14 Extension of solutions of a differential equation (q) and their derivatives

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Special problems of central dispersions

This chapter is devoted to a study of special problems arising in the theory of linear oscillatory differential equations of the second order. We shall be concerned with problems which are related to the concept of central dispersions and which can be solved by application of the theory developed in Chapter A.

14 Extension of solutions of a differential equation (q) and their derivatives

In this paragraph we shall continue to make the previous assumptions, namely that (q) is oscillatory, $j = (a, b)$ and $q < 0$ for all $t \in j$. The last assumption will however not be needed in § 14.1 (on extension of solutions) but is first required in § 14.2 (on extension of derivatives of solutions).

14.1 Extension of solutions of the differential equation (q)

The elementary theory of linear differential equations of the second order shows that for every integral v of (q) the function defined by $v(t) \int_x^t d\sigma/v^2(\sigma)$ is a solution of (q) independent of v, in a neighbourhood of every point $x \in j$ which is not a zero of the integral v. (§ 1.2). We now wish to extend this solution over the entire interval j, in terms of values of the integral v.

Let, therefore, v be an arbitrary integral of the differential equation (q), and let t_0 denote a zero of v. We denote by

$$\cdots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \cdots \tag{14.1}$$

the set of zeros of v. Then in the above notation we have $v(t_\nu) = 0$, $t_\nu = \phi_\nu(t_0)$; $\nu = 0, \pm 1, \pm 2, \ldots$.

Let $j_\nu = (t_\nu, t_{\nu+1})$. Moreover let $x_0 \in j_0$ be an arbitrary number and $x_\nu = \phi_\nu(x_0)$; we have therefore $x_\nu \in j_\nu$.

Now we define, in the interval j, a function u, which we shall conveniently denote by

$$v(t) \int_{(x)}^t \frac{d\sigma}{v^2(\sigma)} \tag{14.2}$$

as follows:

$$u(t) = \begin{cases} v(t) \displaystyle\int_{x_\nu}^t \frac{d\sigma}{v^2(\sigma)} & \text{for} \quad t \in j_\nu, \\ -\dfrac{1}{v'(t_\nu)} & \text{for} \quad t = t_\nu. \end{cases} \tag{14.3}$$

We note first that the function u represents a solution of the differential equation (q) in every interval j_ν, and in fact the solution with the initial values

$$u(x_\nu) = 0, \qquad u'(x_\nu) = \frac{1}{v(x_\nu)}. \tag{14.4}$$

Further, u clearly satisfies the limiting conditions

$$\lim_{t \to t_\nu -} u(t) = -\frac{1}{v'(t_\nu)} = \lim_{t \to t_\nu +} u(t).$$

Hence the function u is everywhere continuous and in every interval j_ν clearly represents the solution of the differential equation (q) determined by the initial values (4).

Now let $U(t)$, $t \in j$, be the integral of the differential equation (q) determined by the initial values

$$U(x_0) = 0, \qquad U'(x_0) = \frac{1}{v(x_0)}.$$

Then, at every point x_ν,

$$U(x_\nu) = 0,$$

and further, from (13.5),

$$U'(x_\nu) = U'[\phi_\nu(x_0)] = (-1)^\nu \frac{U'(x_0)}{\sqrt{\phi_\nu(x_0)}} = \frac{1}{(-1)^\nu v(x_0)\sqrt{\phi_\nu(x_0)}} = \frac{1}{v[\phi_\nu(x_0)]} = \frac{1}{v(x_\nu)}.$$

Consequently the integral U and its derivative U' take the same values at the point x_ν as the functions u, u', hence the functions u, U coincide in every interval j_ν. We have, therefore, $u(t) = U(t)$ in the entire interval j, with the possible exceptions of the points t_ν. But by the continuity of the functions u, U in the interval j it follows that $u(t) = U(t)$ at each point t_ν; hence the function u is an integral of the differential equation (q) in the interval j.

To sum up:

The function

$$u(t) = v(t) \int_{(x)}^{t} \frac{d\sigma}{v^2(\sigma)}$$

represents in the interval j the integral of the differential equation (q) *determined by the initial values* $u(x_0) = 0$, $u'(x_0) = \frac{1}{v(x_0)}$. *The integrals u, v are independent; the Wronskian of the basis* (u, v) *has the value* -1.

A consequence of this result is worth noting. Every (first) phase α of the basis (u, v) satisfies the relationship

$$\tan \alpha(t) = \int_{(x)}^{t} \frac{d\sigma}{v^2(\sigma)}$$

in the interval j, with the exception of the points t_ν. Consider, in particular, the phase α_0 with the zero x_0. This is obviously given by

$$\alpha_0(t) = \text{Arctan} \int_{(x)}^{t} \frac{d\sigma}{v^2(\sigma)}, \qquad \alpha_0(t_\nu) = (2\nu - 1)\frac{\pi}{2}$$

where the symbol Arctan denotes that branch of the function, in the interval j_ν, which takes the value $\nu\pi$ at the point x_ν. The initial values of the phase α_0 at the point x_0 are

$$\alpha_0(x_0) = 0, \qquad \alpha_0'(x_0) = \frac{1}{v^2(x_0)}, \qquad \alpha_0''(x_0) = -2\frac{v'(x_0)}{v^2(x_0)}.$$

The formula (5.18) gives

$$q(t) = -\left\{\int_{(x)} \frac{d\sigma}{v^2(\sigma)}, t\right\}.$$

14.2 Extension of derivatives of solutions of the differential equation (q)

Again, let v be an arbitrary integral of the differential equation (q) and let t_0' be a zero of its derivative v'. Analogously to the above study, we define $v'(t_\nu') = 0$, $t_\nu' = \psi_\nu(t_0')$; $j_\nu' = (t_\nu', t_{\nu+1}')$; $\nu = 0, \pm 1, \pm 2, \ldots$. We choose an arbitrary number $x_0' \in j_0'$ and set $x_\nu' = \psi_\nu(x_0')$. Our supposition that $q < 0$ for all $t \in j$ implies that $x_\nu' \in j_\nu'$.

In the interval j we define the function u', which we conveniently denote by

$$v'(t) \int_{(x')}^{t} \frac{q(\sigma)}{v'^2(\sigma)}\, d\sigma$$

as follows:

$$u'(t) = \begin{cases} v'(t) \displaystyle\int_{x_\nu'}^{t} \frac{q(\sigma)}{v'^2(\sigma)}\, d\sigma & \text{for} \quad t \in j_\nu', \\ -\dfrac{1}{v(t_\nu')} & \text{for} \quad t = t_\nu'. \end{cases}$$

Then we show similarly that:

The function

$$u'(t) = v'(t) \int_{(x')}^{t} \frac{q(\sigma)}{v'^2(\sigma)}\, d\sigma$$

represents in the interval j the derivative of the integral u of (q) *determined by the initial values* $u(x_0') = \dfrac{1}{v'(x_0')}$, $u'(x_\nu') = 0$. *The integrals u, v are independent, the Wronskian of the basis (u, v) being equal to* 1.