4 Centro-affine differential geometry of plane curves

Linear homogeneous differential equations of the second order stand in a close relationship with centro-affine differential geometry of plane curves. This geometry is the theory of those concepts and properties of plane curves which are invariant under transformations of the curve parameter and the plane linear homogeneous group of transformations. Linear homogeneous transformations are for brevity called "centro-affine transformations".

4.1 Representation of plane curves

A plane curve $\mathfrak{K}$ is the set of points which are given by the values of a vector function $x(t) = [u(t), v(t)]$ with two components $u(t)$, $v(t)$. We assume that the function x and consequently its components u, v are defined in an open interval j and $\in C_2$ or, if necessary, a higher class. The determinant $u^{(\mu)}v^{(\nu)} - u^{(\nu)}v^{(\mu)}$ formed from the derivatives $x^{(\mu)}$, $x^{(\nu)}$ we shall usually denote by $(x^{(\mu)}x^{(\nu)})$.

We shall refer to the function $x(t)$ as the *representation* of the curve $\mathfrak{K}$. Its components $u(t)$, $v(t)$ can be interpreted as parametric coordinates of the curve $\mathfrak{K}$ with respect to a coordinate system formed from the two vectors x_1, x_2 with origin O. If the function $x \in C_\mu$, then the curve $\mathfrak{K}$ is said to be also of the class C_μ. The independent variable t is called the parameter of $\mathfrak{K}$ in the representation $x(t)$.

We consider a plane curve $\mathfrak{K}$ with the representation $x(t)$. The curve $\mathfrak{K}$ admits of infinitely many representations, but in what follows we only consider those representations which arise from a single representation (for example, $x(t)$) by means of a transformation of parameter $T = T(t)$. We assume that the function $T(t)$ is defined in j, belongs to the class C_2 or if necessary to a higher class, and that its derivative T' is always non-zero. In this case the range of values of the function $T(t)$ is an open interval J, and in this interval there exists the inverse function, $t(T)$, to $T(t)$. Obviously $J = T(j), j = t(J)$. We call any two numbers $t \in j$, $T \in J$ *homologous* if they are related to each other by the formulae $T = T(t)$, $t = t(T)$. If, now, we are given a representation $x(t)$ and a parameter transformation $T = T(t)$ we define the resulting representation $X(T) = [U(T), V(T)]$ of the curve $\mathfrak{K}$ by specifying that the value of the function X at each point $T \in J$ is precisely the value $x(t)$ of the function x at the homologous point $t \in j$, that is

$$X(T) = x(t) \tag{4.1}$$

for $T = T(t) \in J$, $t = t(T) \in j$. It is clear that each of the two representations x, X of the curve $\mathfrak{K}$ is uniquely determined by the other, and each representation of the curve

$\mathfrak{K}$ arises from the other by a parameter transformation. We stress that the values of two representations of the curve at homologous points give the same point of the curve.

The function $X(T)$ obviously belongs to the class C_2 or to a higher class. Denoting derivatives with respect to T by a dot, the following formulae hold at two homologous points $t \in j$, $T \in J$

$$\left.\begin{aligned} \dot{X}(T) &= x'(t)\dot{t}, & x'(t) &= \dot{X}(T)T', \\ \ddot{X}(T) &= x''(t)\dot{t}^2 + x'(t)\ddot{t}, & x''(t) &= \ddot{X}(T)T'^2 + \dot{X}(T)T'' \end{aligned}\right\} \tag{4.2}$$

and further

$$\left.\begin{aligned} (X\dot{X}) &= (xx')\dot{t}, & (xx') &= (X\dot{X})T', \\ (X\ddot{X}) &= (xx'')\dot{t}^2 + (xx')\ddot{t}, & (xx'') &= (X\ddot{X})T'^2 + (X\dot{X})T'', \\ (\dot{X}\ddot{X}) &= (x'x'')\dot{t}^3, & (x'x'') &= (\dot{X}\ddot{X})T'^3. \end{aligned}\right\} \tag{4.3}$$

If, therefore, at any point $t \in j$ we have $(xx') \neq 0$ or $(xx') = 0$, then at the homologous point $T \in J$ we have, respectively, $(X\dot{X}) \neq 0$ or $(X\dot{X}) = 0$. In the first case we have, moreover,

$$\operatorname{sgn} T' = \operatorname{sgn}(xx') \cdot \operatorname{sgn}(X\dot{X}) = \operatorname{sgn} \dot{t}.$$

The same is true of the functions $(x'x'')$, $(\dot{X}\ddot{X})$.

