9 Relations between first phases of two differential equations (q), (Q)

9.1 Introduction

Let us consider two differential equations (q), (Q) in the intervals $j = (a, b)$, $J = (A, B)$.

Our investigation will depend in general on the type and kind of the differential equations (q), (Q). In the case when these differential equations admit of left or right 1-fundamental sequences, we shall denote them by

$$(a <)\, a_1 < a_2 < \cdots, \qquad (b >)\, b_{-1} > b_{-2} > \cdots$$

and

$$(A <)\, A_1 < A_2 < \cdots, \qquad (B >)\, B_{-1} > B_{-2} > \cdots$$

9.2 Linked phases

Let α, A be arbitrary (first) phases of the differential equations (q), (Q) and let their boundary values be denoted by c, d and C, D respectively.

We shall call the phases α, A *linked* if simultaneously there hold the following relations between them

$$\min(c, d) < \max(C, D); \quad \min(C, D) < \max(c, d). \tag{9.1}$$

We shall show that *the phases α, A have common values if and only if they are linked.* In other words the relation $\alpha(j) \cap \mathrm{A}(J) \neq \varnothing$ implies and is implied by the inequalities (1).

Proof. (a) Let the first inequality (1) be not satisfied; then both the numbers c, d are greater than or equal to each of the numbers C, D. Consequently $\alpha(t) > \mathrm{A}(T)$ for all $t \in j$, $T \in J$.

(b) From (1) it follows that $\min(C, D) \leqslant \min(c, d) < \max(C, D)$ or $\min(c, d) < \min(C, D) < \max(c, d)$. In the first case, we have $\min(C, D) < \alpha(t) < \max(C, D)$ at a certain point $t \in j$, and since in the interval J the function A takes all values between $\min(C, D)$ and $\max(C, D)$, there is a $T \in J$ for which $\alpha(t) = \mathrm{A}(T)$. In the second case, we have $\min(c, d) < \mathrm{A}(T) < \max(c, d)$ at a point $T \in J$ and moreover, as above, $\mathrm{A}(T) = \alpha(t)$ for some $t \in j$, and the proof is complete.

In what follows we shall assume that the phases α, A are linked. Then L, given by

$$L = \alpha(j) \cap \mathrm{A}(J), \tag{9.2}$$

is an open interval. This is obviously the range of the function α in an open interval k $(\subset j)$ and also of $\mathbf{A}$ in an open interval K $(\subset J)$. That is to say, $L = \alpha(k) = \mathbf{A}(K)$.

We now wish to determine the intervals

$$k = \alpha^{-1}(L), \quad K = \mathbf{A}^{-1}(L). \tag{9.3}$$

Let $\bar{c}$, $\bar{d}$ and $\bar{C}$, $\bar{D}$ be the normalized boundary values of the phases α, $\mathbf{A}$ and moreover let $\bar{a}$, $\bar{b}$ and $\bar{A}$, $\bar{B}$ be the normalized end points of the intervals j and J with respect to these phases (§ 7.3). Then we have

$$\left.\begin{array}{ll} \lim\limits_{t\to\bar{a}} \alpha(t) = \bar{c}, & \lim\limits_{t\to\bar{b}} \alpha(t) = \bar{d}, \\ \lim\limits_{T\to\bar{A}} \mathbf{A}(T) = \bar{C}, & \lim\limits_{T\to\bar{B}} \mathbf{A}(T) = \bar{D} \end{array}\right\} \tag{9.4}$$

and moreover

$$\bar{c} < \bar{d}, \qquad \bar{C} < \bar{D}. \tag{9.5}$$

The inequalities (1) can be written as follows

$$\bar{c} < \bar{D}, \qquad \bar{C} < \bar{d}. \tag{9.6}$$

Now, on examining the conditions (5), (6) it is clear that the following five cases, and only these, can occur:

1. $\bar{C} \leqslant \bar{c} < \bar{D} < \bar{d}$, hence $L = (\bar{c}, \bar{D})$;
2. $\bar{C} < \bar{c} < \bar{d} \leqq \bar{D}$, hence $L = (\bar{c}, \bar{d})$;
3. $\bar{c} < \bar{C} < \bar{D} \leqq \bar{d}$, hence $L = (\bar{C}, \bar{D})$;
4. $\bar{c} \leqslant \bar{C} < \bar{d} < \bar{D}$, hence $L = (\bar{C}, \bar{d})$;
5. $\bar{C} = \bar{c} < \bar{D} = \bar{d}$, hence $L = (\bar{c}, \bar{d}) = (\bar{C}, \bar{D})$.

