6 Polar functions

In this paragraph we shall be concerned with the so-called *polar functions* of a differential equation (q). A polar function is given by the difference $\beta - \alpha$ of two phases α, β belonging to a basis of the differential equation (q). Such functions are important principally because of their meaning in problems of a geometrical nature. The introduction of polar functions into the theory of the differential equation (q) provides a notable enrichment of the analysis and finds expression in many elegant relations.

In order to simplify our study, in this section we assume that $q(t) \neq 0$ for all $t \in j$.

6.1 The concept of polar functions

We consider a differential equation (q) ($q \neq 0$). Let (u, v) be a basis of this differential equation.

By a *polar function* of the basis (u, v) we mean the following function formed from a first and a second phase α, β of the basis u, v.

$$\theta = \beta - \alpha \qquad (t \in j). \tag{6.1}$$

We call the phases α, β the *components* of θ; more precisely, α is the *first* and β the *second* component of θ. There is therefore precisely one countable system (θ) of polar functions of the basis (u, v), whose elements differ by a multiple of π. If α_0, β_0 are two neighbouring elements in the ordered mixed phase system of (u, v) (5.30), so that $0 < \beta_0 - \alpha_0 < \pi$ or $-\pi < \beta_0 - \alpha_0 < 0$; and $\theta_0 = \beta_0 - \alpha_0$, then the system (θ) is composed of the functions

$$\theta_\nu(t) = \theta_0(t) + \nu\pi \qquad (\nu = 0, \pm 1, \pm 2, \ldots).$$

Every polar function $\theta \in (\theta)$ obviously has a countable infinity of first and second components: $\alpha = \alpha_0 + n\pi$, $\beta = \beta_0 + n\pi$ (n integral). We see that every first (second) phase α (β) of the basis (u, v) can be expressed by means of a first (second) component of θ, from which the second (first) component is uniquely determined.

For every first component α of θ we have, from (5.34), the relation

$$\theta = \operatorname{Arccot} \frac{1}{2}\left(\frac{1}{\alpha'}\right)'; \tag{6.2}$$

in which Arccot denotes an appropriate branch of this function.

In view of this relation, we call the first component α the *generator* of θ and say that the polar function θ is *generated* by the first phase α.

Conversely, every branch of the above function constructed from an arbitrary first phase α of the basis (u, v) represents a polar function $\theta \in (\theta)$. Every polar function $\theta \in (\theta)$ obviously belongs to the class C_1, and according to (5.29) satisfies for $t \in j$ the relation $n\pi < \theta < (n + 1)\pi$, where n is an integer.

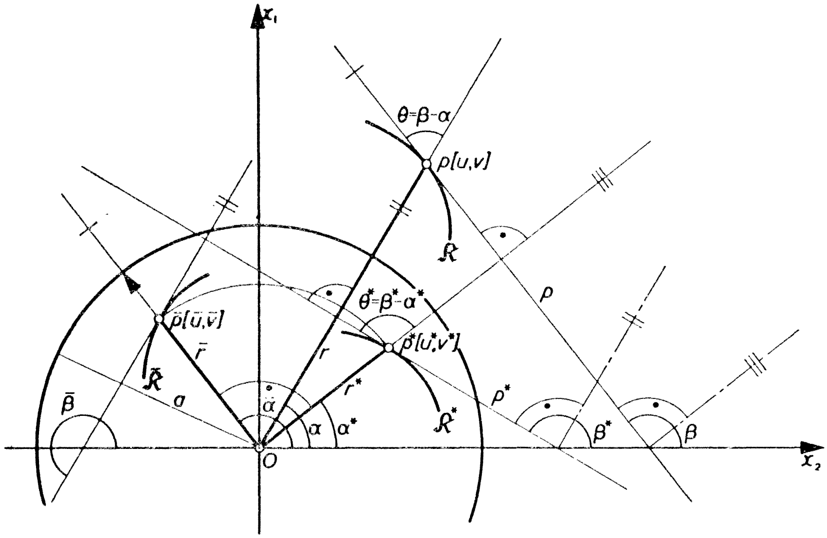

Figure 1

We now wish to explain the geometrical significance of the polar function of the basis (u, v). In order to achieve this we assume that $-w > 0$. Then all first phases of (u, v) are increasing, while the second phases are increasing or decreasing according as $-q > 0$ or $-q < 0$. (§§ 5.3, 5.8).

For example, let $\theta = \beta - \alpha$ be the polar function formed from two proper neighbouring phases α, β; that is to say such that $0 < \beta - \alpha < \pi$. We consider the integral curve $\mathfrak{K}$ of the differential equation (q) with the vector representation $x(t) = [u(t), v(t)]$. Then $x'(t) = [u'(t), v'(t)]$ is the tangent vector to the curve $\mathfrak{K}$ at the point $P[u(t), v(t)]$.

Let $W\alpha(t)$, $W\beta(t)$, $W\theta(t)$ be those numbers lying in the interval $[0, 2\pi)$ which are congruent to $\alpha(t)$, $\beta(t)$, $\theta(t)$ modulo 2π. We know that $W\alpha(t)$ is the angle formed by the vector $x(t)$ and the coordinate vector x_2; $W\beta(t)$ is the angle between the tangent vector $x'(t)$ and the coordinate vector x_2 (§§ 5.3, 5.8).

