# Theory of general dispersions

In this Chapter a constructive theory of the functions known as general dispersions will be developed. In essence, these are solutions of the Kummer non-linear third order differential equation (11.1) for the case of oscillatory differential equations (Q), (q).

# 18 Introduction

### 18.1 Dispersions of the $\kappa$-th kind; $\kappa = 1, 2, 3, 4$

According to the theorem of § 13.11, all central dispersions of the first kind, $\phi_\nu$, of an oscillatory differential equation (q) in the interval $j = (a, b)$ satisfy the non-linear third order differential equation

$$-\{X, t\} + q(X) \cdot X'^2(t) = q(t). \qquad \text{(qq)}$$

Moreover, if the carrier $q\ (< 0) \in C_2$ all central dispersions of the second, third and fourth kinds, $\psi_\nu$, $\chi_\rho$, $\omega_\rho$, satisfy the differential equations $(\hat{q}_1\hat{q}_1)$, $(\hat{q}_1 q)$, $(q\hat{q}_1)$ formed with the first associated carrier $\hat{q}_1$ of $q$.

One of the aims of our further study is to obtain *all* the regular (that is to say, satisfying everywhere the inequality $X' \neq 0$) integrals $X$ of the non-linear third order differential equations (qq), $(\hat{q}_1\hat{q}_1)$, $(\hat{q}_1 q)$, $(q\hat{q}_1)$.

By the term dispersions of the first, second, third and fourth kinds of an oscillatory differential equation (q) in the interval $j = (a, b)$ we mean certain functions of a single variable defined constructively by means of the differential equations (q), $(\hat{q}_1)$. Their significance for the transformation theory under consideration lies in the fact that these functions represent all the regular integrals of the non-linear differential equations of the third order mentioned above.

Central dispersions of the 1st, 2nd, 3rd and 4th kinds are therefore special cases of the dispersions of the corresponding kinds. The theory of dispersions consists essentially in the description of properties of integrals of the above non-linear third order equations, and the connection of these integrals with the particular transformation problem relating to transformations of the differential equations (q), $(\hat{q}_1)$ into themselves and into each other.

In what follows we shall include the theory of dispersions in a broader problem as follows.

### 18.2 General dispersions

Let there be given two oscillatory differential equations (q), (Q) in the intervals $j = (a, b)$ and $J = (A, B)$ respectively:

$$y'' = q(t)y, \qquad \text{(q)}$$

$$\ddot{Y} = Q(T)Y. \qquad \text{(Q)}$$

By general dispersions of the differential equations (q), (Q) (in this order) we mean certain functions of a single variable defined constructively in the interval $j$ by means of the differential equations (q), (Q). The significance of these general dispersions consists in the fact that they represent all the regular integrals $X$ of the non-linear third order differential equation

$$-\{X, t\} + Q(X) \cdot X'^2(t) = q(t). \tag{Qq}$$

Obviously, from the general equation (Qq) we can specialize to the differential equation (qq) by setting $Q = q$, and similarly to the other differential equations $(\hat{q}_1\hat{q}_1)$, $(\hat{q}_1 q)$, $(q\hat{q}_1)$.

The theory of general dispersions consists essentially in the description of the properties of integrals of the differential equation (Qq), and the connection of these integrals with the general transformation problem relating to transformation of the differential equations (Q), (q).

It must be emphasized that the oscillatory character of the differential equations (Q), (q) is of crucial importance for this theory.