Phase theory of ordinary linear homogeneous differential equations of the second order

In this chapter we shall develop the topic of *phase theory* of the differential equations to be considered; this provides the appropriate methodological basis for the transformation theory which we shall build upon it. The reader may perhaps be surprised to discover the richness and breadth into which this phase theory, whose basis depends merely on the properties of a function $q(t)$, can be expanded.

5 Polar coordinates of bases

5.1 Introduction

In this paragraph we start our development with a few elementary facts.

Let (q) be a differential equation in the interval j. We consider a basis (u, v) of (q) and the corresponding integral curve $\mathfrak{K}$ with representation $x(t) = [u(t), v(t)]$, relative to a rectangular coordinate system formed from the vectors x_1, x_2 with origin O. As positive sense of variation of angle we choose that which is given by the rotation from x_2 to x_1.

Let $x_0 = x(t_0)$, $t_0 \in j$, be an arbitrary point of $\mathfrak{K}$ and r_0 (> 0), α_0 be its polar coordinates with respect to the pole O and the polar axis OX_2. Then r_0 represents the modulus of the vector x_0, while the number α_0, which we specify to lie in the interval $[0, 2\pi)$, is the value of the angle (x_2, x_0). Obviously we have $u(t_0) = r_0 \sin \alpha_0$, $v(t_0) = r_0 \cos \alpha_0$. Now we define, in the interval j, the function $r(t)$ by means of the formula

$$r(t) = \sqrt{u^2(t) + v^2(t)}. \tag{5.1}$$

Moreover we let $\alpha(t)$ be the (unique) continuous function, defined in the interval j, which takes the value α_0 at the point t_0 and in the interval j satisfies the equation*

$$\tan \alpha(t) = \frac{u(t)}{v(t)} \tag{5.2}$$

except, of course, at the zeros of v.

Then we have, for all $t \in j$,

$$u(t) = r(t) \sin \alpha(t), \qquad v(t) = r(t) \cos \alpha(t). \tag{5.3}$$

Besides the function α, there are obviously other continuous functions in the interval j which satisfy the equation (2) everywhere in j apart from the zeros of v. Every such function has the form $\alpha_n = \alpha + n\pi$, n being an integer; consequently their totality forms a countable system. Each member is uniquely determined by its initial value $\alpha_n(t_0) = \alpha_0 + n\pi$. For n even the functions r, α_n satisfy the formulae (3), while for n odd the right side of (3) gives the basis $(-u, -v)$, which is proportional to (u, v).

The functions r and α_n constitute the polar coordinates of the basis (u, v). In special cases these ideas can be taken over to the ordered pair (u', v') formed from the derivatives u', v' of the functions u, v.

* This change to polar coordinates is commonly known as the Prüfer substitution (Trans.)

Now we wish to consider the information given here in full detail. We denote the Wronskian of the basis (u, v) by w.

5.2 Amplitudes

The functions r, s defined by the formulae

$$r = \sqrt{u^2 + v^2}, \qquad s = \sqrt{u'^2 + v'^2} \tag{5.4}$$

in the interval j will be called the *first* and *second* amplitudes of the basis (u, v) respectively. These functions are obviously always positive and belong to the classes C_2, C_1 respectively.

Clearly, the amplitudes of the inverse basis (v, u) are also r and s, we can thus refer more briefly to the amplitudes of the (independent) integrals u, v or, if we are concerned with their values, to the amplitudes of the point $x(t) = [u(t), v(t)]$.

We now show that the amplitudes r, s satisfy the following non-linear differential equations of the second order

$$\left.\begin{aligned} r'' &= qr + \frac{w^2}{r^3}, \\ s'' &= qs + \frac{w^2q^2}{s^3} + \frac{q'}{q}s', \end{aligned}\right\} \tag{5.5}$$

the first in the interval j, and the second in every sub-interval $i \subset j$ in which the function q is differentiable and non-zero.

Starting from the formulae

$$r^2 = u^2 + v^2, \qquad rr' = uu' + vv', \qquad w = uv' - u'v, \qquad s^2 = u'^2 + v'^2 \tag{5.6}$$

there follows the relation

$$r^2(s^2 - r'^2) = (uv' - u'v)^2 = w^2,$$

and the further relation

$$s^2 - r'^2 = \frac{w^2}{r^2} \tag{5.7}$$

The second formula (6) leads to the equation

$$rr'' = s^2 - r'^2 + qr^2,$$

and from the last two relations there follows the first of the differential equations (5). To obtain the second, we differentiate the equation (7), use the first equation (5) and so obtain the relation

$$ss' = qrr'. \tag{5.8}$$

Then, assuming the differentiability of q, we get

$$ss'' + s'^2 = qs^2 + q'rr' + q^2r^2,$$

then (8) gives

$$q(ss'' + s'^2) = q^2s^2 + q'ss' + q^3r^2. \tag{5.9}$$

Eliminating r, r' between the formulae (7), (8), (9) we obtain the second equation (5), for $q \neq 0$.

5.3 First phases of a basis

By a *first phase of the basis* (u, v) we mean any function α, continuous in the interval j, which satisfies in this interval the relationship

$$\tan \alpha(t) = \frac{u(t)}{v(t)} \tag{5.10}$$

except at the zeros of v. When convenient we refer to "phases" instead of "first phases" of the basis (u, v).

We note that there is precisely one countable system of phases of the basis (u, v). This system we call the *first phase system of the basis* (u, v), or more briefly the phase system of the basis (u, v). The individual phases of this system differ from each other by integral multiples of π.

