8 Elementary phases

In this section we shall consider phases with certain special properties, known as elementary phases, which we shall often meet in the course of our researches. In order to set out this study as simply as possible, we shall introduce the concept of an elementary phase in a somewhat narrow sense, which however will suffice for our purposes. In this connection we assume that the length of the definition interval $j = (a, b)$ associated with the differential equation (q) is always greater than π. Naturally, we only have to introduce this assumption when j is finite, as it is automatically true for an unbounded interval j. Moreover, whenever a *second* phase is involved we always assume that the corresponding carrier q is negative in the whole interval j.

We consider a differential equation (q).

8.1 Introduction

A first or second phase $\gamma(t)$ of (q) will be called *elementary*, if for any two values t, $t + \pi$ lying in the interval j there holds the relation

$$\gamma(t + \pi) = \gamma(t) + \varepsilon\pi \qquad (\varepsilon = \operatorname{sgn} \gamma'). \tag{8.1}$$

We sometimes speak of elementary phases of the *carrier* q.

For instance, both the first and second phases of the carrier $q(t) = -1$, namely $\alpha(t) = t$ and $\beta(t) = \frac{1}{2}\pi + t$, are elementary.

Let $\gamma(t)$ be an elementary first or second phase of (q). Clearly, the phase γ may be represented in the form

$$\gamma(t) = \varepsilon t + G(t) \qquad (\varepsilon = \operatorname{sgn} \gamma'),$$

where $G(t)$, $t \in j$, is a function with the following properties:

1. G is periodic with period π,
2. $G \in C_3$ or $G \in C_1$ according as γ is a first or second phase,
3. $\operatorname{sgn} [\varepsilon + G'(t)] = \varepsilon$ for all $t \in j$.

Further we obtain from equation (1) and the monotonicity of γ the following result: the values $\gamma(t)$, $\gamma(t + \pi)$ are either both integral multiples of π or neither of them is such a multiple. In the first case, between the numbers t, $t + \pi$ there is no point at which the function γ takes the value of an integral multiple of π, while in the second case there is precisely one such point.

Further properties are:

Every phase of the complete phase system $[\gamma]$, hence every phase of the form $\gamma(t) + \lambda$, λ arbitrary, is also elementary.

The derivative γ' is a periodic function with period π.

8.2 Properties of equations with elementary phases

Now we obtain the following theorem:

Theorem. All first phases of the differential equation (q) *are elementary, if any one first phase of* (q) *possesses this property. The same statement holds also for second phases.*

Proof. Let us assume, for instance, that the first phase α_0 of (q) is elementary. Let α be an arbitrary first phase of (q). Then for every $t \in j$, with the exception of the singular points of the functions $\tan \alpha_0(t)$, $\tan \alpha(t)$, there holds a formula corresponding to (5.39), where α and α_0 are to be read in place of $\bar{\alpha}$ and α respectively, and the c_{ij} are appropriate constants. If we evaluate each side of this formula at two points t, $t + \pi \in j$, we find that $\tan \alpha(t + \pi) = \tan \alpha(t)$ and consequently $\alpha(t + \pi) = \alpha(t) + n\pi$, $n \neq 0$ being an integer. We have to show that $|n| = 1$.

To establish this, we choose two arbitrary values x, $x + \pi \in j$ and consider the following (first) phases of (q):

$$\bar{\alpha}_0(t) = \alpha_0(t) - \alpha_0(x), \quad \bar{\alpha}(t) = \alpha(t) - \alpha(x).$$

We first note that the phases $\bar{\alpha}_0$, $\bar{\alpha}$ have the common zero x; it follows that the two integrals

$$y(t) = \sin \bar{\alpha}_0(t)/\sqrt{|\bar{\alpha}_0'(t)|}, \quad \bar{y}(t) = \sin \bar{\alpha}(t)/\sqrt{|\bar{\alpha}'(t)|}$$

of the differential equation (q) both vanish at the point x and consequently have all their zeros in common.

Moreover, the phase $\bar{\alpha}_0$ is obviously elementary; we therefore have

$$\bar{\alpha}_0(x) = 0, \quad \bar{\alpha}_0(x + \pi) = \pi \cdot \operatorname{sgn} \bar{\alpha}_0'; \quad y(t) \neq 0 \quad \text{for} \quad t \in (x, x + \pi).$$

For the phase $\bar{\alpha}$, there hold the relations

$$\bar{\alpha}(x) = 0, \quad \bar{\alpha}(x + \pi) = |n|\pi \cdot \operatorname{sgn} \bar{\alpha}'.$$

If $|n| \neq 1$, then the function $\bar{\alpha}$ takes the value $\pi \operatorname{sgn} \bar{\alpha}'$ at some point $t_0 \in (x, x + \pi)$. In this case, t_0 is a zero of $\bar{y}$ and consequently also a zero of y. This, however, is impossible since it follows from the above that the integral y does not vanish in the interval $(x, x + \pi)$. We therefore have $|n| = 1$ and the proof is complete.