The intervals, k, K are consequently (taking account of (4)) determined in the individual cases as follows

1. $k = (\bar{a}, \alpha^{-1}(\bar{D}))$,
   $K = (\mathbf{A}^{-1}(\bar{c}), \bar{B})$ or $= (\bar{A}, \bar{B})$, according as $\bar{C} < \bar{c}$ or $\bar{C} = \bar{c}$
2. $k = (\bar{a}, \bar{b})$,
   $K = (\mathbf{A}^{-1}(\bar{c}), \mathbf{A}^{-1}(\bar{d}))$ or $= (\mathbf{A}^{-1}(\bar{c}), \bar{B})$, according as $\bar{d} < \bar{D}$ or $\bar{d} = \bar{D}$
3. $k = (\alpha^{-1}(\bar{C}), \alpha^{-1}(\bar{D}))$ or $= (\alpha^{-1}(\bar{C})), \bar{b})$, according as $\bar{D} < \bar{d}$ or $\bar{D} = \bar{d}$
   $K = (\bar{A}, \bar{B})$;
4. $k = (\alpha^{-1}(\bar{C}), \bar{b})$ or $= (\bar{a}, \bar{b})$, according as $\bar{c} < \bar{C}$ or $\bar{c} = \bar{C}$
   $K = (\bar{A}, \mathbf{A}^{-1}(\bar{d}))$;
5. $k = (\bar{a}, \bar{b})$, $K = (\bar{A}, \bar{B})$.

Combining these, we obtain the following result:

Either: at least one of the end points of the interval k coincides with an end point of j, and at the same time at least one of the end points of the interval K coincides with an end point of J (1; 2, $\bar{d} = \bar{D}$; 3, $\bar{D} = \bar{d}$; 4; 5)

or the interval k coincides with j, and the end points of K lie in the interval J (2, $\bar{d} < \bar{D}$)

or the interval K coincides with J and the end points of k lie in the interval j (3, $\bar{D} < \bar{d}$).

We see in particular that *the interval k coincides with j and simultaneously the interval K coincides with J, if and only if $\bar{c} = \bar{C}$, $\bar{d} = \bar{D}$.* In other words: the ranges of the phases α, A coincide, over their definition intervals j, J, if and only if $C = c$, $D = d$ or $C = d$, $D = c$.

We call the differential equations (q), (Q) *of the same character* if either (i) both are of the same finite type (m), $m \geqslant 1$, and of the same kind (therefore both general or special) or (ii) each is one-sided oscillatory or (iii) both are oscillatory.

We then have (§ 7.2).

The ranges of the phases α, A in their definition intervals j, J can coincide only when the differential equations (q), (Q) *are of the same character.*

In §§ 9.3–9.6 we shall assume this property to hold for the differential equations (q), (Q); that is, (q), (Q) are of the same character.

9.3 Associated numbers

We call two numbers $t_0 \in j$ and $T_0 \in J$ *directly associated* with respect to the differential equations (q), (Q) (or, more shortly, directly associated) if they stand in the same relationship with respect to the numbers a_ν, $b_{-\nu}$ and A_ν, $B_{-\nu}$. Here $\nu = 0, 1, \ldots$; $a_0 = a$, $b_0 = b$; $A_0 = A$, $B_0 = B$.

That is to say:

I. In the case when the differential equations (q), (Q) are of finite type (m) $m \geqslant 1$:

(a) $m = 1$: $t_0 \in j$ arbitrary, $T_0 \in J$ arbitrary;

(b) $m \geqslant 2$:

1. $t_0 = a_{r+1}$, $T_0 = A_{r+1}$;
2. $t_0 = b_{-m+r+1}$, $T_0 = B_{-m+r+1}$;
3. $a_r < t_0 < b_{-m+r+1}$, $A_r < T_0 < B_{-m+r+1}$;
4. $b_{-m+r+1} < t_0 < a_{r+1}$, $B_{-m+r+1} < T_0 < A_{r+1}$;

$(r = 0, 1, \ldots, m-1)$.