Let (α) denote the phase system of (u, v). Let us choose an arbitrary phase $\alpha \in (\alpha)$, then the phase system (α) is composed of the set of functions

$$\alpha_\nu(t) = \alpha(t) + \nu\pi \qquad (\nu = 0, \pm 1, \pm 2, \ldots; \quad \alpha_0 = \alpha) \tag{5.11}$$

and these can clearly be ordered as follows:

$$\cdots < \alpha_{-2} < \alpha_{-1} < \alpha_0 < \alpha_1 < \alpha_2 < \cdots. \tag{5.12}$$

The value of each phase $\alpha_\nu \in (\alpha)$ at a zero of u or v is respectively an even or odd multiple of $\frac{1}{2}\pi$; conversely every point in j, at which a phase $\alpha_\nu \in (\alpha)$ takes the value of an even or odd multiple of $\frac{1}{2}\pi$, is a zero of u or v respectively.

If the integral u does not vanish in j, then there is precisely one phase $\alpha_\nu \in (\alpha)$, whose values lie entirely between 0 and π. If, however, u possesses zeros in j, then corresponding to each of these zeros there is precisely one phase in (α) which vanishes there.

From the first formula (2.1) we deduce that each phase $\alpha_\nu \in (\alpha)$ increases or decreases in j, according as $-w > 0$ or $-w < 0$. The integrals u, v are expressed in terms of the amplitude r, and an arbitrary phase $\alpha_\nu \in (\alpha)$ of the basis (u, v), as follows:

$$u(t) = \varepsilon_\nu r(t) \cdot \sin \alpha_\nu(t), \qquad v(t) = \varepsilon_\nu r(t) \cdot \cos \alpha_\nu(t) \qquad (t \in j); \tag{5.13}$$

in which ε_ν, the so-called *signature of the phase* α_ν, takes the value $+1$ or -1. The phase α_ν is called *proper* or *improper* (with respect to the basis (u, v), according as $\varepsilon_\nu = 1$ or $\varepsilon_\nu = -1$.

Two phases $\alpha_\nu, \alpha_\mu \in (\alpha)$, for which the difference $\nu - \mu$ is even, are both proper or both improper, while if $\nu - \mu$ is odd, then one of them is proper and the other improper. If α_ν, α_μ are both proper or both improper they are said to be *of the same kind*;

otherwise we describe them as *of different kind.* Clearly, in the ordering (12) of the phase system (α) the individual phases are alternately proper and improper: the successor of a proper phase is improper and conversely.

Let t_0 and n be arbitrary numbers, with $t_0 \in j$ and n an integer; then there is precisely one proper and one improper phase whose values at the point t_0 lie in the interval $[2n\pi, (2n + 2)\pi)$. Every proper (improper) phase with respect to the basis (u, v) is improper (proper) with respect to the basis $(-u, -v)$.

The geometrical significance of first phases of the basis (u, v) is as follows:—

Let α be a first phase of the basis (u, v), and let $W\alpha(t)$ be the (unique) number in the interval $[0, 2\pi)$ which is congruent to $\alpha(t)$ modulo 2π; that is to say $\alpha(t) = W\alpha(t) + 2\pi n$, n $(= n(t))$ integral, $0 \leqslant W\alpha(t) < 2\pi$, $t \in j$.

We consider the integral curve $\mathfrak{K}$ with the vector representation $x(t) = [u(t), v(t)]$. Then $W\alpha(t)$ is the angle formed between the vector $x(t)$ or the vector $-x(t)$ and the co-ordinate vector x_2, according as α is proper or improper. In other words $\alpha(t)$ is congruent modulo 2π to that angle in the range 0 to 2π which lies between $x(t)$ or $-x(t)$ and the co-ordinate vector x_2.

5.4 Boundedness of a first phase

Let α be a phase of the basis (u, v); the range of α in the interval j obviously forms an open interval.

We have the following theorem:

Theorem. The phase α is bounded in the interval j if and only if the differential equation (q) *is of finite type.*

Proof. Let J be the range of α in the interval j; in this interval j there hold formulae of the type (13).

(a) Assume that α is bounded in j, then the interval J contains only a finite number, say m $(\geqslant 0)$, of distinct multiples of the number π. From that it follows, using (13), that the integral u vanishes precisely m times in the interval j. Now let $\bar{u}$ be an arbitrary integral of (q). If $\bar{u}$ is linearly dependent upon u, then $\bar{u}$ has precisely m zeros in the interval j, (the same zeros as u). If, however, the integrals u, $\bar{u}$ are linearly independent, then $\bar{u}$ has $m - 1$ or m or $m + 1$ zeros in j, for between every two neighbouring zeros of u there is precisely one zero of $\bar{u}$ and conversely (§ 2.3). Thus the differential equation (q) is of finite type m or $m + 1$.

(b) Let the differential equation (q) be of finite type m $(\geqslant 0)$. Then the integral u has at most m zeros in the interval j. Consequently, from (13), the interval J contains at most m distinct integral multiples of π, so the phase α is bounded, and the proof is complete.

By similar reasoning we can obtain the following result:

If the phase α is increasing (decreasing), then it is unbounded below and bounded above in the interval j if and only if the differential equation (q) is left (right) oscillatory; similarly α is bounded below and unbounded above in the interval j if and only if the differential equation (q) is right (left) oscillatory while α is unbounded both below and above in the interval j if and only if the differential equation (q) is oscillatory.

The theorem of § 3.12 can be supplemented by the following remark:

The values $\alpha(t)$, $\alpha(x)$ of the phase α at two distinct points $t, x \in j$ differ by an integral multiple of π if and only if the numbers t, x are 1-conjugate.

5.5 Continuity property of a first phase

The phase α belongs to the class C_3. To show this we take an arbitrary number $x \in j$, not a zero of v, and form, in the interval j, the function

$$\bar{\alpha}(t) = \alpha(x) + \int_x^t \frac{-w}{r^2}\, d\sigma.$$

Obviously $\bar{\alpha} \in C_3$.

