4. Suppose $\mathfrak{A} \supset \mathfrak{B}$ are subgroups of $\mathfrak{G}$. Consider arbitrary left and right cosets $\bar{a}_l$, $\bar{c}_l$ and $\bar{a}_r$, $\bar{c}_r$ with regard to $\mathfrak{A}$, respectively, and denote:

$$\bar{A}_l = \bar{a}_l \sqcap \mathfrak{G}/_l\mathfrak{B} \; (= \bar{a}_l \sqsubset \mathfrak{G}/_l\mathfrak{B}), \quad \bar{C}_l = \bar{c}_l \sqcap \mathfrak{G}/_l\mathfrak{B} \; (= \bar{c}_l \sqsubset \mathfrak{G}/_l\mathfrak{B}),$$
$$\bar{A}_r = \bar{a}_r \sqcap \mathfrak{G}/_r\mathfrak{B} \; (= \bar{a}_r \sqsubset \mathfrak{G}/_r\mathfrak{B}), \quad \bar{C}_r = \bar{c}_r \sqcap \mathfrak{G}/_r\mathfrak{B} \; (= \bar{c}_r \sqsubset \mathfrak{G}/_r\mathfrak{B}).$$

Each element of the decompositions $\bar{A}_l, \bar{C}_l$ or $\bar{A}_r, \bar{C}_r$ is a left or a right coset with regard to $\mathfrak{B}$, respectively. Moreover, there holds: $\bar{A}_l \simeq \bar{C}_l$, $\bar{A}_r \simeq \bar{C}_r$.

5. Let $\mathfrak{A} \supset \mathfrak{B}$ be subgroups of $\mathfrak{G}$. Consider arbitrary cosets $\bar{a} \in \mathfrak{G}/_l\mathfrak{A}$, $\bar{a}^{-1} \in \mathfrak{G}/_r\mathfrak{A}$ inverse of each other and, on the latter, the decompositions set out below:

$$\bar{A}_l = \bar{a} \sqcap \mathfrak{G}/_l\mathfrak{B} \; (= \bar{a} \sqsubset \mathfrak{G}/_l\mathfrak{B}), \quad \bar{A}_r = \bar{a}^{-1} \sqcap \mathfrak{G}/_r\mathfrak{B} \; (= \bar{a}^{-1} \sqsubset \mathfrak{G}/_r\mathfrak{B}).$$

Either of the decompositions $\bar{A}_l$, $\bar{A}_r$ is, under the extended inversion $\boldsymbol{n}$ of $\mathfrak{G}$, mapped onto the other. $\bar{A}_l$, $\bar{A}_r$ are equivalent sets, hence: $\bar{A}_l \simeq \bar{A}_r$.

6. If $\bar{A}_l$ and $\bar{C}_r$ are the same as in exercise 4, there holds $\bar{A}_l \simeq \bar{C}_r$.

7. Let $p \in \mathfrak{G}$ denote an arbitrary element and $\mathring{\mathfrak{G}}$ the p-group associated with $\mathfrak{G}$ (19.7.11). Moreover, let $\mathfrak{A} \subset \mathfrak{G}$ be a subgroup of $\mathfrak{G}$ and $\mathring{\mathfrak{A}}_l \subset \mathring{\mathfrak{G}}$ ($\mathring{\mathfrak{A}}_r \subset \mathring{\mathfrak{G}}$) the subgroup of $\mathring{\mathfrak{G}}$ on the field $p\mathfrak{A}$ ($\mathfrak{A}p$) (20.3.3). Show that the left (right) decomposition of the group $\mathring{\mathfrak{G}}$ with regard to the subgroup $\mathring{\mathfrak{A}}_l$ ($\mathring{\mathfrak{A}}_r$) coincides with the left (right) decomposition of $\mathfrak{G}$ with regard to $\mathfrak{A}$, that is to say:

$$\mathring{\mathfrak{G}}/_l\mathring{\mathfrak{A}}_l = \mathfrak{G}/_l\mathfrak{A}, \quad \mathring{\mathfrak{G}}/_r\mathring{\mathfrak{A}}_r = \mathfrak{G}/_r\mathfrak{A}.$$

22. Consequences of the properties of decompositions generated by subgroups

22.1. Lagrange's theorem

Assuming $\mathfrak{A} \subset \mathfrak{G}$ to be an arbitrary subgroup of $\mathfrak{G}$, we shall now consider the consequences of the properties of the decompositions $\mathfrak{G}/_l\mathfrak{A}$ and $\mathfrak{G}/_r\mathfrak{A}$.