II. In the case when the differential equations (q), (Q) are of infinite type;

(a) Both differential equations (q), (Q) being right oscillatory:

1. $t_0 = a_{r+1}$, $T_0 = A_{r+1}$;
2. $a_r < t_0 < a_{r+1}$, $A_r < T_0 < A_{r+1}$ $(r = 0, 1, \ldots)$.

(b) Both being left oscillatory:

1. $t_0 = b_{-r-1}$, $T_0 = B_{-r-1}$;

2. $b_{-r-1} < t_0 < b_{-r}$, $B_{-r-1} < T_0 < B_{-r}$ $(r = 0, 1, \ldots)$.

(c) Both being oscillatory: $t_0 \in j$ and $T_0 \in J$ are arbitrary.

Further, we call two numbers $t_0 \in j$ and $T_0 \in J$ *indirectly associated* with respect to the differential equations (q), (Q) (more briefly, indirectly associated) if they stand in converse relationship with respect to the numbers a_ν, $b_{-\nu}$ and A_ν, $B_{-\nu}$. Here $\nu = 0, 1, \ldots$; $a_0 = a$, $b_0 = b$: $A_0 = A$, $B_0 = B$. That is to say:

I. In the case when the differential equations (q), (Q) are of finite type (m) $m \geqslant 1$:

(a) $m = 1$: $t_0 \in j$ arbitrary, $T_0 \in J$ arbitrary;

(b) $m \geqslant 2$:
1. $t_0 = a_{r+1}$, $T_0 = B_{-r-1}$;
2. $t_0 = b_{-m+r+1}$; $T_0 = A_{m-r-1}$;
3. $a_r < t_0 < b_{-m+r+1}$, $A_{m-r-1} < T_0 < B_{-r}$;
4. $b_{-m+r+1} < t_0 < a_{r+1}$, $B_{-r-1} < T_0 < A_{m-r-1}$;

$(r = 0, 1, \ldots, m-1)$.

II. In the case when the differential equations (q), (Q) are of infinite type:

(a) The differential equation (q) being right oscillatory and (Q) being left oscillatory:

1. $t_0 = a_{r+1}$, $T_0 = B_{-r-1}$;

2. $a_r < t_0 < a_{r+1}$, $B_{-r-1} < T_0 < B_{-r}$ $(r = 0, 1, \ldots)$.

(b) The differential equation (q) being left oscillatory and (Q) being right oscillatory:

1. $t_0 = b_{-r-1}$, $T_0 = A_{r+1}$;

2. $b_{-r-1} < t_0 < b_{-r}$, $A_r < T_0 < A_{r+1}$ $(r = 0, 1, \ldots)$.

(c) Both differential equations (q), (Q) being oscillatory: $t_0 \in j$ and $T_0 \in J$ are arbitrary.

If therefore the differential equations (q), (Q) are of type (1) or oscillatory, then every two numbers $t_0 \in j$ and $T_0 \in J$ are both directly and indirectly associated. But it is possible in other cases also to have two numbers $t_0 \in j$, $T_0 \in J$ which possess this property. To be precise, this occurs in special differential equations (q), (Q), when m (> 0) is even and $t_0 = a_{\frac{1}{2}m} = b_{-\frac{1}{2}m}$; $T_0 = A_{\frac{1}{2}m} = B_{-\frac{1}{2}m}$; it also occurs in general differential equations (q), (Q) if m is odd and $a_{\frac{1}{2}(m-1)} < t_0 < b_{-\frac{1}{2}(m-1)}$, $A_{\frac{1}{2}(m-1)} < T_0 < B_{-\frac{1}{2}(m-1)}$ or if m (> 0) is even and $b_{-\frac{1}{2}m} < t_0 < a_{\frac{1}{2}m}$, $B_{-\frac{1}{2}m} < T_0 < A_{\frac{1}{2}m}$.