Now we consider the function $F(t) = \alpha(t) - \bar{\alpha}(t)$. From the definitions of α and $\bar{\alpha}$ it follows that

1. $F(x) = 0$,
2. $F \in C_0$,
3. In every interval $i \subset j$, in which v does not vanish, the derivative of F vanishes identically so that F takes a constant value.

(a) Let there be no zero of v in the interval j, then from 1, 3 we have $F(t) \equiv 0$ and consequently $\alpha(t) = \bar{\alpha}(t) \in C_3$.

(b) Let $t_0 \in j$ be a zero of v. This is an isolated zero of v, consequently there are maximal open intervals $i_1 \subset j$ and $i_2 \subset j$, with right and left end point t_0 respectively, in which v is non-zero. From 3° it follows that $F(t) = k_1$ for $t \in i_1$, $F(t) = k_2$ for $t \in i_2$, k_1, k_2 being constants, and from 2° we obtain $k_1 = F(t_0) = k_2$. There exists therefore a constant k such that $F(t) = k$ for all $t \in j$. Now 1° shows that $k = 0$; it follows that $\alpha(t) = \bar{\alpha}(t) \in C_3$, and the proof is complete.

It is easy to establish the validity of the following formulae in the interval j:

$$\alpha' = \frac{-w}{r^2}, \qquad \alpha'' = 2w\frac{rr'}{r^4}, \qquad \alpha''' = 2w\left(\frac{q}{r^2} - 3\frac{s^2}{r^4} + 4\frac{w^2}{r^6}\right); \tag{5.14}$$

in which naturally r, s denote the first and second amplitudes of the basis (u, v). We easily deduce that in the interval j,

$$\alpha' \neq 0, \tag{5.15}$$

and the phase α satisfies the non-linear differential equation of the third order

$$-\{\alpha, t\} - \alpha'^2(t) = q(t) \tag{5.16}$$

where the symbol $\{\alpha, t\}$ denotes the Schwarzian derivative of α at the point t $(\in j)$ (§ 1.7).

Then a brief calculation gives the relation

$$\{\alpha, t\} + \alpha'^2(t) = \{\tan\alpha, t\}; \tag{5.17}$$

in this relationship, if the function on the right has any singularities we assign to it at such points the corresponding values taken by the left hand side. Hence the non-linear differential equation (16) can be more briefly written as

$$-\{\tan \alpha, t\} = q(t). \tag{5.18}$$

We see that *the phase* α *serves to determine the carrier* q *of the differential equation* (q) *uniquely, through the formulae* (16) *and* (18).

Finally, we note the important first formula (14) which expresses the relation between the phase α and the first amplitude of the basis (u, v).

5.6 First phases of the differential equation (q)

By a *first phase of the differential equation* (q) we mean a first phase of an arbitrary basis of the differential equation (q). Obviously, the above results are valid for all first phases of the differential equation (q) and for the corresponding bases (u, v).

5.7 Phase functions

In the course of our study we shall frequently encounter functions known as "phase functions". By a *phase function* we mean a function α defined in an open interval j with the following properties:

1. $\alpha \in C_1$;
2. $\alpha' \neq 0$ for all $t \in j$.

The following theorem can be established without difficulty:

Each phase function $\alpha \in C_3$ represents, in its interval of definition j, a first phase of the differential equation (q) constructed by formula (16), and the functions

$$u = |\alpha'|^{-\frac{1}{2}} \sin \alpha, \qquad v = |\alpha'|^{-\frac{1}{2}} \cos \alpha$$

are independent integrals of this differential equation (q), α being a first phase of the basis (u, v).

5.8 Second phases of a basis

We now wish to define second phases of the basis (u, v) in a manner analogous to that for first phases. In order to achieve this we assume that the zeros of the first derivative v' of v, in so far as they exist, are isolated. We shall always make this assumption in what follows, when we are concerned with second phases of a basis (u, v) of the differential equation (q). It holds for instance if the carrier q is non-zero in j (§ 2.1).

By a *second phase of the basis* (u, v) we mean any continuous function β in the interval j, which satisfies the relation

$$\tan \beta(t) = \frac{u'(t)}{v'(t)} \tag{5.19}$$

at every point of this interval with the exception of the zeros of v'.

The second phases of the basis (u, v) have, in general, similar properties to those of first phases; we shall therefore only recount them briefly.

The countable system of second phases associated with the basis (u, v) we shall call the *second phase system of the basis* (u, v) and denote it by (β).

If we choose a second phase $\beta \in (\beta)$, then the system (β) comprises the functions

$$\beta_\nu(t) = \beta(t) + \nu\pi \qquad (\nu = 0, \pm 1, \pm 2, \ldots; \beta_0 = \beta), \tag{5.20}$$

and they can clearly be ordered as follows:

$$\cdots < \beta_{-2} < \beta_{-1} < \beta_0 < \beta_1 < \beta_2 < \cdots. \tag{5.21}$$

From the second formula (2.1) we deduce that each second phase $\beta_\nu \in (\beta)$ is increasing or decreasing in j according as $wq > 0$ or < 0.

The derivatives u', v' of the integrals u, v may be expressed in terms of the second amplitude s and an arbitrary second phase $\beta_\nu \in (\beta)$ of the basis (u, v) as follows:

$$u'(t) = \varepsilon'_\nu s(t) \cdot \sin \beta_\nu(t), \qquad v'(t) = \varepsilon'_\nu s(t) \cdot \cos \beta_\nu(t) \qquad (t \in j), \tag{5.22}$$

in which ε'_ν, the so-called *signature of the second phase* β_ν, takes the value $+1$ or -1. The second phase β_ν is called *proper* or *improper* (with respect to the basis (u, v)) according as $\varepsilon'_\nu = 1$ or $\varepsilon'_\nu = -1$. By means of the ordering (21) of the second phase system (β) the individual second phases are alternately proper and improper: the successor of a proper second phase is improper and conversely. Every proper (improper) second phase of a basis (u, v) is improper (proper) with respect to the basis $(-u, -v)$.