Moreover, we have $W\theta(t) = W\beta(t) - W\alpha(t)$ or $= 2\pi + [W\beta(t) - W\alpha(t)]$, according as $W\beta(t) > W\alpha(t)$ or $W\beta(t) < W\alpha(t)$. Consequently $W\theta(t)$ is the angle between the vectors $x(t)$, $x'(t)$.

We see that the value $\theta(t)$ gives modulo 2π the angle formed at the point $P(t)$ between the oriented straight line $OP(t)$ and the oriented tangent p to the curve at the point $P(t)$. The orientation has the same sense of that of the vectors $x(t)$, $x'(t)$ (see Fig. 1; the angles $W\alpha(t)$, $W\beta(t)$, $W\theta(t)$ are denoted by α, β, θ.)

We now wish to spend a moment in the consideration of this figure. It shows that under the inversion $\boldsymbol{K}_a$ with respect to a circle of radius a with centre O, the integral curve $\mathfrak{K}$ is transformed into another curve $\mathfrak{K}^*$. The pair P, p go over into another pair $P^*[u^*, v^*], p^*$, the numbers r; α, β, θ corresponding to numbers r^*; $\alpha^*, \beta^*, \theta^*$, by the relations

$$r^* = \frac{a^2}{r \cdot \sin\theta}; \qquad \alpha^* = \beta - \frac{\pi}{2}, \qquad \beta^* = \alpha + \frac{\pi}{2}, \qquad \theta^* = \pi - \theta;$$

in particular, the angle θ is transformed into its supplement $\pi - \theta$. It is because of this property that we have called the function θ the polar function. We have moreover the following relations (5.28)

$$u^* = r^* \sin\alpha^* = -r^* \cos\beta = -\frac{r^*}{s} v' = -\frac{a^2}{r \cdot s \cdot \sin\theta} v' = -\frac{a^2}{-w} v',$$

$$v^* = r^* \cos\alpha^* = r^* \sin\beta = \frac{r^*}{s} u' = \frac{a^2}{r \cdot s \cdot \sin\theta} u' = \frac{a^2}{-w} u',$$

and from these there follows

$$r^* = \frac{a^2}{-w} s.$$

If we now carry out a rotation through one right angle about O in the positive sense, then the curve $\mathfrak{K}^*$ is transformed into another curve $\bar{\mathfrak{K}}$. At the same time the point $P^* \in \mathfrak{K}^*$ goes over into a point $\bar{P}[\bar{u}, \bar{v}) \in \bar{\mathfrak{K}}$, in which

$$\bar{u} = r^* \sin\left(\alpha^* + \frac{\pi}{2}\right) = v^* = \frac{a^2}{-w} u',$$

$$\bar{v} = r^* \cos\left(\alpha^* + \frac{\pi}{2}\right) = -u^* = \frac{a^2}{-w} v'.$$

The curve $\bar{\mathfrak{K}}$ consequently admits of the vectorial representation

$$\bar{x}(t) = \left[\frac{a^2}{-w} u'(t), \frac{a^2}{-w} v'(t)\right]. \tag{6.3}$$

The tangent vector to the curve $\bar{\mathfrak{K}}$ at the point $\bar{P}(t)$ is obviously

$$\bar{x}'(t) = \left[\frac{a^2}{-w} q(t)u(t), \frac{a^2}{-w} q(t)v(t)\right]. \tag{6.4}$$

This is therefore opposite to or in the same sense as $x(t)$, according as $-q > 0$ or $-q < 0$. The angle formed by the tangent vector $\bar{x}'$ at the point $\bar{P}$ and the coordinate vector x_2 is $\bar{\beta} = \alpha \pm \pi$, in which we take the $+$ or $-$ sign according as $0 \leqslant \alpha < \pi$ or $\pi \leqslant \alpha < 2\pi$.

We have therefore

$$\bar{r} = \frac{a^2}{-w} s, \qquad \bar{\alpha} = \beta, \qquad \bar{\beta} = \alpha \pm \pi. \tag{6.5}$$

Let $\boldsymbol{R}$ be the transformation, arising from the inversion $\boldsymbol{K}_{\sqrt{-w}}$ $(a = \sqrt{-w})$ followed by a rotation about O in the positive sense through one right angle. We see that the integral curve $\mathfrak{K}$ is carried by this transformation into a curve $\overline{\mathfrak{K}}$. At the same time an arbitrary point $P \in \mathfrak{K}$ goes into a point $\bar{P} \in \overline{\mathfrak{K}}$, in such a manner that the radius vectors $x = \overrightarrow{OP}$, $\bar{x} = \overrightarrow{O\bar{P}}$ and the corresponding tangent vectors $x', \bar{x}'$, are related in the following way:

$$\bar{x} = x', \qquad \bar{x}' = q \cdot x, \tag{6.6}$$

while the corresponding amplitudes $\bar{r}$, s and angles α, β; $\bar{\alpha}$, $\bar{\beta}$ are transformed as follows

$$\bar{r} = s, \qquad \bar{\alpha} = \beta, \qquad \bar{\beta} = \alpha \pm \pi. \tag{6.7}$$

If the curve $\mathfrak{K}$ is so specialized that it goes into itself by the transformation $\boldsymbol{R}$, then it obviously has the following "ellipse property": the two straight lines passing through the point O and the corresponding points $P, \bar{P} \in \mathfrak{K}$ are parallel to the tangents to the curve $\mathfrak{K}$ at the points $\bar{P}$, P. Curves with this property are called *Radon curves.* They were first studied by J. Radon [Ber. Verh. Sächs. Akad. Leipzig, 68 (1916), 123–128]; we shall meet them later (§ 16.5) in a quite different connection.