We also observe that if $t_0 \in j$ is a singular number of the differential equation (q) (§ 3.10), then there is precisely one directly associated number or one indirectly associated number $T_0 \in J$ which is a singular number of (Q). Any non-singular number $t_0 \in j$ has always ∞^1 directly or indirectly associated numbers $T_0 \in J$, the set of which represents an open subinterval of J.

9.4 Characteristic triples of two differential equations

Let $t_0 \in j$, $T_0 \in J$ be directly or indirectly associated with respect to the differential equations (q), (Q). Then we have the following:

Theorem. If t_0; c, d is a characteristic triple for the differential equation (q), *then T_0; c, d or T_0; d, c is a characteristic triple for the differential equation* (Q).

Proof. Let t_0; c, d be a characteristic triple for the differential equation (q). The numbers t_0; c, d therefore satisfy one of the relationships (I–IIc) obtained in § 7.13, according to the type and kind of the differential equation (q).

Let $C = c$, $D = d$ or $C = d$, $D = c$ according as t_0, T_0 are directly or indirectly associated.

The theorem will be proved if we can show that the values T_0; A_ν, $B_{-\nu}$; C, D; $\mathbf{E}$ ($= \operatorname{sgn}(D - C)$) satisfy the appropriate conditions I–IIc of § 7.13 corresponding to the type and kind of the differential equation (Q), ($\nu = 0, 1, \ldots$).

(a) Let the numbers t_0, T_0 be directly associated. Then the number T_0 stands in the same relation to A_ν, $B_{-\nu}$ as does t_0 in relation to a_ν, $b_{-\nu}$. Since moreover $C = c$, $D = d$; $\mathbf{E} = \varepsilon$, the condition which has to be satisfied by T_0; A_ν, $B_{-\nu}$; C, D; $\mathbf{E}$ is a consequence of the corresponding condition satisfied by t_0; a_ν, $b_{-\nu}$; c, d; ε.

(b) Let the numbers t_0, T_0 be indirectly associated. Then the number T_0 stands in the converse relationship to A_ν, $B_{-\nu}$ as does t_0 in relation to a_ν, $b_{-\nu}$. Moreover we have $C = d$, $D = c$; $\mathbf{E} = -\varepsilon$. Let us consider, for definiteness, the case I(a), $m \geqslant 2$ and

$$a_r < t_0 < b_{-m+r+1}; \qquad -(r+1)\pi\varepsilon \lesseqgtr c \lesseqgtr -r\pi\varepsilon;$$
$$(m - r - 1)\pi\varepsilon \lesseqgtr d \lesseqgtr (m - r)\pi\varepsilon. \tag{9.7}$$

Since the numbers t_0, T_0 are indirectly associated, we have

$$A_{m-r-1} < T_0 < B_{-r}. \tag{9.8}$$

From the relations (7) it follows that

$$-(r+1)\pi\varepsilon \lesseqgtr D \lesseqgtr -r\pi\varepsilon; \quad (m - r - 1)\pi\varepsilon \lesseqgtr C \lesseqgtr (m - r)\pi\varepsilon.$$

In these formulae, for $\varepsilon = 1$ and $\varepsilon = -1$ (that is, for $\mathbf{E} = -1$ and $\mathbf{E} = 1$), we take the signs $<$ and $>$ respectively.

We have therefore

$$(r+1)\pi\mathbf{E} \gtrless D \gtrless r\pi\mathbf{E}; \quad -(m - r - 1)\pi\mathbf{E} \gtrless C \gtrless -(m - r)\pi\mathbf{E},$$

and these formulae can be written as:

$$-(m - r)\pi\mathbf{E} \lesseqgtr C \lesseqgtr -(m - r - 1)\pi\mathbf{E}; \quad r\pi\mathbf{E} \lesseqgtr D \lesseqgtr (r+1)\pi\mathbf{E}. \tag{9.9}$$

If in (8) and (9) we write r in place of $m - r - 1$, we then have

$$A_r < T_0 < B_{-m+r+1}; \qquad -(r+1)\pi\mathbf{E} \lesseqgtr C \lesseqgtr -r\pi\mathbf{E};$$
$$(m - r - 1)\pi\mathbf{E} \lesseqgtr D \lesseqgtr (m - r)\pi\mathbf{E}.$$

This is precisely the relationship (7), written with capital instead of small letters.

