24 Existence and uniqueness problems for solutions of the differential equation (Qq)

24.1 The existence and uniqueness theorem for solutions of the differential equation (Qq)

At the basis of general transformation theory lies the following theorem:

Theorem. Let $t_0 \in j$, $X_0 \in J$, X_0' $(\neq 0)$, X_0'' *be arbitrary. Then there is precisely one "broadest" solution* $Z(t)$ *of the differential equation* (Qq) *in a certain interval* k $(\subset j)$ *with the Cauchy initial conditions*

$$Z(t_0) = X_0, \qquad Z'(t_0) = X_0', \qquad Z''(t_0) = X_0''; \tag{24.1}$$

where "broadest" is used in the sense that every solution of (Qq) *satisfying the same initial conditions is a portion of* $Z(t)$.

Let α, $\mathbf{A}$ *be arbitrary phases of the differential equations* (q), (Q), *whose values at the points* t_0, X_0 *are linked as follows:*

$$\alpha(t_0) = \mathbf{A}(X_0); \quad \alpha'(t_0) = \dot{\mathbf{A}}(X_0)X_0'; \quad \alpha''(t_0) = \ddot{\mathbf{A}}(X_0)X_0'^2 + \dot{\mathbf{A}}(X_0)X_0''. \tag{24.2}$$

Then $Z(t)$ *is the solution of the differential equation* (Qq) *generated by the linked phases* α, $\mathbf{A}$*:*

$$Z(t) = \mathbf{A}^{-1}\alpha(t). \tag{24.3}$$

Proof. We choose one of the phases α, $\mathbf{A}$, for instance the phase α, arbitrarily; then the other, $\mathbf{A}$, is determined uniquely as in § 7.1 by the values $\mathbf{A}(X_0)$, $\dot{\mathbf{A}}(X_0)$, $\ddot{\mathbf{A}}(X_0)$ given by the formulae (2), (§ 7.1).

The solution $Z(t)$ generated by the phases α, $\mathbf{A}$ obviously satisfies the initial conditions (1). We have therefore to show that every solution $X(t)$ of (Qq) defined in an interval i $(\subset j)$ with the initial values (1) is a portion of $Z(t)$. From § 23.4, 1, the function $\bar{\alpha}(t) = \mathbf{A}[X(t)]$, which is defined in the interval i, is a portion of a phase α_0 of (q); more precisely, of that phase α_0 which is determined by the same initial values (2) as for α. It follows that $\alpha_0(t) = \alpha(t)$ for $t \in j$ and further that $\alpha(t) = \mathbf{A}[X(t)]$ for $t \in i$, thus $X(t)$ is the portion of $Z(t)$ which exists in the interval i. This completes the proof.

From § 23.4, 2 the curve defined by the function $Z(t)$ passes from boundary to boundary of the rectangular region $j \times J$.

24.2 Transformations of given integrals of the differential equations (q), (Q) into each other

We now concern ourselves with the following question; if two integrals y, Y of the differential equations (q), (Q) are given *arbitrarily*, can we transform one of them (say, Y) into a portion $\bar{y}$ of the other integral y, by means of (23.7), using a suitable solution $X(t)$ of the differential equation (Qq), $t \in i$ $(\subset j)$? If the answer is yes, then naturally the integral y is transformed by the solution x of the differential equation (qQ), inverse to X, into a portion $\bar{Y}$ of Y as in (23.10).

The answer to this question is in the affirmative, provided only that we be allowed, if necessary, to change the sign of one of the two integrals y, Y. We can even prescribe arbitrarily the value X_0 taken by the function X at an arbitrary point $t_0 \in j$, $X_0 = X(t_0)$. However, it must be emphasized that the data mentioned above cannot be chosen completely arbitrarily, since at two homologous points $T = X(t)$ $(\in I = X(i))$, and $t = x(T)$ $(\in i)$ the transformation formulae (23.7), (23.10) show that the two integrals y, Y must have the same sign or must both vanish.

We set out the principal result more precisely in the following theorem:

Theorem. Let y, Y be arbitrary integrals of the differential equations (q), (Q). *Moreover, let $t_0 \in j$, $X_0 \in J$ be arbitrary numbers, which satisfy one or other of the following conditions* (a), (b)*:*

(a) $y(t_0) \neq 0 \neq Y(X_0)$, (b) $y(t_0) = 0 = Y(X_0)$.

Then there exist broadest solutions X of the differential equation (Qq), *which take the value X_0 at the point t_0, i.e. $X_0 = X(t_0)$, and in their intervals of definition transform the integral Y into a portion $\bar{y}$ of y:*

$$\bar{y}(t) = \eta \frac{Y[X(t)]}{\sqrt{|X'(t)|}}. \tag{24.4}$$

In case (a) *there is precisely one increasing and precisely one decreasing broadest solution X of the differential equation* (Qq); *in the case* (b) *there are ∞^1 increasing and the same number of decreasing broadest solutions X.*

In both cases (a), (b) *the symbol η denotes the number ± 1, as follows:*

(a) $\eta = \operatorname{sgn} y(t_0) Y(X_0)$

(b) $\eta = \begin{cases} \operatorname{sgn} y'(t_0)\dot{Y}(X_0) & \text{for increasing solutions,} \\ -\operatorname{sgn} y'(t_0)\dot{Y}(X_0) & \text{for decreasing solutions.} \end{cases}$