The geometrical significance of second phases of the basis (u, v) is as follows:

Let β be a second phase of the basis (u, v). Moreover let $W\beta(t)$ be that value lying in the interval $[0, 2\pi)$ which is congruent to $\beta(t)$ modulo 2π: that is $\beta(t) = W\beta(t) + 2\pi n$, $n\ (= n(t))$ integral, $0 \leqslant W\beta(t) < 2\pi$; $t \in j$.

We consider the integral curve $\mathfrak{K}$ with the vectorial representation $x(t) = [u(t), v(t)]$. Then $x'(t) = [u'(t), v'(t)]$ is the tangent vector to the curve κ at the point $P[u(t), v(t)]$, and $W\beta(t)$ is the angle between the tangent vector $x'(t)$ or the opposite vector $-x'(t)$ and the co-ordinate vector x_2, according as β is proper or improper. In other words $\beta(t)$ is congruent modulo 2π to that angle in the range $[0, 2\pi)$ between the vector $x'(t)$ or $-x'(t)$ and the coordinate vector x_2.

5.9 Boundedness of a second phase

Let β be a second phase of the basis (u, v). If the carrier q of (q) is non-zero in the interval j, then the second phase β is bounded or not according to the type of the differential equation (q) in the interval j, and similar statements can be made as for a first phase (§ 5.4). The theorem of § 3.12 can be extended as follows: the values $\beta(t)$,

$\beta(x)$ of the phase β at two distinct points $t, x \in j$ differ by an integral multiple of π if and only if the numbers t, x are 2-conjugate.

5.10 Continuity property of a second phase

The second phase β belongs to the class C_1. The proof of this statement follows the lines of that in § 5.5 for the first phase. In this case we consider the function

$$\bar{\beta}(t) = \beta(x) + \int_x^t \frac{wq}{s^2}\, d\sigma.$$

Assuming the existence of the appropriate derivatives of q, β belongs to a higher class than C_1 and we have the formulae

$$\left.\begin{aligned} \beta' &= w\,\frac{q}{s^2}, \qquad \beta'' = w\left(\frac{q'}{s^2} - 2\,\frac{qs'}{s^3}\right), \\ \beta''' &= w\left(\frac{q''}{s^2} - 2\,\frac{2q's' + qs''}{s^3} + 6\,\frac{qs'^2}{s^4}\right). \end{aligned}\right\} \tag{5.23}$$

The first formula shows that the zeros of the function β' coincide with those of the carrier q.

5.11 Connection of a second phase with the associated differential equation

Now we assume that the carrier q of (q) does not vanish in j, and that $q \in C_2$. Then the functions

$$u_1 = \frac{u'}{\sqrt{|q|}}, \qquad v_1 = \frac{v'}{\sqrt{|q|}}$$

form a basis of the differential equation $(\hat{q}_1)$, the associated differential equation of (q) (§ 1.9). From the relation $u'/v' = u_1/v_1$ we see that the second phase system of the basis (u, v) coincides with the first phase system of the basis (u_1, v_1). There follows the relationship, valid for $t \in j$,

$$-\tan\{\beta, t\} = \hat{q}_1(t). \tag{5.24}$$

Moreover

The differential equation (q) *and the associated differential equation* $(\hat{q}_1)$ *have the same oscillatory character; that is, both are simultaneously of finite type or are oscillatory of the same kind.*

For, let β be an increasing second phase of the basis (u, v) and consequently also an increasing first phase α_1 of (u_1, v_1): $\beta = \alpha_1$. If the differential equation (q) is of finite type, then (by § 5.9) the function β i.e. α_1 is bounded; § 5.4 then shows that the differential equation $(\hat{q}_1)$ is of finite type. If the differential equation (q) is of infinite type and is left or right oscillatory or oscillatory, then by § 5.9 the function β is respectively unbounded below and bounded above, or bounded below and unbounded above, or unbounded on both sides, and the phase α_1 naturally has the same properties. Then from § 5.4 we conclude that the differential equation ($\hat{q}_1$) is of infinite type, and respectively left oscillatory, or right oscillatory, or oscillatory.

By a similar argument, a given oscillatory character of ($\hat{q}_1$) implies the same character of (q).

5.12 Second phases of the differential equation (q)

By a *second phase of the differential equation* (q) we mean a second phase of any basis of (q). Obviously the results obtained in §§ 5.8–5.10 are valid for any second phase of the differential equation (q) and the corresponding basis (u, v). We observe that the problem of determining the differential equation (q) with a non-vanishing carrier $q \in C_2$, when given one of its second phases β is equivalent to the problem of integrating the non-linear second-order differential equation

$$X'' = -\{\tan \beta, t\} \cdot X + \frac{\varepsilon}{X} \qquad (\varepsilon = \pm 1).$$

This may be seen when we write the formula (1.18) in the following way:

$$-\{\tan \beta, t\} \frac{1}{\sqrt{|q(t)|}} = \frac{\operatorname{sgn} q(t)}{\dfrac{1}{\sqrt{|q(t)|}}} + \left(\frac{1}{\sqrt{|q(t)|}}\right)''.$$

5.13 Integrals of the differential equation (q) and their derivatives expressed in polar coordinates

Let (u, v) be a basis of the differential equation (q), r, s be its amplitudes and α, β a first and second phase of the basis (u, v). We have already seen ((13), (22)) that the integrals u, v and their derivatives u', v' can be represented in the interval j by the formulae

$$\left.\begin{aligned} u(t) &= \varepsilon r(t) \cdot \sin \alpha(t), & v(t) &= \varepsilon r(t) \cdot \cos \alpha(t), \\ u'(t) &= \varepsilon' s(t) \cdot \sin \beta(t), & v'(t) &= \varepsilon' s(t) \cdot \cos \beta(t) \end{aligned}\right\} \quad (\varepsilon, \varepsilon' = \pm 1) \tag{5.25}$$

in which the values of $\varepsilon, \varepsilon'$ depend on the choice of the phases α, β.

