25 Physical application of general transformation theory

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25.1 Straight line motion in physical space

We consider two physical spaces I and II. In these spaces let time be measured during certain (open) time intervals k, K respectively on clocks [I] and [II]. We assume that the clocks [I], [II] are coordinated by means of two inverse functions $X(t)$, $t \in k$ and $x(T)$, $T \in K$, defined in the intervals k, K: that is, at any instant $t \in k$ measured on clock [I], clock [II] shows the time $T = X(t)$ $(\in K)$, and at any instant when clock [II] indicates the time $T \in K$ clock [I] shows the time $t = x(T)$ $(\in k)$. We call X and x the *time functions* for the spaces II and I respectively. With a view to the following application we assume that these functions belong to the class C_3 and their derivatives X', $\dot{x}$ are always positive. Then it is meaningful to speak of the velocity of time $X'(t)$ and acceleration of time $X''(t)$ in the space II at the instant t $(\in k)$, and in the same way of the velocity of time $\dot{x}(T)$ and acceleration of time $\ddot{x}(T)$ in the space I at the instant T $(\in K)$. Two homologous instants $t \in k$ and $T = X(t) \in K$, or $T \in K$ and $t = x(T) \in k$, we shall call *simultaneous*.

Now let oriented straight lines G_{I}, G_{II} be given in the spaces I, II respectively, upon which two points P_{I}, P_{II} are moving. On these straight lines we take fixed points O_{I} and O_{II} respectively, the origin of each line; the instantaneous distances of the moving points P_{I} and P_{II} are measured from these fixed points and are positive and negative in the positive and negative directions of the corresponding straight lines (§ 1.5).

We assume that the motions of the points P_{I}, P_{II} are governed by arbitrary differential equations

$$y'' = q(t)y, \tag{q}$$

$$\ddot{Y} = Q(T)Y \tag{Q}$$

where $t \in j$, $T \in J$, as follows: At arbitrary instants $t_0 \in j$, $T_0 \in J$ let us choose the positions of the points P_{I}, P_{II} on the straight lines G_{I}, G_{II} (that is, their distances y_0, Y_0 from the origins O_{I}, O_{II}) and let us choose also their velocities y_0', $\dot{Y}_0$. Then the subsequent motions of the points P_{I}, P_{II} follow the integrals $y(t)$, $Y(T)$ of the differential equations (q), (Q) as determined by the initial values $y(t_0) = y_0$, $y'(t_0) = y_0'$ and $Y(T_0) = Y_0$, $\dot{Y}(T_0) = \dot{Y}_0$. The position of the point P_{I} at any instant $t \in j$ is therefore given by its distance $y(t)$ from the origin O_{I}; moreover, $y(t) > 0$ or $y(t) < 0$ or $y(t) = 0$ according as the point P_{I} lies in the positive or negative direction from the origin O_{I} or is passing through this point, and similarly for the point P_{II}. If the differential equations (q), (Q) are oscillatory, then the points P_{I}, P_{II} are at all times vibrating about the origins O_{I}, O_{II}.

We assume, for definiteness, that $y(t_0) > 0$, $Y(T_0) > 0$. From the theorem of § 24.2, there is precisely one increasing broadest solution X of the differential equation (Qq), which takes the value T_0 at the point t_0 and in its interval of definition k ($\subset j$) transforms the integral Y into the portion of y defined in the interval k. Simultaneously, the function x inverse to X represents in its interval of definition $X(k) = K (\subset J)$ the increasing broadest solution of (qQ), which takes the value t_0 at the point T_0 and which transforms the integral y over the interval K into the portion of Y defined in the interval K. These transformations may be expressed by means of the formula

$$\sqrt[4]{X'(t)}y(t) = \sqrt[4]{\dot{x}(T)}\, Y(T). \tag{25.1}$$

We now choose the functions X, x during the time intervals k and K as time functions for the spaces II and I respectively. Moreover, we choose the unit of length in space I at any instant t ($\in k$) as the fourth root of the corresponding velocity in space II, that is to say $\sqrt[4]{X'(t)}$, and analogously we choose that in space II as $\sqrt[4]{\dot{x}(T)}$. Then, by the formula (1), the instantaneous distances of the points P_{I}, P_{II} from the origins O_{I}, O_{II} are always the same at any instant, that is to say the motion of the points P_{I}, P_{II} are the same during the time intervals k, K.

To summarize:

In physical spaces, for appropriate measures of time and length all straight line motions governed by differential equations of the second order are the same.

25.2 Harmonic motion

We now apply the above theory to the case of harmonic motion assuming that the motion of the points P_{I}, P_{II} are governed by the differential equations formed with arbitrary constants $\omega > 0$, $\Omega > 0$

$$y'' = -\omega^2 y, \tag{q}$$

$$\ddot{Y} = -\Omega^2 Y \tag{Q}$$

in the time interval $(-\infty, \infty)$.

The initial positions and velocities of the points P_{I}, P_{II} we shall choose as follows:

$$y_0 = \frac{c}{\sqrt{\omega}}, \qquad y_0' = 0; \qquad Y_0 = \frac{c}{\sqrt{\Omega}}, \qquad Y_0' = 0 \quad (c = \text{const} > 0).$$

Then the motion of the points P_{I}, P_{II} is given by the following integrals of the differential equations (q), (Q):

$$y(t) = \frac{c}{\sqrt{\omega}} \sin\left[\omega(t - t_0) + \frac{\pi}{2}\right], \qquad Y(T) = \frac{c}{\sqrt{\Omega}} \sin\left[\Omega(T - T_0) + \frac{\pi}{2}\right].$$

$$t, T \in (-\infty, \infty).$$

The increasing broadest solution $X(t)$ of the differential equation

$$-\{X, t\} - \Omega^2 X'^2(t) = -\omega^2, \tag{Qq}$$

which takes the value T_0 at the point t_0, and its inverse function $x(T)$ are both linear and have the following forms:

$$X(t) = \frac{\omega}{\Omega}(t - t_0) + T_0; \quad x(T) = \frac{\Omega}{\omega}(T - T_0) + t_0 \quad (t, T \in (-\infty, \infty)).$$

These functions transform the integral Y into y and y into Y, over the interval $(-\infty, \infty)$, and we have

$$\sqrt[4]{\frac{\omega}{\Omega}} \cdot \frac{c}{\sqrt{\omega}} \sin\left[\omega(t - t_0) + \frac{\pi}{2}\right] = \sqrt[4]{\frac{\Omega}{\omega}} \cdot \frac{c}{\sqrt{\Omega}} \sin\left[\Omega(T - T_0) + \frac{\pi}{2}\right]. \tag{25.2}$$

Following the ideas described above, we now take the functions X, x to be our time functions for the spaces II and I respectively in the time interval $(-\infty, \infty)$. The linearity of these functions expresses the linear passage of time in the spaces considered. Moreover, we choose the units of length in the spaces I, II to be constants, having the values $\sqrt[4]{(\omega/\Omega)}$ and $\sqrt[4]{(\Omega/\omega)}$. Then, by (2), the motions of the points P_{I}, P_{II} are the same in the time interval $(-\infty, \infty)$.

To summarize:

Straight line harmonic motions of two points in physical spaces are the same in each space if the time functions are appropriately chosen linear functions and the units of length are appropriately chosen constants.