Proof. We first assume that there is a solution X of the differential equation (Qq) defined in an interval k $(\subset j)$ and which is broadest in the sense of this theorem. Then the following relations hold in the interval k

$$\left.\begin{aligned} \bar{y}(t) &= \eta \frac{Y[X(t)]}{\sqrt{|X'(t)|}}, \\ \bar{y}'(t) &= \eta \left[\frac{\dot{Y}[X(t)]}{\sqrt{|X'(t)|}} X'(t) - \frac{1}{2} \frac{Y[X(t)]}{\sqrt{|X'(t)|}} \cdot \frac{X''(t)}{X'(t)}\right]. \end{aligned}\right\} \tag{24.5}$$

It is easy to verify that the functions X, X', X'' take the following values at the point t_0 in the two cases (a), (b):

$$\left.\begin{aligned} &\text{(a)}\quad X(t_0) = X_0, \qquad X'(t_0) = \varepsilon \frac{Y^2(X_0)}{y^2(t_0)}, \\ &X''(t_0) = 2 \frac{Y^2(X_0)}{y^4(t_0)} [Y(X_0)\dot{Y}(X_0) - \varepsilon y(t_0) y'(t_0)]; \\ &\text{(b)}\quad X(t_0) = X_0, \qquad X'(t_0) = \varepsilon \frac{y'^2(t_0)}{\dot{Y}^2(X_0)}; \end{aligned}\right\} \tag{24.6}$$

where $\varepsilon = \pm 1$. In case (b) the value $X''(t_0)$ is not determined by the conditions (5). Obviously, $\varepsilon = 1$ or $\varepsilon = -1$ according as X is increasing or decreasing in the interval k.

In case (a), therefore, the initial values $X(t_0) = X_0$, $X'(t_0)$ $(\neq 0)$ and $X''(t_0)$ are uniquely determined by (i) the integrals y, Y (ii) the choice of the values $t_0 \in j$, $X_0 \in J$ and (iii) whether the function X is increasing or decreasing. In case (b) this holds only for the initial values $X(t_0)$, $X'(t_0)$. From the theorem of § 24.1, it follows that the number of broadest solutions X of the differential equation (Qq) satisfying the condition of the theorem cannot exceed the number stated in this theorem.

Now let X be the broadest solution of the differential equation (Qq) determined by the initial conditions (6) (a) or (b); in the case (b) let X_0'' be arbitrary. The existence of this solution X is ensured by the theorem of § 24.1; let the interval of definition of X be k $(\subset j)$.

According to § 23.2, 1, the function

$$\bar{y}(t) = \frac{Y[X(t)]}{\sqrt{|X'(t)|}}, \tag{24.7}$$

which is defined in the interval k, is a solution of the differential equation (q) and it is in fact the portion contained in k of the integral $\bar{y}$ of (q) determined by the Cauchy initial conditions

$$\bar{y}(t_0) = \frac{Y(X_0)}{\sqrt{|X'(t_0)|}},$$

$$\bar{y}'(t_0) = \frac{\dot{Y}(X_0)}{\sqrt{|X'(t_0)|}} X'(t_0) - \frac{1}{2} \frac{Y(X_0)}{\sqrt{|X'(t_0)|}} \cdot \frac{X''(t_0)}{X'(t_0)}.$$

If we replace $X'(t_0)$, $X''(t_0)$ by the values given in the formulae (6), then in both cases (a), (b) we have

$$\bar{y}(t_0) = \eta y(t_0); \quad \bar{y}'(t_0) = \eta y'(t_0),$$

and it follows that for $t \in k$

$$\bar{y}(t) = \eta y(t).$$

Consequently the solution X of the differential equation (Qq) transforms (by (7)) the integral ηY into the portion of the integral y defined in the interval k. This completes the proof.

One remark needs to be added. The formula (7) can also be expressed as:

$$y(t) = \eta \frac{Y[X(t)]}{\sqrt{|X'(t)|}}, \tag{24.8}$$

where, however, validity is limited to the interval k $(\subset j)$. In special cases it can happen that (8) is valid in the whole interval j and at the same time the range of the function X coincides with the interval J. Then the function X transforms (by (8)) the integral ηY in its whole domain into the integral y. Naturally, this situation only occurs if the interval of definition k of X is identical with j and also the interval of definition, K, of the function x inverse to X is identical with J. This occurs, in particular, if the differential equations (q), (Q) are oscillatory. Then any two arbitrary phases

α, **A** of these differential equations are similar to each other; consequently the intervals k and j coincide and the intervals K, J coincide also (§ 9.2).

For example, the function $\sin t$ (arising from the carrier $q = -1$), is transformed into the integral $\sqrt{T}J_\nu(T)$ of the Bessel differential equation (1.24) over the whole range $t \in (-\infty, \infty)$, by means of a suitable increasing function $x_\nu(T)$ ($\in C_3$), $T \in (0, \infty)$. Hence we have the following representation of the Bessel function $J_\nu(T)$:

$$J_\nu(T) = \frac{\sin x_\nu(T)}{\sqrt{T \cdot \dot{x}_\nu(T)}}.$$