Making use of (14), (23) we can also write

$$\left.\begin{aligned} u(t) &= \varepsilon \sqrt{|w|} \frac{\sin \alpha(t)}{\sqrt{|\alpha'(t)|}}, & v(t) &= \varepsilon \sqrt{|w|} \frac{\cos \alpha(t)}{\sqrt{|\alpha'(t)|}}, \\ \sqrt{|\beta'(t)|}\, u'(t) &= \varepsilon' \sqrt{|wq(t)|} \sin \beta(t), & \sqrt{|\beta'(t)|}\, v'(t) &= \varepsilon' \sqrt{|wq(t)|} \cos \beta(t). \end{aligned}\right\} \tag{5.26}$$

Consequently we have, for the general integral y of the differential equation (q) and its derivative y', the expressions

$$\left.\begin{aligned} y(t) &= k_1 \frac{\sin[\alpha(t) + k_2]}{\sqrt{|\alpha'(t)|}}, \\ \sqrt{|\beta'(t)|}\, y'(t) &= \pm k_1 \sqrt{|q(t)|} \sin[\beta(t) + k_2], \end{aligned}\right\} \tag{5.27}$$

in which k_1, k_2 are arbitrary constants. In the second formula (27) we take the sign $+$ or $-$ according as the signatures ε, ε' of the phases α, β are the same or different.

If $k_2 = n\pi + k_2'$, $0 \leqslant k_2' < \pi$, n being an integer, then the value of the right hand side of (27) is not changed if we replace k_1 by $k_1' = (-1)^n k_1$ and k_2 by k_2'. Consequently, in the formula (27) we can assume without loss of generality that $0 \leqslant k_2 < \pi$.

5.14 Ordering relations between first and second phases of the same basis

We consider an arbitrary basis (u, v) of the differential equation (q) with the Wronskian w $(= uv' - u'v)$. Let $\alpha \in (\alpha)$, $\beta \in (\beta)$ be respectively a first and second phase of the basis (u, v) and ε, ε' be the corresponding signatures.

From the definition of w and the formula (25) there follows (for $t \in j$) the relation

$$r \cdot s \cdot \sin(\beta - \alpha) = \varepsilon\varepsilon'(-w). \tag{5.28}$$

Since the right side of this equation is everywhere non-zero, there is an integer n such that the difference $\beta - \alpha$ lies between $n\pi$ and $(n + 1)\pi$ $\forall\, t \in j$,

$$n\pi < \beta - \alpha < (n + 1)\pi. \tag{5.29}$$

We set $\alpha_0 = \alpha + n\pi$, $\beta_0 = \beta$ and define the phases $\alpha_\nu \in (\alpha)$, $\beta_\nu \in (\beta)$ as in the formulae (11), (20). It is clear that the system formed from all first and second phases of the basis (u, v), which we call the *mixed phase system* of the basis (u, v), can be ordered in the following way:

$$\cdots < \alpha_{-1} < \beta_{-1} < \alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \cdots. \tag{5.30}$$

This ordering is obviously such that in the interval j two neighbouring phases α_ν, β_ν or β_ν, $\alpha_{\nu+1}$ satisfy respectively the relations

$$0 < \beta_\nu - \alpha_\nu < \pi, \qquad -\pi < \beta_\nu - \alpha_{\nu+1} < 0.$$

It follows from (28) that, respectively,

$$\operatorname{sgn} \varepsilon_\nu \varepsilon_\nu'(-w) = 1, \qquad \operatorname{sgn} \varepsilon_\nu' \varepsilon_{\nu+1}(-w) = -1.$$

We have, clearly, to consider two cases, according as $-w > 0$ or $-w < 0$; correspondingly, the first phase α_ν is increasing or decreasing.

In the case $-w > 0$ we have

$$\operatorname{sgn} \varepsilon_\nu \varepsilon_\nu' = 1; \qquad \operatorname{sgn} \varepsilon_\nu' \varepsilon_{\nu+1} = -1.$$

Then the ordering (30) of the mixed phase system of the basis (u, v) has the effect that each proper (improper) first phase α_ν is followed by a proper (improper) second phase

β_ν, while after each proper (improper) second phase β_ν there follows an improper (proper) first phase $\alpha_{\nu+1}$.

In the case $-w < 0$ we have

$$\operatorname{sgn} \varepsilon_\nu \varepsilon'_\nu = -1, \qquad \operatorname{sgn} \varepsilon'_\nu \varepsilon_{\nu+1} = 1,$$

and then each proper (improper) first phase α_ν is followed by an improper (proper) second phase β_ν, while each proper (improper) second phase β_ν is followed by a proper (improper) first phase $\alpha_{\nu+1}$.

This provides a relationship between first and second phases of the same basis of the differential equation (q).