6.2 General polar form of the carrier q

Let (u, v) be a basis of the differential equation (q) and θ a polar function of (u, v).

The function $\cot\theta$ constructed from the function θ is obviously the same for all polar functions of the basis (u, v), it is therefore determined uniquely by this basis. From (2) and (5.14) we have the formulae

$$-w \cdot \cot\theta = rr', \qquad 2\cot\theta = \left(\frac{1}{\alpha'}\right)', \qquad \alpha' \cdot \cot\theta = (\log r)'; \tag{6.8}$$

in this r, α naturally represent the first amplitude and a first phase of the basis (u, v).

Let $t_0 \in j$ be an arbitrary number. The value which any function takes at the point t_0 we shall as a rule indicate by the suffix $_0$; for instance $r(t_0)$ will be denoted by r_0.

From (8) it follows that for $t \in j$

$$r^2 = r_0^2 - 2w\int_{t_0}^{t} \cot\theta(\tau)\, d\tau$$

and moreover, taking account of (5.14),

$$r^2 = r_0^2\left(1 + 2\alpha_0'\int_{t_0}^{t} \cot\theta(\tau)\, d\tau\right). \tag{6.9}$$

This formula gives

$$1 + 2\alpha_0'\int_{t_0}^{t} \cot\theta(\tau)\, d\tau > 0.$$

Clearly, in the interval j the inequality

$$\int_{t_0}^{t} \cot\theta(\tau)\, d\tau \gtrless -\frac{1}{2\alpha_0'} \tag{6.10}$$

holds, according as $\alpha_0' > 0$ or $\alpha_0' < 0$. Consequently the function $\int_{t_0}^{t} \cot\theta(\tau)\,d\tau$ is bounded in j at least on one side by the number $-\frac{1}{2}(\alpha_0')^{-1}$, this bound being below or above according as $\alpha_0' > 0$ or $\alpha_0' < 0$. From (8) we have, for $t \in j$,

$$\alpha = \alpha_0 + \alpha_0' \int_{t_0}^{t} \frac{d\sigma}{1 + 2\alpha_0' \int_{t_0}^{\sigma} \cot\theta(\tau)\,d\tau} \tag{6.11}$$

and further, by (5.16)

$$q = -\frac{\alpha_0'}{\sin^2\theta} \cdot \frac{\alpha_0' + \theta'\left(1 + 2\alpha_0' \int_{t_0}^{t} \cot\theta(\tau)\,d\tau\right)}{\left(1 + 2\alpha_0' \int_{t_0}^{t} \cot\theta(\tau)\,d\tau\right)^2}. \tag{6.12}$$

We call the expression on the right of this formula the *general polar form of the carrier* q. We speak also of the general polar form of the *differential equation* (q), when the carrier q is put into the general polar form (12).

Consequently the inequality

$$\theta' \gtrless -\frac{\alpha_0'}{1 + 2\alpha_0' \int_{t_0}^{t} \cot\theta(\tau)\,d\tau} \tag{6.13}$$

holds in j, according as $-q\alpha_0' > 0$ or $-q\alpha_0' < 0$.

6.3 Determination of the carrier from a polar function

By a *polar function* of the carrier q or of the differential equation (q) we mean a polar function of any basis of the differential equation (q). Clearly, every polar function θ of the differential equation (q) has the following properties in the interval j:

1. $\theta \in C_1$:
2. $n\pi < \theta < (n + 1)\pi$ (n being an integer); (6.14)
3. the function $\int_{t_0}^{t} \cot\theta(\tau)\,d\tau$ ($t_0 \in j$ fixed) is bounded at least on one side.

The function $\theta(t) = c$ (= constant) in the interval $j = (-\infty, \infty)$ cannot, for instance, represent a polar function of the differential equation (q) if $n\pi < c < (n + 1)\pi$, $c \neq (n + \frac{1}{2})\pi$, n integral.

Now we wish to consider how far a given polar function determines the differential equation (q).

Let θ be a function, defined in j, with the above properties 1–3. We choose a number $t_0 \in j$. From property 2, the function $\int_{t_0}^{t} \cot\theta(\tau)\,d\tau$ exists in j, and from property 3 it is bounded at least on one side; for instance, let it be bounded below. Then we have, for a certain constant $A > 0$,

$$1 + 2A \int_{t_0}^{t} \cot \theta(\tau)\, d\tau > 0 \qquad (t \in j). \tag{6.15}$$

We now choose an arbitrary number α_0, and also a number α_0' such that $0 < \alpha_0' < A$; then there holds an inequality of the form (10) with the sign $>$. We shall now establish the following result:

Precisely one function q defined in the interval j can be constructed, so that the differential equation (q) *admits of the function θ as polar function with the initial values $\alpha(t_0) = \alpha_0$, $\alpha'(t_0) = \alpha_0'$ of its generator α.*

Proof. First we observe that there can exist at most one function q with the designated properties. For, given any function q of this kind the generator α of θ is uniquely determined by θ, α_0, α_0' by means of formula (11). From (5.16) there is precisely one carrier q which admits the function α as first phase.