4.2 Centro-affine representatives of a plane curve and its representations

We now consider a linear homogeneous, (hence centro-affine) transformation of the plane

$$\left.\begin{aligned} \bar{x}_1 &= c_{11}x_1 + c_{12}x_2, \\ \bar{x}_2 &= c_{21}x_1 + c_{22}x_2, \end{aligned}\right\} \tag{4.4}$$

with the matrix $C = (c_{ik})$ $(|C| =)$ $|c_{ik}| \neq 0, i, k = 1, 2$; such transformations obviously leave the point (0, 0), the so-called *centre* of the centro-affine plane, invariant. By the transformation (4) the function $x(t)$ goes over into the function $(Cx(t) =)\bar{x}(t)$ with the components $\bar{u}(t) = c_{11}u(t) + c_{12}v(t)$, $\bar{v}(t) = c_{21}u(t) + c_{22}v(t)$. We describe the curve $C\mathfrak{K} = \bar{\mathfrak{K}}$, determined by the function $\bar{x}(t)$, as a centro-affine *representative* or, more precisely, the *C-representative* of $\mathfrak{K}$; we also associate this term with the function $\bar{x}(t)$ itself and call $\bar{\mathfrak{K}}$ the C-representative of $x(t)$. Two points $x(t) \in \mathfrak{K}$, $\bar{x}(t) \in \bar{\mathfrak{K}}$, which are defined by the same value $t \in j$ of the parameter, we call (mutually) *associated*. In particular, the curve $\mathfrak{K}$ and its representation $x(t)$ is its own representative ($c_{11} = c_{22} = 1$, $c_{12} = c_{21} = 0$), while the curve $\mathfrak{K}$ and its representation $x(t)$ is a representative, more precisely the C^{-1}-representative, of $\bar{\mathfrak{K}}$ and $\bar{x}(t)$.

Let $X(T)$ be the representation of the curve $\mathfrak{K}$ arising from $x(t)$ by means of the parameter transformation $T = T(t)$. The C-representatives Cx, CX of the representations x, X of the curve $\mathfrak{K}$ obviously take the same value at two homologous points t, T. It follows that CX is the representation of the curve $C\mathfrak{K}$ arising from Cx by means of the parameter transformation $T = T(t)$. The values taken by two representations x, X of the curve $\mathfrak{K}$ at two homologous points, and the corresponding representatives Cx, CX at the same points, give associated points of the curves $\mathfrak{K}$ and $C\mathfrak{K}$. The determinants

$(\bar{x}\bar{x}')$, $(\bar{x}'\bar{x}'')$ constructed from a representative $\bar{x}(t) = Cx(t)$ of the function $x(t)$ differ from (xx'), (x', x'') by the same non-zero factor, namely the determinant $|C|$. It follows that if at a point $t \in j$ the relation $(xx') \neq 0$ or $(xx') = 0$ holds, then for every representative $\bar{x}$ of x at the same point t the analogous relation $(\bar{x}\bar{x}') \neq 0$ or $(\bar{x}\bar{x}') = 0$ holds. Moreover sgn $(\bar{x}\bar{x}) =$ sgn $|C| \cdot$ sgn (xx'). The same is true for the functions $(x'x'')$, $(\bar{x}'\bar{x}'')$.