5.15 Some consequences

We now wish to develop some further relations. First, from (28), (14) and (23) there follows:

$$\frac{\alpha'\beta'}{\sin^2(\beta - \alpha)} = -q. \tag{5.31}$$

Moreover from the first formulae (14), (23) we obtain

$$\frac{\beta' \cdot s^2}{\alpha' \cdot r^2} = -q. \tag{5.32}$$

If at a point $t \in j$, we have $-q > 0$, or $-q < 0$, then the functions α', β' have the same or opposite signs at this point respectively. Hence if the function q is everywhere non-zero in the interval j, the phases α, β have the property that in the case $-q > 0$ both phases α, β increase or decrease together, while in the case $-q < 0$ one increases while the other decreases.

Consequently if the function q is everywhere non-zero in the interval j, then the mixed phase system (30) behaves as follows: in the case $-q > 0$ both phases α_ν, β_ν increase or decrease together, while in the case $-q < 0$ one of the phases α_ν, β_ν increases while the other decreases.

We can supplement the theorem of § 3.12 as follows: the values $\alpha(t)$, $\beta(x)$ of the phases α, β at two points t, $x \in j$ differ by an integral multiple of π if and only if x is 3-conjugate with t and consequently t is 4-conjugate with x.

5.16 Explicit connection between first and second phases of the same basis

By §§ 5.5, 5.10, arbitrary phases α, β of the basis (u, v) are such that, in the interval j

$$\alpha \in C_3, \qquad \beta \in C_1, \qquad \alpha' \neq 0. \tag{5.33}$$

Now we prove the following theorem:

Theorem. Two functions α, β *in an open interval* j, *with the properties* (33), *represent a first and second phase of a basis* (u, v) *of a differential equation* (q) *if and only if*

$$\beta = \alpha + \operatorname{Arccot} \frac{1}{2} \left(\frac{1}{\alpha'}\right)' . \tag{5.34}$$

Here Arccot denotes a particular or an arbitrary branch of the function.

If the relation (34) *is satisfied then the functions defined in* j *by the following relations*

$$u = |\alpha'|^{-\frac{1}{2}} \sin \alpha, \qquad v = |\alpha'|^{-\frac{1}{2}} \cos \alpha \tag{5.35}$$

have the desired property and the corresponding carrier q *is determined by the formula* (16).

Proof. (a) Let (u, v) be a basis of a differential equation (q) and α, β a first and second phase respectively of (u, v). The functions α, β have the properties (33) and they satisfy formulae such as (10), (19), (29). From the relation

$$\cot(\beta - \alpha) = -\frac{\sin \alpha \sin \beta + \cos \alpha \cos \beta}{\sin \alpha \cos \beta - \sin \beta \cos \alpha} = -\frac{uu' + vv'}{uv' - u'v} = -\frac{1}{w} rr',$$

and the first formula (14), there follows the relation (34).

(b) Let α, β be arbitrary functions in an open interval j with the properties (33), (34). In j we define the functions q, u, v by means of formulae (16) and (35). Obviously the functions $u, v \in C_2$ and at every point $t \in j$ there hold the formulae

$$\left.\begin{aligned} u' &= \varepsilon|\alpha'|^{\frac{1}{2}} \left[\cos \alpha + \frac{1}{2}\left(\frac{1}{\alpha'}\right)' \sin \alpha\right], \\ v' &= \varepsilon|\alpha'|^{\frac{1}{2}} \left[-\sin \alpha + \frac{1}{2}\left(\frac{1}{\alpha'}\right)' \cos \alpha\right] \end{aligned} \quad (\varepsilon = \operatorname{sgn} \alpha'). \right\} \tag{5.36}$$

Moreover,

$$\begin{aligned} u'' &= [-\{\alpha, t\} - \alpha'^2]|\alpha'|^{-\frac{1}{2}} \sin \alpha, \\ v'' &= [-\{\alpha, t\} - \alpha'^2]|\alpha'|^{-\frac{1}{2}} \cos \alpha. \end{aligned}$$

The functions u, v thus form a basis of the differential equation (q) with the Wronskian $w = -\varepsilon$.

From (35), $u/v = \tan \alpha$ throughout the interval j, except for zeros of v, while (36) and (34) give

$$\frac{u'}{v'} = \frac{\cos \alpha + \frac{1}{2}\left(\frac{1}{\alpha'}\right)' \sin \alpha}{-\sin \alpha + \frac{1}{2}\left(\frac{1}{\alpha'}\right)' \cos \alpha} = \frac{\cos \alpha + \sin \alpha \cdot \cot(\beta - \alpha)}{-\sin \alpha + \cos \alpha \cdot \cot(\beta - \alpha)} = \tan \beta$$

except at the zeros of v'.

Clearly, α is a first and β a second phase of the basis (u, v) and the proof is complete.

Finally we remark that arbitrary phases α, β of the basis (u, v) in the interval j are related by means of the following "bracket formula"

$$\{\tan \alpha, t\} - \{\tan \beta, t\} + \left\{\int_{t_0}^{t} q \, d\sigma, t\right\} = 0 \quad (t_0 \in j), \tag{5.37}$$

this relation being obtained from (18), (24) and (1.20).

5.17 Phases of different bases of the differential equation (q)

We now wish to study relationships between the first and second phases of two different bases of the differential equation (q).