We now show that there is at least one function q with the above properties. With this aim we construct the function $\alpha(t)$ $(t \in j)$ in accordance with formula (11). According to property 1, $\alpha \in C_3$, and by (15) we have $\alpha'(t) > 0$. Consequently α represents a phase function (§ 5.7) and by application of (5.16) a calculation shows that α is a first phase of the carrier q determined by a formula similar to (12). Moreover formula (11) yields, when differentiated, a relationship such as (2).

By § 5.15, the function

$$\beta = \alpha + \operatorname{Arccot} \frac{1}{2} \left(\frac{1}{\alpha'}\right)' \tag{6.16}$$

represents a second phase of each basis, determined from α, of the carrier q; in this formula Arccot is that branch of the function lying between $n\pi$ and $(n+1)\pi$. From (2), (16) we have $\beta = \alpha + \theta$. Consequently θ is a polar function of the carrier q, and the proof is complete.

6.4 Radon functions

We now go back to the situation of § 6.1 and for convenience assume that $q(t) < 0$ for all $t \in j$. Then the functions α', β' always have the same sign in the interval j (§ 5.14), and in fact $\alpha' > 0$, $\beta' > 0$ or $\alpha' < 0$, $\beta' < 0$, according as $-w > 0$ or $-w < 0$.

By a *Radon function* of the basis (u, v) we mean a function

$$\zeta = \beta + \alpha \qquad (t \in j) \tag{6.17}$$

constructed from a first phase α and a second phase β of the basis (u, v). A Radon function ζ of the basis (u, v) is also called a *Radon parameter* of the basis (u, v). We make use of this designation in recognition of the elegant study made by J. Radon of the curves described above (§ 6.1), possessing the ellipse property, in which functions of this kind appear.

Obviously there is a countable system of Radon functions of the basis (u, v) and the individual functions of this system differ from each other by an integral multiple of π. Clearly $\zeta \in C_1$ and moreover $\zeta' > 0$ or $\zeta' < 0$ according as $-w > 0$ or $-w < 0$, for $t \in j$.

Two functions θ, ζ constructed from the same phases α, β of the basis (u, v)

$$\theta = \beta - \alpha, \qquad \zeta = \beta + \alpha \tag{6.18}$$

we call *associated*. The formulae (18) yield the results

$$\alpha = \frac{1}{2}(\zeta - \theta), \qquad \beta = \frac{1}{2}(\zeta + \theta). \tag{6.19}$$

Clearly, $\zeta - \theta \in C_3$. From $\alpha'\beta' > 0$, $\operatorname{sgn} \alpha' = \operatorname{sgn}(-w)$, we have

$$-\zeta' < \theta' < \zeta' \qquad \text{or} \qquad \zeta' < \theta' < -\zeta', \tag{6.20}$$

according as $-w > 0$ or $-w < 0$.

Now let us start from the formulae

$$\beta' \cot \theta = (\alpha' + \theta') \cot \theta = \alpha' \cot \theta + (\log |\sin \theta|)'.$$

By making use of (8) we obtain

$$\zeta' \cot \theta = (\log r^2 |\sin \theta|)',$$

and it follows that for $t \in j$

$$r^2 = r_0^2 \frac{\sin \theta_0}{\sin \theta} \exp \int_{t_0}^{t} \zeta'(\tau) \cot \theta(\tau)\, d\tau; \tag{6.21}$$

moreover, by (5.28),

$$s^2 = \alpha_0'^2 r_0^2 \frac{1}{\sin \theta_0 \cdot \sin \theta} \exp\left(-\int_{t_0}^{t} \zeta'(\tau) \cot \theta(\tau)\, d\tau\right). \tag{6.22}$$

Finally, formula (5.32) gives

$$q = -\frac{\alpha_0'^2}{\sin^2 \theta_0} \cdot \frac{\zeta' + \theta'}{\zeta' - \theta'} \exp\left(-2\int_{t_0}^{t} \zeta'(\tau) \cot \theta(\tau)\, d\tau\right). \tag{6.23}$$

The expression on the right of (23) is called the *Radon polar form* of the carrier q. We speak of the *Radon polar form of the differential equation* (q), if the carrier q is put in the Radon polar form.

6.5 Normalized polar functions

In the following study we wish to relate the values of a polar function to its components or to the Radon parameter, regarding the latter as an independent variable. A polar function transformed in this way we call a normalized polar function. We assume $q(t) \neq 0$ for all $t \in j$.

Let $\theta = \beta - \alpha$ be a polar function of the basis (u, v) of the differential equation (q). We choose an arbitrary number $t_0 \in j$ and denote, as before, by the suffix $_0$ the values taken by the functions concerned at the point t_0. The function θ naturally has the properties (14) above.