Suppose $\mathfrak{G}$ is finite.

Let us denote by N and n the order of $\mathfrak{G}$ and $\mathfrak{A}$, respectively, so that N is the number of the elements of G and n the number of the elements of $\mathfrak{A}$. One of the elements of $\mathfrak{G}/_l\mathfrak{A}$ is the field A of $\mathfrak{A}$. This element therefore consists of n elements of $\mathfrak{G}$ and, consequently (by 20.2.5), each element of $\mathfrak{G}/_l\mathfrak{A}$ consists of n elements of $\mathfrak{G}$. Hence $N = qn$, q denoting the number of the elements of $\mathfrak{G}/_l\mathfrak{A}$. Thus we have arrived at the following result:

The order of each subgroup $\mathfrak{A}$ of an arbitrary finite group $\mathfrak{G}$ is a divisor of the order of $\mathfrak{G}$.

This is *Lagrange's theorem*, considered to be one of the most important in the theory of finite groups. The number q, i.e., the number of the elements of the decomposition $\mathfrak{G}/_l\mathfrak{A}$ and, at the same time, even the quotient of N and n is called the *index of* $\mathfrak{A}$ *in* $\mathfrak{G}$. Since $\mathfrak{G}/_l\mathfrak{A}$ and $\mathfrak{G}/_r\mathfrak{A}$ are equivalent sets, the index of $\mathfrak{A}$ in $\mathfrak{G}$ simultaneously indicates the number of the elements of $\mathfrak{G}/_r\mathfrak{A}$. According to Lagrange's theorem, e.g., an arbitrary finite group whose order is a prime number does not contain any proper subgroup different from the least subgroup.

Lagrange's theorem applies even if $\mathfrak{G}$ is infinite ($N = 0$).

Example. Consider the group $\mathfrak{S}_3$ whose elements are denoted by $\underline{1}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{f}$, as in 11.4. From the multiplication table of the group $\mathfrak{S}_3$ (11.4) we see that the elements $\underline{1}, \boldsymbol{f}$ form a subgroup of $\mathfrak{S}_3$. Let us denote it by $\mathfrak{A}$.

The left cosets of the individual elements with respect to $\mathfrak{A}$ are:

$$\underline{1}\mathfrak{A} = \boldsymbol{f}\mathfrak{A} = \{\underline{1}, \boldsymbol{f}\}, \quad \boldsymbol{a}\mathfrak{A} = \boldsymbol{c}\mathfrak{A} = \{\boldsymbol{a}, \boldsymbol{c}\}, \quad \boldsymbol{b}\mathfrak{A} = \boldsymbol{d}\mathfrak{A} = \{\boldsymbol{b}, \boldsymbol{d}\}.$$

The right cosets are:

$$\mathfrak{A}\underline{1} = \mathfrak{A}\boldsymbol{f} = \{\underline{1}, \boldsymbol{f}\}, \quad \mathfrak{A}\boldsymbol{a} = \mathfrak{A}\boldsymbol{d} = \{\boldsymbol{a}, \boldsymbol{d}\}, \quad \mathfrak{A}\boldsymbol{b} = \mathfrak{A}\boldsymbol{c} = \{\boldsymbol{b}, \boldsymbol{c}\}.$$

The left decomposition of the group $\mathfrak{S}_3$, generated by $\mathfrak{A}$, therefore consists of the elements $\{\underline{1}, \boldsymbol{f}\}$, $\{\boldsymbol{a}, \boldsymbol{c}\}$, $\{\boldsymbol{b}, \boldsymbol{d}\}$, whereas the right decomposition comprises the elements $\{\underline{1}, \boldsymbol{f}\}$, $\{\boldsymbol{a}, \boldsymbol{d}\}$, $\{\boldsymbol{b}, \boldsymbol{c}\}$. Note that these two decompositions are different. The order of $\mathfrak{S}_3$ is 6, the order of $\mathfrak{A}$ is 2, the index of $\mathfrak{A}$ in $\mathfrak{S}_3$ is $6 : 2 = 3 =$ the number of the elements of the left and, simultaneously, even of the right decomposition of $\mathfrak{S}_3$ generated by $\mathfrak{A}$.