Let (u, v), $(\bar{u}, \bar{v})$ be bases of the differential equation (q) and w, $\bar{w}$ their Wronskians; moreover let α, $\bar{\alpha}$ and β, $\bar{\beta}$ be first and second phases of these bases and $c_{11}, c_{12}, c_{21}, c_{22}$ be constants such that the determinant $\Delta = |c_{ik}|$ is not zero. Then we have the following theorem

Theorem. If

$$\left.\begin{aligned} \bar{u} &= c_{11}u + c_{12}v, \\ \bar{v} &= c_{21}u + c_{22}v \end{aligned}\right\} \tag{5.38}$$

then

$$\tan \bar{\alpha} = \frac{c_{11} \tan \alpha + c_{12}}{c_{21} \tan \alpha + c_{22}}, \qquad \tan \bar{\beta} = \frac{c_{11} \tan \beta + c_{12}}{c_{21} \tan \beta + c_{22}}. \tag{5.39}$$

Conversely, from the first relation (39), *or from the second relation if $q \neq 0$ in j, it follows that*

$$\left.\begin{aligned} \bar{u} &= \pm \sqrt{\frac{\bar{w}}{w\Delta}} (c_{11}u + c_{12}v), \\ \bar{v} &= \pm \sqrt{\frac{\bar{w}}{w\Delta}} (c_{21}u + c_{22}v). \end{aligned}\right\} \tag{5.40}$$

The relations (39) are understood to hold throughout j except for singular points of the functions involved.

Proof. The first part of the theorem is obviously valid. We assume therefore that the first relation (39) holds; then $\bar{u}/\bar{v} = (c_{11}u + c_{12}v)/(c_{21}u + c_{22}v)$ and moreover, on taking account of (2.1) we have also $\bar{w}/\bar{v}^2 = w\Delta/(c_{21}u + c_{22}v)^2$. Then (40) follows immediately.

Now we assume that the second relation (39) holds, and that $q \neq 0$ for all $t \in j$. We then have $\bar{u}'/\bar{v}' = (c_{11}u' + c_{12}v')/(c_{21}u' + c_{22}v')$, then (2.1) and the hypothesis $q \neq 0$ yield the relation $\bar{w}/\bar{v}'^2 = w\Delta/(c_{21}u' + c_{22}v')^2$. From this we obtain the formulae (40).

We now have the following corollaries:

1. The first phase systems of two proportional bases of the differential equation (q) coincide; the second phase systems also coincide. Conversely, if two bases of (q) have

a common first or second phase, and q is non-zero in the interval j, then these bases are proportional.

2. If $\alpha(\beta)$ is a first (second) phase of the basis (u, v), then $\frac{1}{2}\pi - \alpha$ $(\frac{1}{2}\pi - \beta)$ is a first (second) phase for the inverse basis (v, u) of (u, v).

That is to say, one obtains the elements of the first or second phase system of the inverse basis (v, u) by multiplying the corresponding elements of the basis (u, v) by -1 and increasing by $\frac{1}{2}\pi$.

3. If $\alpha(\beta)$ is a first (second) phase of the basis (u, v) and λ is arbitrary then $\alpha + \lambda$, $(\beta + \lambda)$ is a first (second) phase of the basis (u, v) transformed by the orthogonal substitution

$$\begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix} \tag{5.41}$$

that is to say the basis

$$\bar{u} = u \cdot \cos\lambda + v \cdot \sin\lambda,$$
$$\bar{v} = -u \cdot \sin\lambda + v \cdot \cos\lambda.$$

When we have a first (second) phase α (β) of the differential equation (q) then the system formed from this by means of the formula $\bar{\alpha} = \alpha + \lambda$ $(\bar{\beta} = \beta + \lambda)$ with arbitrary λ, we call the *complete phase system* of the phase $\alpha(\beta)$. For this we use the notation $[\alpha]$, $([\beta])$. Obviously, given any number $t_0 \in j$ there is precisely one first (second) phase of the differential equation (q) in the system $[\alpha]$ $([\beta])$, which vanishes at the point t_0.

4. Two first or second phases α, $\bar{\alpha}$ or β, $\bar{\beta}$ of the differential equation (q) are connected by means of the formulae (39). If conversely $\alpha(\beta)$ is a first (second) phase of (q) and there holds a formula similar to (39) for a function $\bar{\alpha}$ $(\bar{\beta})$ defined in the interval j, then $\bar{\alpha}$ $(\bar{\beta})$ is also a first (second) phase of (q).

5.18 Calculation of the integrals $\int_{x_0}^{x_1} g(\sigma)d\sigma$, $\int_{x_0}^{x_1} h(\sigma)d\sigma$ in the neighbourhood of singular points

As an application of the concept of phases, and of their properties obtained so far, we show how to evaluate the integrals considered in § 2.4.

We revert to the situation described there, using the same notation. In particular y denotes an integral of the differential equation (q) with a zero c; j_{-1} or j_0 denotes a left or right neighbourhood of c, in which the integral y does not vanish, and $g(\sigma)$ for $\sigma \in j_{-1}$ or $\sigma \in j_0$ denotes the function

$$g(\sigma) = \frac{1}{y^2(\sigma)} - \frac{1}{y'^2(c)} \cdot \frac{1}{(\sigma - c)^2}.$$

We know that for $x_0 \in j_{-1}$, $x_1 \in j_0$ the integral $\int_{x_0}^{x_1} g(\sigma)\, d\sigma$ exists. We now wish to determine its value.

Let $t \in (x_0, c)$ be arbitrary. Then we have

$$\int_{x_0}^{t} g(\sigma)\, d\sigma = \int_{x_0}^{t} \frac{d\sigma}{y^2(\sigma)} + \frac{1}{y'^2(c)}\left[\frac{1}{t-c} - \frac{1}{x_0 - c}\right].$$

Let α be the first phase, vanishing at c, of that basis (u, v) of the differential equation (q) determined by the initial values $u(c) = 0$, $u'(c) = 1$; $v(c) = 1$, $v'(c) = 0$. Then we have

$$\alpha(c) = 0, \qquad \alpha'(c) = 1, \qquad \alpha''(c) = 0,$$

and the first formula (27) gives

$$y(\sigma) = y'(c)\, \frac{\sin \alpha(\sigma)}{\sqrt{\alpha'(\sigma)}}.$$

Thus

$$y'^2(c) \int_{x_0}^{t} \frac{d\sigma}{y^2(\sigma)} = \int_{x_0}^{t} \frac{\alpha'(\sigma)\, d\sigma}{\sin^2 \alpha(\sigma)} = \int_{\alpha(x_0)}^{\alpha(t)} \frac{d\sigma}{\sin^2 \sigma} = -\cot \alpha(t) + \cot \alpha(x_0),$$

and consequently

$$y'^2(c) \int_{x_0}^{t} g(\sigma)\, d\sigma = -\cot \alpha(t) + \frac{1}{t-c} + \cot \alpha(x_0) - \frac{1}{x_0 - c}.$$