6.6 Normalized polar functions of the first kind

First we shall represent the polar function θ as a function of the independent variable α.

Since the function $\alpha(t)$, in the interval j, is strictly increasing ($\alpha' > 0$) or decreasing ($\alpha' < 0$), the inverse function α^{-1} exists. This is defined in the range J_1 of the function α in the interval j. Consequently J_1 is an open interval, and $\alpha_0 \in J_1$.

Now we observe that if the differential equation (q) is of finite type, then the interval J_1 is bounded. If (q) is left (right) oscillatory, and $\alpha' > 0$ ($\alpha' < 0$), then the interval J_1 is of the form $(-\infty, A)$, A being finite; if on the other hand the differential equation (q) is left (right) oscillatory and $\alpha' < 0$ ($\alpha' > 0$), then the interval J_1 is of the type (A, ∞), A being finite. Finally if the differential equation (q) is oscillatory then the interval J_1 is unbounded on both sides.

The function α^{-1} obviously belongs to the class C_3. Moreover it forms a one-to-one mapping of the interval J_1 on j; in particular we have $\alpha^{-1}(\alpha_0) = t_0$. From the definition of α^{-1} it follows that for $\alpha \in J_1$, $\alpha^{-1}(\alpha) = t \in j$ is that number for which $\alpha(t) = \alpha$. We again use the term *homologous* to describe two such numbers $t = \alpha^{-1}(\alpha) \in j$ and $\alpha = \alpha(t) \in J_1$.

Now we define, in the interval J_1, the function $h(\alpha)$ or more briefly $h\alpha$, by assigning to it at each point $\alpha \in J_1$ the value of the function θ at the homologous point $t \in j$, that is

$$h(\alpha) = \theta\alpha^{-1}(\alpha) = \theta(t). \tag{6.24}$$

Consequently $h\alpha$ is the polar function θ regarded as a function of the independent variable α. We call h a *normalized polar function of the first kind of the basis* (u, v), more briefly a *first normalized* (or 1-*normalized*) *polar function of the basis* (u, v)

Obviously the function h in the interval J_1 has the following properties:

1. $h \in C_1$;
2. $n\pi < h < (n + 1)\pi$, n being an integer.

We emphasize that if the differential equation (q) is oscillatory then the interval of definition J_1 of the first normalized polar function h is the interval $(-\infty, \infty)$.

From (8) and (5.14) we have the following formulae, valid in the interval j

$$r(t) = r_0 \exp \int_{\alpha_0}^{\alpha} \cot h(\rho)\, d\rho, \tag{6.25}$$

$$\alpha'(t) = \alpha_0' \exp \left(-2 \int_{\alpha_0}^{\alpha} \cot h(\rho)\, d\rho\right). \tag{6.26}$$

When we have a function of α ($\in J_1$) we shall denote its derivative with respect to α by means of an oblique dash ($\grave{}$).

From (26) it follows that

$$t^{\backprime}(\alpha) = \frac{1}{\alpha_0'} \exp 2 \int_{\alpha_0}^{\alpha} \cot h(\rho)\, d\rho, \tag{6.27}$$

and this relationship gives

$$t = t_0 + \frac{1}{\alpha_0'} \int_{\alpha_0}^{\alpha} \left(\exp 2 \int_{\alpha_0}^{\sigma} \cot h(\rho)\, d\rho \right) d\sigma. \tag{6.28}$$

Obviously this expression represents the inverse function $t = \alpha^{-1}(\alpha)$ of the first phase $\alpha(t)$.

From (5.16) and (26) we obtain the following formula relating any two homologous points $t \in j$, $\alpha \in J_1$:

$$q(t) = -\alpha_0'^2 \frac{1 + h^{\backprime}(\alpha)}{\sin^2 h(\alpha)} \exp \left(-4 \int_{\alpha_0}^{\alpha} \cot h(\rho)\, d\rho \right). \tag{6.29}$$

The expression on the right of this formula is called *the first polar form of the carrier q*. We speak of the *first polar form of the differential equation* (q), when the carrier q is in its first polar form.

Obviously, in the interval J_1 we have the inequality

$$h^{\backprime}(\alpha) \gtrless -1,$$

according as $-q > 0$ or $-q < 0$.

We observe that the formula (25) represents the equation of the integral curve $x(t) = [u(t), v(t)]$ in polar coordinates.

6.7 Determination of the carrier from a first normalized polar function

By a *normalized polar function of the first kind of the carrier q* or of the *differential equation* (q), which we shall more briefly call the first normalized polar function of the carrier q or of the differential equation (q), we mean a first normalized polar function of any basis of the differential equation (q).

Every first normalized polar function h of the differential equation (q) has the following properties in its interval of definition J_1.

$$\begin{aligned} &1.\ h \in C_1; \\ &2.\ n\pi < h < (n+1)\pi \quad (n \text{ integral}) \\ &3.\ h^{\backprime} > -1 \quad \text{or} \quad h^{\backprime} < -1. \end{aligned} \tag{6.30}$$

We now study the question of how far the differential equation (q) is determined from a first normalized polar function.

Let h be a function defined in an open interval J_1 with the above properties (30). We choose a number t_0 and further numbers $\alpha_0 \in J_1$, $\alpha_0' \neq 0$. Now we have the following theorem.