The proof in other cases proceeds similarly.

9.5 Similar phases

We call two phases α, $\mathbf{A}$ of the differential equations (q), (Q) *similar* if their normalized boundary values $\bar{c}$, $\bar{d}$ and $\bar{C}$, $\bar{D}$ coincide: $\bar{c} = \bar{C}$, $\bar{d} = \bar{D}$. This obviously occurs if and only if the boundary values c, d of α and C, D of $\mathbf{A}$ are related by either $C = c$, $D = d$ or $C = d$, $D = c$.

If $C = c$, $D = d$ then to be more precise we call the phases α, $\mathbf{A}$ *directly similar*; in this case we have $\operatorname{sgn} \alpha' \operatorname{sgn} \dot{\mathbf{A}} = 1$. If $C = d$, $D = c$, we call the phases α, $\mathbf{A}$ *indirectly similar*; in this case we have $\operatorname{sgn} \alpha' \operatorname{sgn} \dot{\mathbf{A}} = -1$.

If, for instance, the differential equations (q), (Q) are oscillatory, then for all their phases α, $\mathbf{A}$: $\bar{c} = \bar{C} = -\infty$, $\bar{d} = \bar{D} = \infty$. From this it follows that every two phases α, $\mathbf{A}$ of the differential equations (q), (Q) are similar; more precisely, they are directly similar if both phases increase or both decrease, and indirectly similar if one increases and the other decreases.

In particular we have (§ 9.2)

The ranges of two phases α, $\mathbf{A}$ *of the differential equations* (q), (Q) *coincide in their intervals of definition if and only if the phases* α, $\mathbf{A}$ *are similar.*

Now let α, $\mathbf{A}$ be directly or indirectly similar phases. We prove the following results:

1. *The phases* α, $\mathbf{A}$ *take the same value at two directly or indirectly associated singular points of the differential equations* (q), (Q).

Proof. We apply the formulae (7.10) to the left or right null phases $\alpha - c$, $\mathbf{A} - C$ or $\alpha - d$, $\mathbf{A} - D$ of the differential equations (q), (Q) and obtain

$$\left.\begin{array}{ll} \alpha(a_\nu) = c + \varepsilon\nu\pi, & \alpha(b_{-\nu}) = d - \varepsilon\nu\pi; \\ \mathbf{A}(A_\nu) = C + \mathbf{E}\nu\pi, & \mathbf{A}(B_{-\nu}) = D - \mathbf{E}\nu\pi \\ (\nu = 1, 2, \ldots; \quad \varepsilon = \operatorname{sgn} \alpha', & \mathbf{E} = \operatorname{sgn} \dot{\mathbf{A}}). \end{array}\right\} \tag{9.10}$$

(a) Let the phases α, $\mathbf{A}$ be directly similar:

$$C = c, \qquad D = d; \qquad \mathbf{E} = \varepsilon. \tag{9.11}$$

From § 9.3 any two directly associated singular points t_0, T_0 of the differential equations (q), (Q) must be either $t_0 = a_\nu$, $T_0 = A_\nu$ or $t_0 = b_{-\nu}$, $T_0 = B_{-\nu}$ ($\nu = 1, 2, \ldots$). In both cases there follows from (10) and (11) the relationship $\alpha(t_0) = \mathbf{A}(T_0)$.

(b) Let the phases α, $\mathbf{A}$ be indirectly similar:

$$C = d, \qquad D = c; \qquad \mathbf{E} = -\varepsilon. \tag{9.12}$$

From § 9.3 any two indirectly associated singular points t_0, T_0 of the differential equations (q), (Q) must be either $t_0 = a_\nu$, $T_0 = B_{-\nu}$ or $t_0 = b_{-\nu}$ and $T_0 = A_\nu$ ($\nu = 1, 2, \ldots$). In both cases, from (10) and (11) it follows that $\alpha(t_0) = \mathbf{A}(T_0)$. This completes the proof.

We now assume that α, $\mathbf{A}$ are similar normal phases. We denote their zeros by t_0, T_0.