22.2. Relations between interchangeable subgroups

The result arrived at in 21.6 and the properties of complementary decompositions (5) lead to a number of consequences as regards interchangeable subgroups. We shall restrict our attention to a few of them and leave further initiative to the reader. The formulae we shall obtain can mostly be verified directly. Owing to our method we can not only find them but even get a better understanding of their structure.

1. *Let* $\mathfrak{A} \supset \mathfrak{B}, \mathfrak{D}$ *be arbitrary subgroups of* $\mathfrak{G}$ *and suppose* $\mathfrak{B}$ *and* $\mathfrak{D}$ *are interchangeable. Then even the subgroups* $\mathfrak{B}, \mathfrak{A} \cap \mathfrak{D}$ *are interchangeable and there holds*

$$\mathfrak{A} \cap \mathfrak{D}\mathfrak{B} = (\mathfrak{A} \cap \mathfrak{D})\mathfrak{B}. \tag{1}$$

In fact, by 21.6, the decompositions $\mathfrak{G}/_l\mathfrak{B}$, $\mathfrak{G}/_l\mathfrak{D}$ are complementary. Since $\mathfrak{A} \supset \mathfrak{B}$, we have $\mathfrak{G}/_l\mathfrak{A} \geqq \mathfrak{G}/_l\mathfrak{B}$ (21.3). In accordance with 5.3, $\mathfrak{G}/_l\mathfrak{B}$ is complementary to $(\mathfrak{G}/_l\mathfrak{A}, \mathfrak{G}/_l\mathfrak{D})$ and, by 21.4, there holds $(\mathfrak{G}/_l\mathfrak{A}, \mathfrak{G}/_l\mathfrak{D}) \coloneq \mathfrak{G}/_l(\mathfrak{A} \cap \mathfrak{D})$. We observe that the decompositions $\mathfrak{G}/_l\mathfrak{B}$, $\mathfrak{G}/_l(\mathfrak{A} \cap \mathfrak{D})$ are complementary; from 21.6 we conclude that the subgroups $\mathfrak{B}$, $\mathfrak{A} \cap \mathfrak{D}$ are interchangeable.

By 5.4, $\mathfrak{G}/_l\mathfrak{D}$ is modular with regard to $\mathfrak{G}/_l\mathfrak{A}$, $\mathfrak{G}/_l\mathfrak{B}$ and so

$$(\mathfrak{G}/_l\mathfrak{A}, [\mathfrak{G}/_l\mathfrak{B}, \mathfrak{G}/_l\mathfrak{D}]) = [\mathfrak{G}/_l\mathfrak{B}, (\mathfrak{G}/_l\mathfrak{A}, \mathfrak{G}/_l\mathfrak{D})].$$

Hence, on taking account of 21.4 and 21.5, there follows

$$(\mathfrak{G}/_l\mathfrak{A}, \mathfrak{G}/_l\mathfrak{D}\mathfrak{B}) = \left[\mathfrak{G}_l\mathfrak{B}, \mathfrak{G}/_l(\mathfrak{A} \cap \mathfrak{D})\right]$$

as well as

$$\mathfrak{G}/_l(\mathfrak{A} \cap \mathfrak{D}\mathfrak{B}) = \mathfrak{G}/_l(\mathfrak{A} \cap \mathfrak{D})\mathfrak{B}.$$

It is easy to see that the element of this decomposition, containing the unit $\underline{1}$ of $\mathfrak{G}$, is the field of the subgroup $\mathfrak{A} \cap \mathfrak{D}\mathfrak{B}$ and, at the same time, the field of the subgroup $(\mathfrak{A} \cap \mathfrak{D})\mathfrak{B}$. The above formula is therefore correct.