According to this result a differential equation (q) either has all its first (second) phases elementary or none of them are elementary. A differential equation (q) whose first (second) phases are elementary we shall call a *differential equation with elementary first* (*second*) *phases*, and apply the same terminology also to the corresponding carriers. It is convenient also to speak of differential equations and carriers as elementary with respect to their first (second) phases.

Now let (q) be a differential equation with elementary first (second) phases. The question arises: what properties have the second (first) phases of (q)?

In order to answer this, consider a first and a second phase α and β of the same basis of (q). Between the phases α, β there holds therefore a relationship similar to (5.34), and it follows that for any two values t, $t + \pi \in j$

$$\beta(t + \pi) - \beta(t) - \varepsilon\pi = \alpha(t + \pi) - \alpha(t) - \varepsilon\pi + \operatorname{Arccot}\left[\frac{1}{2}(1/\alpha'(t + \pi))'\right] - \operatorname{Arccot}\left[\frac{1}{2}(1/\alpha'(t))'\right] \tag{8.2}$$
$$(\varepsilon = \operatorname{sgn}\alpha' = \operatorname{sgn}\beta').$$

If the phase α is elementary, then α' is a periodic function with period π and so, consequently, is Arccot $[\frac{1}{2}(1/\alpha')']$. Formula (2) then shows that the phase β is also elementary.

If, conversely, the phase β is elementary, then the left side of (2) is identically zero and it is clear that the phase α is elementary if and only if the function $(1/\alpha')'$ has period π.

To sum up:

In a differential equation (q) *with elementary first phases, the second phases are also elementary.*

In a differential equation (q) *with elementary second phases the first phases* α *are also elementary if and only if the functions* $(1/\alpha')'$ *formed from them have period* π.

8.3 Properties of integrals, and their derivatives, of differential equations (q) with elementary phases

Integrals of differential equations (q) with elementary first or second phases, and their derivatives, are distinguished by particular properties which we now examine. Our investigation will be mainly concerned with differential equations (q) possessing elementary first phases; with regard to equations with elementary second phases we shall content ourselves with an indication of the results.

We show that

If the differential equation (q) *is elementary with respect to its first phases, then the values taken by every integral* y *of* (q) *at two arbitrary points* t, $t + \pi \in j$ *are equal and opposite in sign, that is*

$$y(t + \pi) = -y(t). \tag{8.3}$$

To show this, we assume that the first phases of (q) are elementary. Consider an integral y and a first phase α of (q). Then for an appropriate choice of the constants k_1, k_2 there holds a formula like the first of the formulae (5.27). Since the phase α is elementary, α' has period π, so, at two arbitrary points t, $t + \pi \in j$ the relation (3) holds.

Moreover

Theorem. The differential equation (q) *is elementary with respect to its first phases if and only if any two numbers* t, $t + \pi \in j$ *are neighbouring* 1*-conjugate numbers.*

Proof. (a) Let the differential equation (q) be elementary with respect to its first phases.

Let $t, t + \pi \in j$ be arbitrary numbers; let us choose a first phase α and an integral y of (q) which vanishes at t. Then we have a formula such as (5.27), in which $\alpha(t) + k_2 = n\pi$, n integral. Since the phase α is elementary, we have $\alpha(t + \pi) + k_2 = (n + \varepsilon)\pi$, $\varepsilon = \operatorname{sgn} \alpha'$, and between the numbers $t, t + \pi$ there is no point at which the value of the function $\alpha + k_2$ is an integral multiple of π. Consequently, $t + \pi$ is the first zero of y following t.

(b) Let two arbitrary numbers $t, t + \pi \in j$ be neighbouring 1- conjugate numbers.

We consider a first phase α of (q). Let $t, t + \pi \in j$ be arbitrary numbers, and let y be an integral of (q) which vanishes at t. Then we have a formula such as (5.27) in which $\alpha(t) + k_2 = n\pi$, where n is integral. According to our assumption, $t + \pi$ is a zero of y, and indeed the first zero of y following t; it follows that $\alpha(t + \pi) + k_2 = (n + \varepsilon)\pi$, $\varepsilon = \operatorname{sgn} \alpha'$. We have therefore $\alpha(t + \pi) = \alpha(t) + \varepsilon\pi$, which establishes our result.