2. *According as the phases* α, $\mathbf{A}$ *are directly or indirectly similar, their zeros* t_0, T_0 *are directly or indirectly associated.*

Proof. (a) Let the phases α, $\mathbf{A}$ be directly similar. In this case formulae (11) hold and we have relationships similar to those in the theorem of § 7.13 for t_0; c, d; ε and similarly for T_0; C, D; $\mathbf{E}$. Consequently the numbers t_0 and T_0 stand in the same relationship with respect to the numbers a_ν, $b_{-\nu}$ and A_ν, $B_{-\nu}$ respectively. ($\nu = 0, 1, \ldots$).

(b) Let the phases α, $\mathbf{A}$ be indirectly similar; then formulae (12) hold and the reader will easily convince himself that in all possible cases, whether the differential equations (q), (Q) are of finite or infinite type, the numbers t_0 and T_0 stand respectively in converse relationship with respect to the numbers a_ν, $b_{-\nu}$ and A_ν, $B_{-\nu}$ ($\nu = 0, 1, \ldots$).

For instance, consider the case when the differential equations (q), (Q) are of a finite type (m), $m \geqslant 2$, and $t_0 = a_{r+1}$. Then from the theorem of § 7.13, $-(r + 1)\pi\varepsilon = c$. From this, and (12), it follows that $(r + 1)\pi\mathbf{E} = D$ and moreover (from the same theorem) $T_0 = B_{-r-1}$.

9.6 Existence of similar phases

We now consider the question whether the differential equations (q), (Q), of the same character possess similar phases and, if so, how many such there are. We shall establish the following theorem:

Theorem. Let α *be a normal phase of the differential equation* (q) *and* t_0 *its zero. Let* T_0 *be a number which is directly or indirectly associated with* t_0 *with respect to the differential equations* (q), (Q). *Then there always exist normal phases* $\mathbf{A}$ *of the differential equation* (Q) *with the zero* T_0 *which are directly or indirectly similar to the phase* α. *According to the type and kind of the differential equations* (q), (Q) *and according to whether the numbers* t_0, T_0 *are singular or not, there is either one normal phase* $\mathbf{A}$ *or there is one* 1- *or* 2-*parameter system of normal phases* $\mathbf{A}$.

Proof. Let t_0; c, d be the boundary characteristic of α. Then from § 9.4 T_0; c, d or T_0; d, c is a characteristic triple for the differential equation (Q). From § 7.15 there exist normal phases $\mathbf{A}$ of (Q) with the boundary characteristic T_0; c, d or T_0; d, c. According to the type and kind of the differential equation (Q) and according as the number T_0 is singular or not, there is either one normal phase $\mathbf{A}$ or one 1- or 2-parameter system of normal phases of the differential equation (Q) with the boundary characteristic mentioned. Obviously, each normal phase $\mathbf{A}$ is directly or indirectly similar to the phase α and T_0 is its zero. This completes the proof.

More precisely (from § 7.15) the situation is as follows:

There is one normal phase $\mathbf{A}$ if the differential equations (q), (Q) are general either of type (1) or of type (m), $m \geqslant 2$, the numbers t_0, T_0 not being singular.

There are precisely ∞^1 normal phases $\mathbf{A}$, if the differential equations (q), (Q) are general of type (m), $m \geqslant 2$, and the numbers t_0, T_0 are singular; also, if (q), (Q) are special of type (1) or of type (m), $m \geqslant 2$, the numbers t_0, T_0 not being singular, and finally, if (q), (Q) are 1-sided oscillatory and the numbers t_0, T_0 are not singular.

There are precisely ∞^2 normal phases **A**, if the differential equations (q), (Q) are special of type (m), $m \geqslant 2$, the numbers t_0, T_0 being singular; also if (q), (Q) are 1-sided oscillatory and t_0, T_0 are singular, and finally, if (q), (Q) are oscillatory.

In § 9.2 we saw that the ranges of two phases α, **A** in the definition intervals j, J of the latter can only coincide when the differential equations (q), (Q) are of the same character. This observation, when taken together with the above result, shows that *the differential equations* (q), (Q) *admit of similar phases if and only if they are of the same character.*