive and all negative numbers respectively, and let $I = I_1 \cup I_{-1}$.

Corresponding to every number $\rho \in I$ we denote by X_ρ the complete solution of the differential equation (Qq) which is generated by the normal phase $\alpha_\rho \in P_1(a_r)$ using the phase $\mathbf{A}$, or by the normal phase $\alpha_\rho \in P_{-1}(a_r)$ using the phase $\bar{\mathbf{A}}$.

Let $\boldsymbol{K}$ be the mapping $\rho \to X_\rho$ of I on M. Obviously, the mapping $\boldsymbol{K}$ maps the interval I_ε on the set M_ε $(\varepsilon = \pm 1)$.

The mapping $\boldsymbol{K}$ is simple; this follows from § 27.2, 1 and § 7.12, 2.

For $\rho, \bar{\rho} \in I_\varepsilon$ and $\rho < \bar{\rho}$, in the interval j_μ or j'_ν respectively we have the relations

$$X_\rho < X_{\bar{\rho}} \qquad \text{or} \qquad X_\rho > X_{\bar{\rho}}.$$

This follows from § 27.2, 2 and § 7.12, 3.

The set M_ε admits of the following ordering relation $\prec$: for $X, \bar{X} \in M_\varepsilon$, we have $X \prec \bar{X}$ if and only if, in every interval j_μ or j'_ν the relation $X < \bar{X}$ or $X > \bar{X}$, respectively, holds.

The mapping $\boldsymbol{K}$ is order-preserving with respect to this ordering.

We assume that the values of the function $\mathbf{A}$ lie between positive bounds λ, Λ;

$$\lambda \leqslant \dot{\mathbf{A}} \leqslant \Lambda \quad (\lambda, \Lambda > 0).$$

Then we have (from (2)) the following relationship holding in the interval j for every two elements $X_\rho, X_{\bar{\rho}} \in M_\varepsilon$

$$\lambda|X_\rho - X_{\bar{\rho}}| \leqslant \alpha_\rho - \alpha_{\bar{\rho}}| \geqslant \Lambda|X_\rho - X_{\bar{\rho}}|.$$

From the first inequality and (7.32) it follows that the difference $X_\rho - X_{\bar{\rho}}$ is bounded.

In the set M_ε we define a metric, d, by means of the formula

$$d(X_\rho, X_{\bar{\rho}}) = \sup_{t \in j} |X_\rho(t) - X_{\bar{\rho}}(t)|.$$

In the interval I_ε we take the Euclidean metric. We now show that:

The mapping $\boldsymbol{K}$ is homeomorphic.

Proof. From (2) and the relations (7.33), (7.34) we have

$$d(X_\rho, X_{\bar{\rho}}) \leqslant \frac{1}{2\lambda} \frac{|\rho - \bar{\rho}|}{\sqrt{\rho\bar{\rho}}},$$

$$\frac{|\rho - \bar{\rho}|}{1 + \rho\bar{\rho}} \leqslant \tan\,[\Lambda \cdot d(X_\rho, X_{\bar{\rho}})].$$

The first relation shows that $\boldsymbol{K}$ is continuous at every point $\bar{\rho} \in I_\varepsilon$; the second relationship shows the continuity of the mapping $\boldsymbol{K}^{-1}$ at every point $X_{\bar{\rho}} \in M_\varepsilon$, which completes the proof.

27.4 Canonical forms of the differential equation (q)

We now make use of the theory of complete transformations to express the differential equation (q) in certain canonical forms. For simplicity we shall call a function $X(t)$, $t \in j = (a, b)$ a *canonical phase function* if, throughout the interval j, $X \in C_3$ and $X' > 0$ and when, moreover, the numbers $C = \lim_{t \to a+} X$, $D = \lim_{t \to b-} X$ are as set out in one of the following five cases:

I. (a) $C = 0$, $D = (m - \frac{1}{2})\pi$, $m \geqslant 1$, integral;
   (b) $C = 0$, $D = m\pi$, $m \geqslant 1$, integral;

II. (a) $C = 0$, $D = \infty$;
   (b) $C = -\infty$, $D = 0$;
   (c) $C = -\infty$, $D = \infty$.

Theorem. The carrier q of every differential equation (q), $j = (a, b)$ *can be represented by means of one of the canonical phase functions X defined in the interval j, in the form*

$$q(t) = -\{X, t\} - X'^2(t). \tag{27.3}$$

According as the differential equation (q) *is of the following types*

I. *of finite type* (m), $m \geqslant 1$ *and*
   (a) *general or*
   (b) *special,*

or

II. (a) *right oscillatory or*
   (b) *left oscillatory or*
   (c) *oscillatory,*

then the function X has the corresponding property I. (a)–II. (c).

Proof. We shall confine ourselves, as an example, to the proof in the case I. (a).

Let (q) be a general differential equation of finite type (m), $m \geqslant 1$. The differential equation (Q) with $Q = -1$ in the interval $J = (0, (m - \frac{1}{2})\pi)$ is also general of finite type (m). For the function $\mathrm{A}(T) = T$ $(T \in J)$ is obviously a first phase of the differential equation (Q) and its boundary values are $C = 0$, $D = (m - \frac{1}{2})\pi$. We have therefore $O(\mathrm{A}|J) = (m - \frac{1}{2})\pi$ and our statement follows (§ 7.16).

We see that the differential equations (q), (Q) are of the same character. It follows that there exists a first phase α of the differential equation (q) which is directly similar to A. For this phase, we have $\lim_{t \to a+} \alpha(t) = 0$, $\lim_{t \to b-} \alpha(t) = (m - \frac{1}{2})\pi$. Clearly, α is a canonical phase function with the property I. (a). Now, the solution $X(t)$ of the functional equation $\mathrm{A}(X) = \alpha(t)$, i.e. the function $\alpha(t)$, is a complete solution of the differential equation (Qq), so α satisfies the condition (3), and this completes the proof.