Theorem. Precisely one function q, defined and never zero in an open interval $j(t_0 \in j)$, can be constructed so that the differential equation (q) *admits of the function h as first normalized polar function, and this polar function is generated by a first phase α of the differential equation* (q) *with initial values $\alpha(t_0) = \alpha_0$, $\alpha'(t_0) = \alpha'_0$.*

We shall only sketch the proof, which is essentially similar to that of § 6.3.

If there exist a differential equation (q) and a first phase $\alpha(t)$ as described in the theorem, then the latter is uniquely determined as the inverse of the function $t(\alpha)$ defined by formula (28). There can therefore be at most one differential equation (q) as specified by the theorem. Now we define the function $t(\alpha)$ in terms of the function h and the numbers α_0, α'_0 by the formula (28). This functions maps the interval J_1 onto an open interval j, and $t_0 \in j$. Let $\alpha(t)$ be the inverse function to $t(\alpha)$ defined in the interval j; this is obviously a phase function. Let $q(t)$ be the carrier of the differential equation (q) (defined in the sense of (5.16)), for which the function $\alpha(t)$ represents a first phase. Then there is an analogous formula to (29), and by (30), 3 it follows that $q(t) \neq 0$ for all $t \in j$. Further, let θ be the polar function of the differential equation (q), which is generated by the first phase α and lies between $n\pi$ and $(n + 1)\pi$. By (26) we have, at two homologous points $t \in j$, $\alpha \in J_1$,

$$\cot \theta(t) = \frac{1}{2}\left(\frac{1}{\alpha'}\right)' = \frac{1}{2}\frac{1}{\alpha'_0}\left[\exp\left(2\int_{\alpha_0}^{\alpha} \cot h(\rho)\, d\rho\right)\right]^{\cdot} \cdot \alpha'_0 \exp\left(-2\int_{\alpha_0}^{\alpha} \cot h(\rho)\, d\rho\right)$$
$$= \cot h(\alpha).$$

Consequently $\theta(t) = h(\alpha)$ and the proof is complete.

6.8 Normalized polar functions of the second kind

In the second place we shall regard the polar function θ as a function of the independent variable β.

Following the lines of our study in § 6.6 we consider the function β^{-1} inverse to the function β; this exists in an open interval J_2, and we have $\beta_0 \in J_2$. With regard to the character of the interval J_2, analogous remarks can be made to those in § 6.6 about the interval J_1.

Now we define, in the interval J_2, the function $-k(\beta)$, or more shortly $-k\beta$, in such a way that to every point $\beta \in J_2$ there corresponds the value of the function θ at the homologous point $\beta^{-1}(\beta) = t \in j$, that is:

$$-k(\beta) = \theta\beta^{-1}(\beta) = \theta(t). \tag{6.31}$$

Consequently $-k\beta$ is the polar function θ regarded as a function of the independent variable β. We call $-k$ *a normalized polar function of the second kind of the basis* (u, v) or more shortly *a second normalized* (or *2-normalized*) *polar function of the basis* (u, v). Obviously the function $-k$ has the following properties in the interval J_2:

1. $-k \in C_1$;
2. $n\pi < -k < (n + 1)\pi$, n integral.

We emphasize that if the differential equation (q) is oscillatory then the definition interval J_2 of the second normalized polar function $-k$ is the interval $(-\infty, \infty)$. The derivative of a function of β ($\in J_2$) with respect to β we shall similarly denote by an oblique dash $^{\backslash}$. Now, from (1), (31) we have the following relation holding at two homologous points $t \in j$, $\beta \in J_2$

$$\alpha(t) = \beta + k(\beta), \tag{6.32}$$

and consequently also

$$\alpha'(t) = [1 + k^{\backslash}(\beta)]\beta'(t). \tag{6.33}$$

Moreover, we obviously have

$$\int_{t_0}^{t} \alpha'(\tau) \cot \theta(\tau)\, d\tau = \int_{t_0}^{t} [1 + k^{\backslash}\beta(\tau)] \cot \theta(\tau) \cdot \beta'(\tau)\, d\tau = -\int_{\beta_0}^{\beta} [1 + k^{\backslash}(\rho)] \cot k(\rho)\, d\rho$$

and hence, by (8), (33),

$$t^{\backslash}(\beta) = \frac{1}{\alpha_0'} [1 + k^{\backslash}(\beta)] \exp \left(-2 \int_{\beta_0}^{\beta} [1 + k^{\backslash}(\rho)] \cot k(\rho)\, d\rho\right) \cdot \tag{6.34}$$

This relationship gives

$$t = t_0 + \frac{1}{\alpha_0'} \int_{\beta_0}^{\beta} [1 + k^{\backslash}(\sigma)] \exp \left(-2 \int_{\beta_0}^{\sigma} [1 + k^{\backslash}(\rho)] \cot k(\rho)\, d\rho\right) d\sigma. \tag{6.35}$$

Obviously this formula represents the inverse function $t = \beta^{-1}(\beta)$ to the second phase $\beta(t)$.

