Linear Differential Transformations of the Second Order

11 The transformation problem

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Theory of central dispersions

In the theory of central dispersions we make contact for the first time in this book with transformations of linear differential equations of the second order (§ 13.5). For this reason we begin by setting out the transformation problem itself, which may appear rather isolated at this point but as our studies continue will come more and more into the foreground.

11 The transformation problem

11.1 Historical background

The transformation problem for ordinary linear differential equations of the second order originated with the German mathematician E. E. Kummer and so can conveniently be referred to as the Kummer Transformation Problem.

In his exposition "*De generali quadam aequatione differentiali tertii ordinis*", which was first given in the year 1834 in the programme of the Evangelical Royal and State Gymnasium in Liegnitz and later, in the year 1887, was re-issued in the J. für die reine und angewandte Math. (Vol. 100), Kummer considered the non-linear third order differential equation

$$2\frac{d^3z}{dz\,dx^2} - 3\left(\frac{d^2z}{dz\,dx}\right)^2 - Z\frac{dz^2}{dx^2} + X = 0. \tag{11.1}$$

It may perhaps be interesting to reproduce (in translation) the starting point of Kummer's study,

"We first notice that our equation, which is of the third order, can be reduced to two linear equations of the second order

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0, \tag{11.2}$$

$$\frac{d^2v}{dz^2} + P\frac{dv}{dz} + Qv = 0, \tag{11.3}$$

in which p and q are functions of the variable x, P and Q functions of the variable z. However, we shall grasp this inherently difficult problem more clearly, if we derive instead the equation (1) from the equations (2) and (3). To achieve this, let us consider z as a function of the variable x and assume that the variable $y = wv$, where w is a given function of the variable x, satisfies equation (2). Then by differentiation it follows, when we hold the differential dx constant, that

$$y = wv,$$

$$\frac{dy}{dx} = \frac{dw}{dx}v + w\frac{dv}{dz}\frac{dz}{dx},$$

$$\frac{d^2y}{dx^2} = \frac{d^2w}{dx^2}v + 2\frac{dw}{dx}\frac{dz}{dx}\frac{dv}{dz} + w\frac{d^2z}{dx^2}\frac{dv}{dz} + w\frac{dz^2}{dx^2}\frac{d^2v}{dz^2},$$

and when we substitute these values in the equation (2) we obtain

$$w\frac{dz^2}{dx^2}\frac{d^2v}{dz^2}+\left(2\frac{dw}{dx}\frac{dz}{dx}+w\frac{d^2z}{dx^2}+pw\frac{dz}{dx}\right)\frac{dv}{dz}+\left(\frac{d^2w}{dx^2}+p\frac{dw}{dx}+qw\right)v=0. \tag{11.4}$$

This is a second order linear equation in the variable v, and must be identical with equation (3) which has the same form; this will be the case if we set

$$2\frac{dw}{dx}\frac{dz}{dx}+w\frac{d^2z}{dx^2}+pw\frac{dz}{dx}-Pw\frac{dz^2}{dx^2}=0, \tag{11.5}$$

$$\frac{d^2w}{dx^2}+p\frac{dw}{dx}+\left(q-Q\frac{dz^2}{dx^2}\right)w=0. \tag{11.6}$$

From these equations (5) and (6) there follows by elimination of the variable w and its derivatives the third order equation:

$$2\frac{d^3z}{dz\,dx^2}-3\left(\frac{d^2z}{dz\,dx}\right)^2-\left(2\frac{dP}{dz}+P^2-4Q\right)\frac{dz^2}{dx^2}+\left(2\frac{dp}{dx}+p^2-4q\right)=0, \tag{11.7}$$

which for

$$2\frac{dP}{dz}+P^2-4Q=Z; \qquad Q=\frac{1}{4}\left(2\frac{dP}{dz}+P^2-Z\right), \tag{11.8}$$

$$2\frac{dp}{dx}+p^2-4q=X; \qquad q=\frac{1}{4}\left(2\frac{dp}{dx}+p^2-X\right) \tag{11.9}$$

goes over into our equation.

We find, therefore, that equation (1) gives the relationship necessary between the variables z and x in order that $y = wv$ should be an integral of equation (2), in the case when the variables y and v are determined by means of equations (2) and (3) and q and Q by equations (8) and (9).

Moreover, the quantity w, which we shall call the multiplier, is obtained from equation (5); when we divide the latter by $w\,dz/dx$, thus separating the three variables w, z and x, and integrate, it gives the formula

$$w^2=c\cdot e^{\int P dz}\cdot e^{-\int p dz}\cdot\frac{dx}{dz}; \tag{11.10}$$

in this, e denotes the base of natural logarithms and c an arbitrary constant".

11.2 Formulation of the transformation problem

The transformation problem which we wish to consider is as follows:

Let two linear differential equations of the second order be given, namely

$$y''=q(t)y, \tag{q}$$

$$\ddot{Y}=Q(T)Y \tag{Q}$$

in the (open) bounded or unbounded intervals $j = (a, b)$, $J = (A, B)$. We assume that the carriers q, Q of these differential equations are continuous in their intervals of definition j, J.

By a *transformation* of the differential equation (Q) into the differential equation (q) we mean an ordered pair $[w, X]$ of functions $w(t)$, $X(t)$, defined in an open interval i ($i \subset j$), such that for every integral Y of the differential equation (Q) the function

$$y(t) = w(t) \cdot Y[X(t)] \tag{11.11}$$

is a solution of the differential equation (q).

We make the following assumptions regarding the functions w, X:

1. $w \in C_2, \quad X \in C_3$;
2. $wX' \neq 0$ for all $t \in i$;
3. $X(i) \subset J$.

The function X we call the *transformation function* of the differential equations (q), (Q) (note the order), or more shortly the transformation; we also conveniently call it the *kernel of the transformation* $[w, X]$. The function w we shall call the *multiplier* of the transformation $[w, X]$. Naturally, these definitions comprise also the concept of transformation of the differential equation (q) into (Q).

The transformation problem which we have described above in an introductory fashion can now be formulated as follows:

To determine all reciprocal transformations of the differential equations (q), (Q) *and to describe their properties.*

Let $[w, X]$ be a transformation of (Q) into (q) and Y an integral of (Q), then we shall designate the solution of (q) defined in the interval $i \subset j$ by means of formula (11) as the *image* and the integral of the differential equation (q) including this image as the *image integral* of Y under the transformation $[w, X]$. More briefly we call these simply the *image* and *image integral* of Y.

Turning back to the above study by E. E. Kummer, we take formulae (7) and (10) with $p = P = 0$ and write t, T, $-q$, $-Q$ in place of x, z, q, Q; this then yields the following result:

Every transformation function X of the differential equations (q), (Q) *is, in its definition interval i, a solution of the non-linear third order differential equation*

$$-\{X, t\} + Q(X)X'^2 = q(t). \tag{Qq}$$

The multiplier w of each transformation $[w, X]$ of the differential equations (q), (Q), *is determined uniquely by means of its kernel X up to a multiplicative constant $k \neq 0$:*

$$w(t) = \frac{k}{\sqrt{|X'(t)|}}. \tag{11.12}$$