On account of the fact that

$$\lim_{t \to c-} \left[-\cot \alpha(t) + \frac{1}{t-c}\right] = 0$$

we have

$$y'^2(c) \int_{x_0}^{c} g(\sigma)\, d\sigma = \cot \alpha(x_0) - \frac{1}{x_0 - c}. \tag{5.42}$$

Similarly we obtain, for arbitrary $x_1 \in j_0$

$$y'^2(c) \int_{c}^{x_1} g(\sigma)\, d\sigma = -\cot \alpha(x_1) + \frac{1}{x_1 - c}. \tag{5 43}$$

Clearly therefore, for arbitrary numbers $x_0 \in j_{-1}$, $x_1 \in j_0$ we have

$$\int_{x_0}^{x_1} \left[\frac{1}{y^2(\sigma)} - \frac{1}{y'^2(c)}\, \frac{1}{(\sigma - c)^2}\right] d\sigma$$
$$= \frac{1}{y'^2(c)}\left[-\cot \alpha(x_1) + \cot \alpha(x_0) + \frac{1}{c - x_0} + \frac{1}{x_1 - c}\right]. \tag{5 44}$$

If the numbers x_0, x_1 are 1-conjugate, then the quantities $\alpha(x_0)$, $\alpha(x_1)$ differ by an integral multiple of π (§ 5.4) In this case we have

$$\int_{x_0}^{x_1} \left[\frac{1}{y^2(\sigma)} - \frac{1}{y'^2(c)}\, \frac{1}{(\sigma - c)^2}\right] d\sigma = \frac{1}{y'^2(c)}\left[\frac{1}{c - x_0} + \frac{1}{x_1 - c}\right]. \tag{5.45}$$

If we apply these results to the calculation of the integral $\int_{x_0}^{x_m} g_m(\sigma)\, d\sigma$ considered in § 2.4, then we first obtain the formula

$$\int_{x_0}^{x_m} g_m(\sigma)\, d\sigma = \sum_{\nu=1}^{m} \frac{1}{y'^2(c_\nu)} [-\cot \alpha_\nu(x_\nu) + \cot \alpha_\nu(x_{\nu-1})] + \sum_{\nu=1}^{m} \frac{1}{y'^2(c_\nu)} \left[\frac{1}{c_\nu - x_0} + \frac{1}{x_m - c_\nu}\right]; \tag{5.46}$$

in which α_ν is naturally the first phase, vanishing at the point c_ν, of that basis (u, v) of the differential equation (q) determined by the initial values $u(c_\nu) = 0$, $u'(c_\nu) = 1$; $v(c_\nu) = 1$, $v'(c_\nu) = 0$; $\nu = 1, \ldots, m$.

If, in particular, the numbers x_0, x_m are 1-conjugate, then

$$\int_{x_0}^{x_m} g_m(\sigma)\, d\sigma = \sum_{\nu=1}^{m} \frac{1}{y'^2(c_\nu)} \left[\frac{1}{c_\nu - x_0} + \frac{1}{x_m - c_\nu}\right]. \tag{5.47}$$

Similarly, we can express the values of the integrals $\int_{x_0}^{x_1} h(\sigma)\, d\sigma$, $\int_{x_0}^{x_m} h_m(\sigma)\, d\sigma$ considered in § 2.5, in terms of appropriate second phases β, β_ν of the differential equation (q), as follows:

$$\int_{x_0}^{x_1} \left[\frac{q(\sigma)}{y'^2(\sigma)} - \frac{1}{q(e)y^2(e)} \cdot \frac{1}{(\sigma - e)^2}\right] d\sigma = \frac{1}{q(e)y^2(e)} \left[-\cot \beta(x_1) + \cot \beta(x_0) + \frac{1}{e - x_0} + \frac{1}{x_1 - e}\right]; \tag{5.48}$$

$$\int_{x_0}^{x_m} h_m(\sigma)\, d\sigma = \sum_{\nu=1}^{m} \frac{1}{q(e_\nu)y^2(e_\nu)} [-\cot \beta_\nu(x_\nu) + \cot \beta_\nu(x_{\nu-1})] + \sum_{\nu=1}^{m} \frac{1}{q(e_\nu)y^2(e_\nu)} \left[\frac{1}{e_\nu - x_0} + \frac{1}{x_m - e_\nu}\right]. \tag{5.49}$$

If the numbers x_0, x_1 or x_0, x_m are 2-conjugate, then there hold the simpler formulae (§ 5.9).

$$\int_{x_0}^{x_1} \left[\frac{q(\sigma)}{y'^2(\sigma)} - \frac{1}{q(e)y^2(e)} \cdot \frac{1}{(\sigma - e)^2}\right] d\sigma = \frac{1}{q(e)y^2(e)} \left[\frac{1}{e - x_0} + \frac{1}{x_1 - e}\right], \tag{5.50}$$

$$\int_{x_0}^{x_m} h_m(\sigma)\, d\sigma = \sum_{\nu=1}^{m} \frac{1}{q(e_\nu)y^2(e_\nu)} \left[\frac{1}{e_\nu - x_0} + \frac{1}{x_m - e_\nu}\right] \tag{5.51}$$