From (5.31) and (33), (34) we obtain the following formula, valid at two homologous points $t \in j$, $\beta \in J_2$

$$q(t) = -\frac{\alpha_0'^2}{\sin^2 k(\beta)} \cdot \frac{\exp 4 \int_{\beta_0}^{\beta} [1 + k^{\backslash}(\rho)] \cot k(\rho)\, d\rho}{1 + k^{\backslash}(\beta)}. \tag{6.36}$$

The expression on the right of this formula is called the *second polar form of the carrier q*. We speak of the *second polar form of the differential equation* (q), when the carrier q is in the second polar form.

In the interval J_2 there hold therefore the inequalities

$$-k^{\backslash}(\beta) \lessgtr 1,$$

according as $-q > 0$ or $-q < 0$.

6.9 Determination of the carrier from a second normalized polar function

By a normalized polar function of the second kind *of the carrier q* or of the *differential equation* (q), (more shortly, a second normalized polar function of the carrier q or the differential equation (q)) we mean a normalized polar function of the second kind of any basis of the differential equation (q). Every second normalized polar function

$-k$ of the differential equation (q) has the following properties in its interval of definition J_2

$$1.\ -k \in C_1;$$
$$2.\ n\pi < -k < (n+1)\pi \quad (n \text{ integral})$$
$$3.\ -k^{\setminus} < 1 \quad \text{or} \quad > 1. \tag{6.37}$$

As in the case of first normalized polar functions (§ 6.7), one can consider the problem of determining the differential equation (q) from its second normalized polar functions. There is, indeed a relevant theorem, analogous to that of § 6.7 but we shall not formulate it since no essentially new ideas are involved and there are no applications of it in our subsequent studies.

6.10 Normalized polar functions of the third kind

In the third place we go back to the situation examined in § 6.4 and set ourselves the object of representing the polar function $\theta = \beta - \alpha$ as a function of the Radon parameter $\zeta = \beta + \alpha$. We know that $\zeta \in C_1$ and that $\zeta' > 0$ or $\zeta' < 0$ according as $-w > 0$ or $-w < 0$.

Consider the inverse function ζ^{-1} of ζ. This is defined in an open interval J_3, and we have $\zeta_0 \in J_3$. With regard to the nature of the interval J_3, similar remarks apply to those on the interval J_1 in § 6.6.

In the interval J_3 we define the function $p(\zeta)$, more shortly $p\zeta$, by associating with every point $\zeta \in J_3$ the value of the function θ at the point homologous to ζ, namely $\zeta^{-1}(\zeta) = t \in j$;

$$p(\zeta) = \theta\zeta^{-1}(\zeta) = \theta(t). \tag{6.38}$$

Consequently $p\zeta$ is the polar function θ regarded as a function of the independent variable ζ. We call p a *normalized polar function of the third kind of the basis* (u, v) or more briefly a *third normalized* (or *3-normalized*) *polar function of the basis* (u, v).

The third normalized polar function p clearly has, in the interval J_3, the following properties:

$$1.\ p \in C_1;$$
$$2.\ n\pi < p < (n+1)\pi,\ n \text{ integral}.$$

The derivative of a function of $\zeta (\in J_3)$ with respect to ζ will be denoted by a $^{\setminus}$.

We have, in the interval j,

$$\zeta = \theta + 2\alpha$$

and consequently, from (21) and (5.14)

$$\zeta' = \theta' + 2\,\frac{\alpha_0'}{\sin\theta_0}\sin\theta\cdot\exp\left(-\int_{t_0}^{t}\zeta'(\tau)\cot\theta(\tau)\,d\tau\right).$$

This formula, together with (38), shows that any two homologous points $t = \zeta^{-1}\zeta \in j$, $\zeta = \zeta(t) \in J_3$ are so related that

$$\zeta'(t) = 2\,\frac{\alpha_0'}{\sin p_0}\,\frac{\sin p(\zeta)}{1 - p^{\setminus}(\zeta)}\exp\left(-\int_{\zeta_0}^{\zeta}\cot p(\rho)\,d\rho\right). \tag{6.39}$$

From (39) we find that

$$t = t_0 + \frac{1}{2}\frac{\sin p_0}{\alpha_0'}\int_{\zeta_0}^{\zeta}\frac{1 - p^{\setminus}(\sigma)}{\sin p(\sigma)}\left(\exp\int_{\zeta_0}^{\sigma}\cot p(\rho)\,d\rho\right)d\sigma; \tag{6.40}$$

this formula gives the inverse function $\zeta^{-1}(\zeta)$ to the Radon parameter $\zeta(t)$. From (23) we obtain the following result, valid for two homologous points $t \in j$, $\zeta \in J_3$

$$q(t) = -\frac{\alpha_0'^2}{\sin^2 p_0}\frac{1 + p^{\setminus}(\zeta)}{1 - p^{\setminus}(\zeta)}\exp\left(-2\int_{\zeta_0}^{\zeta}\cot p(\rho)\,d\rho\right). \tag{6.41}$$

The expression on the right of this formula is called the *third* or *Radon polar form* of the carrier q. We speak of the *third* or *Radon polar form of the differential equation* (q) when the carrier q is in this form.