The theorem which has just been proved can obviously be formulated as follows:

The differential equation (q) is elementary with respect to its first phases if and only if the zeros of all its integrals are situated at a distance π apart; that is, if two neighbouring zeros always have the same separation π.

The following property of a differential equation (q) with elementary first phases, in the case $b - a > 2\pi$, is worth mentioning: the integrals of such an equation are periodic functions with the fundamental period 2π.

For, by the relation (3), the integrals of (q) have period 2π, and for the fundamental period $p > 0$ of an integral y of (q) we have $0 < p \leqslant 2\pi$. If $p < 2\pi$, then for appropriate values $t, t + p \in j$ we have the inequality $y(t)\, y(t + p) < 0$, which conflicts with the definition of p.

Analogous properties are possessed by differential equations (q) with negative carriers and elementary second phases:

If the differential equation (q) *is elementary with respect to its second phases, then for the values of the derivative y' of every integral y of* (q) *at two arbitrary points $t, t + \pi$ there holds the relationship*

$$\frac{y'(t + \pi)}{\sqrt{|q(t + \pi)|}} = - \frac{y'(t)}{\sqrt{|q(t)|}}. \tag{8.4}$$

The differential equation (q) *is elementary with respect to its second phases if and only if every two numbers $t, t + \pi \in j$ are neighbouring 2-conjugate numbers.*

The differential equation (q) is elementary with respect to its second phases if and only if the zeros of the derivatives of all its integrals are separated by a distance π.

For a differential equation (q) with elementary second phases, in the case when $b - a > 2\pi$, the function $y'(t)/\sqrt{|q(t)|}$ constructed from an arbitrary integral y of the differential equation (q) is periodic with fundamental period 2π.

We know that for a differential equation (q) with elementary first phases, the second phases are also elementary (§ 8.2). From that, and the above results, we deduce that:

For a differential equation (q) *with a negative carrier q and elementary first phases, the successive zeros of any integral of* (q) *are separated by a distance π; so also are the successive turning points of such an integral.*

8.4 Determination of all carriers with elementary first phases

Our knowledge of elementary phases now makes it possible to determine explicitly all carriers with elementary first phases. For brevity, in §§ 8.4–8.7, we shall speak of phases and elementary carriers instead of first phases and carriers with elementary first phases.

From § 8.1, every elementary phase α of a differential equation (q) may be expressed in the form

$$\alpha(t) = \varepsilon t + A(t) \quad (\varepsilon = \operatorname{sgn} \alpha') \tag{8.5}$$

involving a function $A(t)$, $t \in j$, with the following properties:

1. A has period π,
2. $A \in C_3$,
3. $\operatorname{sgn} [\varepsilon + A'(t)] = \varepsilon$ for all $t \in j$.

From § 5.5, we know that the carrier q is determined uniquely by the phase α, being given precisely by the formula (5.16), so that

$$q(t) = -\frac{1}{2}\frac{A'''(t)}{\varepsilon + A'(t)} + \frac{3}{4}\frac{A''^2(t)}{(\varepsilon + A'(t))^2} - (\varepsilon + A'(t))^2. \tag{8.6}$$

If, conversely, we choose $\varepsilon = +1$ or $\varepsilon = -1$ and an arbitrary function $A(t)$, $t \in j$, with the above properties 1–3, then the function α defined by (5) represents a phase function (§ 5.7) with the property that $\alpha(t + \pi) = \alpha(t) + \varepsilon\pi$. Consequently, this function is an elementary phase of the carrier q determined by (6).

In this way we have determined all elementary carriers in the interval j:

All elementary carriers q in the interval j are given by the formula (6)*; ε denotes* $+1$ *or* -1, *and A represents an arbitrary function in the interval j with the above properties* 1–3.

8.5 Equations with elementary phases, defined over $(-\infty, \infty)$

With subsequent applications in mind, we now consider differential equations (q), with elementary phases, on the interval j, $j = (-\infty, \infty)$. From (5) we see that each phase of such a differential equation (q) is unbounded above and below. It follows (§ 5.4) that all differential equations (q) with elementary phases in the interval $j = (-\infty, \infty)$ are oscillatory. The corresponding carriers q are naturally given by formula (6).

An example, which we shall need, of a system of elementary carriers q in the interval $j = (-\infty, \infty)$ is given by the following formula, due to F. Neuman [53] (c is an arbitrary constant):

$$q(t|c) = \frac{\sin 4(t - c) + \frac{1}{3}\sin^4 (t - c)}{\left(1 - \frac{1}{3}\sin 2(t - c) \cdot \sin^2 (t - c)\right)^2} - 1. \tag{8.7}$$

This system is obtained from formula (6) by the following choice of the function $A(t)$:

$$A(t) = \begin{cases} \arctan\left(\frac{1}{6}\cos 2(t-c) - \cot(t-c)\right) - t + \nu\pi & \text{for} \quad t \in (c + \nu\pi, c + (\nu+1)\pi), \\ -\frac{\pi}{2} - c \quad \text{for} \quad t = \nu\pi; \quad (\nu = 0, \pm 1, \pm 2, \ldots). \end{cases}$$

Here the symbol arctan denotes that branch of the function which lies in the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. Later we shall obtain this system of carriers by another method (§ 15.8).