2. *Let* $\mathfrak{A} \supset \mathfrak{B}$, $\mathfrak{C} \supset \mathfrak{D}$ *be arbitrary subgroups of* $\mathfrak{G}$ *and suppose* $\mathfrak{B}$, $\mathfrak{D}$ *are interchangeable. Then* $\mathfrak{B}$, $\mathfrak{A} \cap \mathfrak{D}$ *and* $\mathfrak{D}$, $\mathfrak{C} \cap \mathfrak{B}$ *are interchangeable as well. Simultaneously, even* $\mathfrak{A} \cap \mathfrak{D}$, $\mathfrak{C} \cap \mathfrak{B}$ *have the same property and there holds*

$$\mathfrak{A} \cap \mathfrak{C} \cap \mathfrak{D}\mathfrak{B} = (\mathfrak{A} \cap \mathfrak{D})\,(\mathfrak{C} \cap \mathfrak{B}). \tag{2}$$

Indeed, the first part of this statement immediately follows from the above theorem. Moreover, there holds (by 5.6.1)

$$\big((\mathfrak{G}/_l\mathfrak{A}, \mathfrak{G}/_l\mathfrak{C}), [\mathfrak{G}/_l\mathfrak{B}, \mathfrak{G}/_l\mathfrak{D}]\big) = [(\mathfrak{G}/_l\mathfrak{A}, \mathfrak{G}/_l\mathfrak{D}), (\mathfrak{G}/_l\mathfrak{C}, \mathfrak{G}/_l\mathfrak{B})] \tag{3}$$

and the decompositions $(\mathfrak{G}/_l\mathfrak{A}, \mathfrak{G}/_l\mathfrak{D})$, $(\mathfrak{G}/_l\mathfrak{C}, \mathfrak{G}/_l\mathfrak{B})$, i.e., $\mathfrak{G}/_l(\mathfrak{A} \cap \mathfrak{D})$, $\mathfrak{G}/_l(\mathfrak{C} \cap \mathfrak{B})$ are complementary. Consequently (21.6), the subgroups $\mathfrak{A} \cap \mathfrak{D}$, $\mathfrak{C} \cap \mathfrak{B}$ are interchangeable and so (3) yields (2).

3. *In the situation described by Theorem* 2 *there also holds*:

$$(\mathfrak{A} \cap \mathfrak{D})\mathfrak{B} \cap \mathfrak{C} = (\mathfrak{C} \cap \mathfrak{B})\mathfrak{D} \cap \mathfrak{A} = (\mathfrak{A} \cap \mathfrak{D})(\mathfrak{C} \cap \mathfrak{B}). \tag{4}$$

We know that $\mathfrak{G}/_l\mathfrak{B}$ and $\mathfrak{G}/_l\mathfrak{D}$ are complementary; moreover, there holds $\mathfrak{G}/_l\mathfrak{A} \geqq \mathfrak{G}/_l\mathfrak{B}$, $\mathfrak{G}/_l\mathfrak{C} \geqq \mathfrak{G}/_l\mathfrak{D}$. Note that the fields of the subgroups $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ are elements of the corresponding left decompositions and contain the unit $\underline{1}$ of $\mathfrak{G}$.

Let us now use the result of 5.5 by which the decompositions

$$\mathfrak{A}/_l\mathfrak{B} = \mathfrak{A} \sqsubset \mathfrak{G}/_l\mathfrak{B}\ (= \mathfrak{G}/_l\mathfrak{B} \sqcap \mathfrak{A}),$$

$$\mathfrak{C}/_l\mathfrak{D} = \mathfrak{C} \sqsubset \mathfrak{G}/_l\mathfrak{D}\ (= \mathfrak{G}/_l\mathfrak{D} \sqcap \mathfrak{C})$$

are adjoint with respect to $\mathfrak{B}$, $\mathfrak{D}$. Thus there holds

$$\mathbf{s}(\mathfrak{D} \sqsubset \mathfrak{A}/_l\mathfrak{B} \sqcap \mathfrak{C}) = \mathbf{s}(\mathfrak{B} \sqsubset \mathfrak{C}/_l\mathfrak{D} \sqcap \mathfrak{A}). \tag{5}$$