In the interval J_3 we have the inequalities

$$-1 < p^{\setminus}(\zeta) < 1.$$

6.11 Determination of the carrier from a third normalized polar function

By a normalized polar function of the third kind of the carrier q or of the differential equation (q) we mean a normalized polar function of the third kind of any basis of (q). More briefly we call this a third normalized polar function of the carrier q or of the differential equation (q).

Every third-normalized polar function p of the differential equation (q) obviously has the following properties in its definition interval J_3:

1. $p \in C_1$;
2. $n\pi < p < (n + 1)\pi$ (n integral);
3. $-1 < p^{\setminus} < 1$. (6.42)

With regard to the problem of determining the differential equation (q) from its third-normalized polar functions, similar remarks hold as in the case of second-normalized polar functions (§ 6.9).

6.12 Some applications of polar function

First let us apply the above results to answering the following question:

What are the possible carriers q determined by a *constant* polar function with the value $(2n + 1)\pi/2$, n being an integer?

Each of the formulae (12), (23), (29), (36), (41) gives the answer

$$q(t) = -\alpha_0'^2.$$

The differential equations (q) with constant negative carriers q are therefore the only ones which admit of a constant polar function with a value congruent to $\frac{\pi}{2}$ (mod π). For such an equation $(-k^2)$ whose carrier $-k^2$ is a negative constant, there are precisely two first phases of $(-k^2)$ passing through any point (t_0, α_0); these have the directions $\alpha_0' = k$ and $\alpha_0' = -k$ in such a way that the polar functions generated by them have a constant value congruent to $\frac{\pi}{2}$ (mod π).

Because of their geometrical significance (§ 6.1), polar functions, and particularly normalized polar functions, of differential equations (q) occur in the study of curves with special properties of a centro-affine nature. We develop the comment in the following study.

Let $(P =) P(t)$, $(\bar{P} =) \bar{P}(\bar{t})$, $t \neq \bar{t}$, be arbitrary points of the integral curve $\mathfrak{K}$ with the vectorial representation $x = [u, v]$, and $p, \bar{p}$ be the corresponding tangents to the curve. The straight lines OP, $O\bar{P}$ we shall denote by $g, \bar{g}$. Moreover, let $\theta = \beta - \alpha$ be a polar function of the basis (u, v). We know (§ 6.1) that the values $\theta(t)$, $\theta(\bar{t})$ represent the angles (mod 2π) between the corresponding directed straight lines g, p and $\bar{g}, \bar{p}$.

If the points P, $\bar{P}$ lie on the same straight line g, (i.e. $\bar{g} = g$) and so are points of intersection of the integral curve $\mathfrak{K}$ with g, then the values $(\alpha =)\ \alpha(t)$ $(\bar{\alpha} =)\ \alpha(\bar{t})$ differ by an integral multiple of π, $\bar{\alpha} = \alpha + n\pi$ (n integral), and conversely.

If the tangents $p, \bar{p}$ are parallel then the values $(\beta =)\ \beta(t)$, $(\bar{\beta} =)\ \beta(\bar{t})$ differ by an integral multiple of π, $\bar{\beta} = \beta + n\pi$ (n integral) and conversely. If the straight lines $\bar{g}, p$ or $g, \bar{p}$ are parallel then $\bar{\alpha} = \beta + m\pi$ or $\bar{\beta} = \alpha + n\pi$ (m, n integral), and conversely.

If the first normalized polar function h of the basis (u, v) is periodic with period π, then we have the following situation: through the point O there pass straight lines which cut the curve $\mathfrak{K}$ in at least two points; moreover the tangents at all points of intersection of such a line with the curve $\mathfrak{K}$ are parallel to each other, and conversely.

If the second normalized polar function $-k$ of the basis (u, v) is periodic with period π then there are tangents to the curve $\mathfrak{K}$ to which parallel tangents exist; moreover the points of contact with the curve $\mathfrak{K}$ of all parallel tangents lie on a straight line passing through O, and conversely.

The third normalized polar function p of the basis (u, v) needs rather fuller consideration. We assume that the function p is defined in an interval J_3 of length $> \pi$ and satisfies the functional equation

$$p(\zeta) + p(\zeta + \pi) = \pi. \tag{6.43}$$

Naturally we are only considering points ζ, $\zeta + \pi$ such that both of them lie in J_3. At points ζ and $\bar{\zeta} = \zeta + \pi$ the components of p have certain values α, β and $\bar{\alpha}, \bar{\beta}$ and obviously

$$\bar{\beta} + \bar{\alpha} = \beta + \alpha + \pi. \tag{6.44}$$

Moreover it follows from (43) that

$$\bar{\beta} - \bar{\alpha} = -\beta + \alpha + \pi. \tag{6.45}$$

From these two equations we obtain

$$\bar{\alpha} = \beta; \qquad \bar{\beta} = \alpha + \pi. \tag{6.46}$$

Conversely (46) yields the relations (44), (45).

If the third normalized polar function p of the basis (u, v) satisfies the functional equation (43) then we have the following situation: there are tangents p to the curve $\mathfrak{K}$ such that the straight line $\bar{g}$ passing through the point O and parallel to p cuts the curve $\mathfrak{K}$ at least once; moreover the tangents to $\mathfrak{K}$ at its points of intersection with $\bar{g}$ are parallel to the straight line passing through O and the point of contact of p. This is the "ellipse property".