4.3 Centro-affine invariants of plane curves

If we have a scalar function $f[x(t), x'(t), x''(t)]$ constructed from a representation $x(t)$ of $\mathfrak{K}$ and some of its derivatives such as $x'(t)$, $x''(t)$, then we call it a (centro-affine) *relative invariant* of the curve $\mathfrak{K}$, if for arbitrary choice of the representatives $\bar{x}(t)$, $\bar{X}(T)$ of two representations $x(t)$, $X(T)$ of the curve $\mathfrak{K}$ at two homologous points t, T, the values taken by this function are the same or the same with opposite sign, i.e.

$$f[\bar{x}(t), \bar{x}'(t), \bar{x}''(t)] = \pm f[\bar{X}(T), \dot{\bar{X}}(T), \ddot{\bar{X}}(T)].$$

We call the function f a (centro-affine) *absolute invariant*, if it takes the same value at two homologous points t, T for arbitrary choice of $\bar{x}(t)$, $\bar{X}(T)$. Obviously the absolute value of a relative invariant is an absolute invariant.

Among the simplest relative invariants of the curve $\mathfrak{K}$ are the functions mentioned above, namely sgn (xx'), sgn $(x'x'')$. Let us observe the geometrical significance of these, without going into details (for which see [80]). At a point $t \in j$, according as sgn $(xx') = \pm 1$, or sgn $(xx') = 0$, the centre of the centro-affine plane lies off or on the tangent to the curve $\mathfrak{K}$ at the point $x(t)$. According as sgn $(x'x'') = \pm 1$ or sgn $(x'x'') = 0$, the curve $\mathfrak{K}$ in the neighbourhood of the point $x(t)$ lies on one side of the tangent at the point $x(t)$, or the point $x(t)$ is a turning point of $\mathfrak{K}$, respectively.

4.4 Regular curves

We call the curve $\mathfrak{K}$ *regular*, if the two relationships sgn $(xx') = \pm 1$, sgn $(x'x'') = \pm 1$ hold everywhere in the interval j.

If therefore the curve $\mathfrak{K}$ is regular, then none of its tangents pass through the centre of the centro-affine plane, the curve is locally convex and has no turning points.

Now let $\mathfrak{K}$ be a regular curve with the representation $x(t)$, $t \in j$. We associate with this representation the functions

$$a(t) = \frac{(xx'')}{(xx')}, \quad b(t) = -\frac{(x'x'')}{(xx')} \tag{4.5}$$

which belong to the class C_0 or a higher class.

It is easy to verify that the function $x(t)$ satisfies the second order differential equation

$$x'' = a(t)x' + b(t)x, \tag{4.6}$$

constructed with the coefficients (5); that is to say, each of the components $u(t)$, $v(t)$ of the function $x(t)$ satisfies the scalar differential equation (6). Conversely, if the function $x(t)$ satisfies a differential equation of the form (6), then the coefficients $a(t)$, $b(t)$ are uniquely determined by $x(t)$ through the formulae (5). Further, the functions $\bar{a}(t)$, $\bar{b}(t)$ associated with a representative $\bar{x}(t)$ of $x(t)$ coincide with $a(t)$, $b(t)$; $\bar{a}(t) = a(t)$, $\bar{b}(t) = b(t)$; $t \in j$. Thus the functions $a(t)$, $b(t)$ are invariant no matter what choice is made of a representative of x, so every representative of the representation $x(t)$ provides a solution of the differential equation (6). We call the functions $a(t)$, $b(t)$ the *first* and *second centro-affine semi-invariants* of the representation $x(t)$.

Now let $x(t)$, $t \in j$ and $X(T)$, $T \in J$ be two representations of the curve $\mathfrak{K}$ and $a(t)$, $A(T)$ and $b(t)$, $B(T)$ be their first and second centro-affine semi-invariants. From (5) and (3) we find that the values of these functions at two homologous points $t \in j$, $T \in J$ are related as follows

$$\left.\begin{aligned} A(T) &= a(t)\dot{t} + \frac{\ddot{t}}{\dot{t}}\,, \quad B(T) = b(t)\dot{t}^2, \\ a(t) &= A(T)T' + \frac{T''}{T'}, \quad b(t) = B(T)T'^2. \end{aligned}\right\} \tag{4.7}$$