It is easy to show that any two elementary carriers $q(t|c_1)$, $q(t|c_2)$ of the system (7) obtained from different values $c_1, c_2 \in [0, \frac{1}{4}\pi)$ represent different functions. It follows that:

The system of elementary carriers (7) *has the power of the continuum,* $\aleph$.

8.6 Power of the set of elementary carriers

We now seek to determine the power of the set of all elementary carriers in the interval $j = (-\infty, \infty)$. Let E denote this set. Since the elements of E can be obtained from an arbitrary π-periodic function A of class C_3 by means of the formula (6), it is reasonable to suppose that the power of E is $\aleph$. This can be proved formally as follows:

The system of elementary carriers given by the formula (7) represents a subset of E. Since this possesses the power $\aleph$, we have card $E \geqslant \aleph$. Moreover, the elements of E are continuous functions in the interval j, so that card $E \leqslant \aleph$; consequently:

The power of the set of all elementary carriers in the interval $j = (-\infty, \infty)$ *is precisely the power of the continuum,* $\aleph$.

8.7 Generalization of the concept of elementary phases

We now close this section with the following remarks. Consider a differential equation $(\bar{q})$ in the interval $\bar{j} = (\bar{a}, \bar{b})$. Let $c > 0$, $k > 0$ be arbitrary numbers and $\bar{b} - \bar{a} > c$.

A first or second phase $\bar{\gamma}(t)$ of $(\bar{q})$ will be called *quasi-elementary* if for every two values t, $t + c$ lying in the interval $\bar{j}$ there holds the relation

$$\bar{\gamma}(t + c) = \bar{\gamma}(t) + \varepsilon k \quad (\varepsilon = \operatorname{sgn} \bar{\gamma}').$$

It is easy to verify that the function

$$\gamma(t) = \frac{\pi}{k}\bar{\gamma}\left(\frac{c}{\pi}t\right)$$

defined by means of a quasi-elementary phase $\bar{\gamma}$ of $(\bar{q})$ in the interval (a, b), $a = (c/\pi)\bar{a}$, $b = (c/\pi)\bar{b}$ satisfies the relationship

$$\gamma(t + \pi) = \gamma(t) + \varepsilon\pi \quad (\varepsilon = \operatorname{sgn} \gamma').$$

If $\bar{\gamma}$ $(= \bar{\alpha})$ is a first quasi-elementary phase of $(\bar{q})$, then the function γ $(= \alpha) \in C_3$, and taking account of the formula (1.17), we have

$$\{\alpha, t\} = \frac{c^2}{\pi^2}\left\{\bar{\alpha}, \frac{c}{\pi} t\right\}.$$

It follows that

$$(q(t) =) -\{\alpha, t\} - \alpha'^2(t) = \frac{c^2}{\pi^2}\left[-\left\{\bar{\alpha}, \frac{c}{\pi} t\right\} - \bar{\alpha}'^2\left(\frac{c}{\pi} t\right)\right]$$
$$+ c^2\left(\frac{1}{\pi^2} - \frac{1}{k^2}\right)\bar{\alpha}'^2\left(\frac{c}{\pi} t\right) = \frac{c^2}{\pi^2}\bar{q}\left(\frac{c}{\pi} t\right) + c^2\left(\frac{1}{\pi^2} - \frac{1}{k^2}\right)\bar{\alpha}'^2\left(\frac{c}{\pi} t\right).$$

Hence, if $\bar{\alpha}(t)$ is a first quasi-elementary phase of the carrier $\bar{q}(t)$ in the interval $(\bar{a}, \bar{b})$ then the function

$$\alpha(t) = \frac{\pi}{k}\,\bar{\alpha}\left(\frac{c}{\pi} t\right)$$

represents in the interval (a, b), $a = (c/\pi)\bar{a}$, $b = (c/\pi)\bar{b}$ a first elementary phase of the carrier

$$q(t) = \frac{c^2}{\pi^2}\bar{q}\left(\frac{c}{\pi} t\right) + c^2\left(\frac{1}{\pi^2} - \frac{1}{k^2}\right)\bar{\alpha}'^2\left(\frac{c}{\pi} t\right).$$