From these relations it follows that the two centro-affine semi-invariants $b(t)$, $B(T)$ of the representations $x(t)$, $X(T)$ at two homologous points $t \in j$, $T \in J$, have the same sign:

$$(\varepsilon =)\ \operatorname{sgn} b(t) = \operatorname{sgn} B(T).$$

We see that this sign ε is an absolute centro-affine invariant of the curve $\mathfrak{K}$. Since the curve $\mathfrak{K}$ is regular it follows from (5) that ε takes the same value for all representations of $\mathfrak{K}$ and its representatives for all values of the parameter. Regarding the geometrical significance of ε, we remark that according as $\varepsilon = -1$ or $= +1$, the centre of the centro-affine plane and the curve $\mathfrak{K}$ lie on the same or different sides of each tangent to the curve.

4.5 Centro-affine curvature

Let $t_0 \in j$, $T_0 \in J$ be two arbitrary homologous values. Then at two homologous points $t \in j$, $T \in J$ we have (from (7))

$$\operatorname{sgn}(xx') \int_{t_0}^{t} \sqrt{|b(\sigma)|}\, d\sigma = \operatorname{sgn}(X\dot{X}) \int_{T_0}^{T} \sqrt{|B(\sigma)|}\, d\sigma \tag{4.8}$$

and moreover

$$\frac{\operatorname{sgn}(xx')}{\sqrt{|b(t)|}}\left[a(t) - \frac{1}{2}\frac{b'(t)}{b(t)}\right] = \frac{\operatorname{sgn}(X\dot{X})}{\sqrt{|B(T)|}}\left[A(T) - \frac{1}{2}\frac{\dot{B}(T)}{B(T)}\right]. \tag{4.9}$$

Now we consider the following functions, defined in the interval j,

$$s(t) = \operatorname{sgn}(xx') \int_{t_0}^{t} \sqrt{|b(\sigma)|}\, d\sigma, \qquad k(t) = \frac{\operatorname{sgn}(xx')}{\sqrt{|b(t)|}}\left[a(t) - \frac{1}{2}\frac{b'(t)}{b(t)}\right]. \tag{4.10}$$

By (8), (9) it follows that each of the functions $s(t)$, $k(t)$ takes the same value at two homologous points t, T in an arbitrary representation $X(T)$ of $\mathfrak{K}$. For an arbitrary choice of a representative $\bar{x}(t) = Cx(t)$ of $x(t)$, it obviously takes the same value at every point t or takes the same value with opposite sign, according as sgn $|C| = 1$ or -1. It follows that the functions $s(t)$, $k(t)$ are relative invariants of the curve $\mathfrak{K}$. The function s is called the *centro-affine oriented arc-length* of the curve $\mathfrak{K}$; its value $s(t)$ gives the length of the centro-affine oriented arc from the point $x(t_0)$ to the point $x(t)$. The function k is called the *centro-affine curvature* of the curve $\mathfrak{K}$; its value $k(t)$ gives the centro-affine curvature of $\mathfrak{K}$ at the point $x(t)$.

From (5) we have

$$k(t) = \frac{\operatorname{sgn}(xx')}{2}\sqrt{\frac{|(xx')|}{|(x'x'')|}}\left[3\,\frac{(xx'')}{(xx')} - \frac{(x'x''')}{(x'x'')}\right]. \tag{4.11}$$

The function $s(t)$, $t \in j$, belongs to the class C_1 or to a higher class, and its derivative s' is always non-zero. It maps the interval j onto an interval i including zero, i.e. $0 \in i$.

Let $Y(s)$, $s \in i$ be the representation of the curve $\mathfrak{K}$ arising from the parametric transformation $s = s(t)$. Then at two homologous points $s \in i$, $t \in j$ we have

$$A(s) = k(t), \qquad B(s) = \varepsilon.$$

The value $A(s)$ $(= K(s))$ of the first semi-invariant A at each point $s \in i$ gives the centro-affine curvature of the curve $\mathfrak{K}$ at the point $Y(s)$, while the second semi-invariant B takes the constant value ε $(= \pm 1)$. The representation $Y(s)$ of $\mathfrak{K}$ satisfies the linear differential equation of the second order

$$\ddot{Y} = K(s)\dot{Y} + \varepsilon Y. \tag{4.12}$$

If we write $\dot{Y}(s) = Z(s)$, then this differential equation can be replaced by the system of (vector) differential equations of the first order

$$\left.\begin{aligned} \dot{Y} &= \qquad\quad\;\; Z, \\ \dot{Z} &= \varepsilon Y + K(s)Z. \end{aligned}\right\} \tag{4.13}$$

These differential equations (13) are called the *Serret-Frenet formulae* of centro-affine plane curve theory, and the curves with representations $\dot{Y}(s)$ and $\ddot{Y}(s)$, $s \in i$, are designated the *tangent* and *curvature curves*, respectively, of $\mathfrak{K}$.

4.6 Application of the above theory to integral curves of the differential equation (q)

We consider a differential equation (q) in which we specify that $q \in C_1$. Let (u, v) be a basis of the differential equation (q) and $\mathfrak{K}$ the integral curve of (q) determined by this basis. The curve $\mathfrak{K}$ therefore admits of the representation $x(t) = [u(t), v(t)]$, $t \in j$. The various integral curves of (q) naturally constitute the representatives of the curve $\mathfrak{K}$. Obviously we have

$$(xx') = w\ (\neq 0); \qquad (xx'') = 0, \qquad (x'x'') = -qw, \qquad (x'x''') = -q'w.$$

The curve $\mathfrak{K}$ is then regular if and only if $q(t) \neq 0$ for $t \in j$.

Now we assume that the curve $\mathfrak{K}$ is regular, so that $q(t) \neq 0$ for $t \in j$. In the two cases $q < 0$ and $q > 0$ the centre of the centro-affine plane and the curve $\mathfrak{K}$ lie respectively on the same and on opposite sides of each tangent to the curve. The centro-affine oriented arc of the curve $\mathfrak{K}$ and its centro-affine curvature are given by the formulae

$$\left.\begin{aligned} s(t) &= \operatorname{sgn} w \cdot \int_{t_0}^{t} \sqrt{|q(\sigma)|}\, d\sigma, \\ k(t) &= \operatorname{sgn} w \left(\frac{1}{\sqrt{|q(t)|}}\right)', \end{aligned}\right\} \tag{4.14}$$

The carrier q can be expressed in terms of the functions s, k as follows: $q(t) = \operatorname{sgn} q(t_0) \cdot s'^2(t)$,

$$q(t) = \frac{q(t_0)}{\left[1 + \operatorname{sgn} w \cdot \sqrt{|q(t_0)|} \int_{t_0}^{t} k(\sigma)\, d\sigma\right]^2}. \tag{4.15}$$

The representation $Y(s)$ of the curve $\mathfrak{K}$ given by the parameter transformation $s = s(t)$ satisfies a linear differential equation of the second order of the form (12) in which $\varepsilon = \operatorname{sgn} q$.

Let $Q(s)$ be the function which the carrier $q(t)$ of (q) becomes on transforming to the variable s. The function Q is therefore defined by the relation $Q(s) = q(t)$ at every two homologous points $s \in i$, $t \in j$.

From (14) we have

$$\dot{Q}(s) = \operatorname{sgn} w \cdot \frac{q'(t)}{\sqrt{|q(t)|}},$$

and hence

$$K(s) = -\frac{1}{2} \cdot \frac{\dot{Q}(s)}{Q(s)}.$$

We see that if the centro-affine oriented arc of the curve $\mathfrak{K}$ is chosen as parameter for the representation of $\mathfrak{K}$, then the centro-affine curvature K of this curve is given by the formula

$$K(s) = \frac{d}{ds} \log \frac{1}{\sqrt{|Q(s)|}}. \tag{4.16}$$

From (16) it follows that, at every two homologous points $t \in j$, $s \in i$,

$$q(t) = q(t_0) \exp\left(-2 \int_0^s K(\sigma)\, d\sigma\right). \tag{4.17}$$