By 2.6.5 a) we have

$$\mathfrak{D} \sqsubset \mathfrak{A}/_l\mathfrak{B} \sqcap \mathfrak{C} = (\mathfrak{D} \sqsubset \mathfrak{A}/_l\mathfrak{B}) \sqcap \mathfrak{C} = \mathfrak{D} \sqsubset (\mathfrak{A}/_l\mathfrak{B} \sqcap \mathfrak{C}),$$
$$\mathfrak{B} \sqsubset \mathfrak{C}/_l\mathfrak{D} \sqcap \mathfrak{A} = (\mathfrak{B} \sqsubset \mathfrak{C}/_l\mathfrak{D}) \sqcap \mathfrak{A} = \mathfrak{B} \sqsubset (\mathfrak{C}/_l\mathfrak{D} \sqcap \mathfrak{A})$$

and the results from 21.2.1 yield the formulas

$$(\mathfrak{D} \sqsubset \mathfrak{A}/_l\mathfrak{B}) \sqcap \mathfrak{C} = \big((\mathfrak{A} \cap \mathfrak{D})\mathfrak{B} \cap \mathfrak{C}\big)/_l(\mathfrak{C} \cap \mathfrak{B}),$$
$$\mathfrak{D} \sqsubset (\mathfrak{A}/_l\mathfrak{B} \sqcap \mathfrak{C}) = (\mathfrak{A} \cap \mathfrak{D})(\mathfrak{C} \cap \mathfrak{B})/_l(\mathfrak{C} \cap \mathfrak{B}),$$
$$(\mathfrak{B} \sqsubset \mathfrak{C}/_l\mathfrak{D}) \sqcap \mathfrak{A} = \big((\mathfrak{C} \cap \mathfrak{B})\mathfrak{D} \cap \mathfrak{A}\big)/_l(\mathfrak{A} \cap \mathfrak{D}),$$
$$\mathfrak{B} \sqsubset (\mathfrak{C}/_l\mathfrak{D} \sqcap \mathfrak{A}) = (\mathfrak{C} \cap \mathfrak{B})(\mathfrak{A} \cap \mathfrak{D})/_l(\mathfrak{A} \cap \mathfrak{D}).$$

So we have

$$\mathbf{s}(\mathfrak{D} \sqsubset \mathfrak{A}/_l\mathfrak{B} \sqcap \mathfrak{C}) = (\mathfrak{A} \cap \mathfrak{D})\mathfrak{B} \cap \mathfrak{C} = (\mathfrak{A} \cap \mathfrak{D})(\mathfrak{C} \cap \mathfrak{B}),$$
$$\mathbf{s}(\mathfrak{B} \sqsubset \mathfrak{C}/_l\mathfrak{D} \sqcap \mathfrak{A}) = (\mathfrak{C} \cap \mathfrak{B})\mathfrak{D} \cap \mathfrak{A} = (\mathfrak{C} \cap \mathfrak{B})(\mathfrak{A} \cap \mathfrak{D})$$

which, together with (5), yield the formulas (4).

22.3. Modular lattices of subgroups and of decompositions generated by subgroups

Consider an arbitrary nonempty system O of subgroups of the group $\mathfrak{G}$. Assume every two subgroups of the system O to be interchangeable and O to be closed with regard to the intersections and the products of the pairs of subgroups: the intersection and the product of any pair of subgroups $\mathfrak{A}$, $\mathfrak{B} \in O$ also belong to O, hence $\mathfrak{A} \cap \mathfrak{B}$, $\mathfrak{A}\mathfrak{B} \in O$.

Let us associate, with every two-membered sequence of subgroups $\mathfrak{A}$, $\mathfrak{B} \in O$, first, the intersection $\mathfrak{A} \cap \mathfrak{B}$ and, next, the product $\mathfrak{A}\mathfrak{B}$ of $\mathfrak{A}$ and $\mathfrak{B}$. Thus we have defined two multiplications in the system O, hence a pair of groupoids on the field O. Each of the two groupoids is Abelian (1.6), associative (1.10.4; 18.1.1) and all its elements are idempotent (1.10.1; 15.6.4). Moreover, the multiplications in both groupoids are connected by the formulae:

$$\mathfrak{A}(\mathfrak{A} \cap \mathfrak{B}) = \mathfrak{A}, \quad \mathfrak{A} \cap \mathfrak{A}\mathfrak{B} = \mathfrak{A}.$$

It follows that the above pair of groupoids is a lattice, Ω.

Let us now choose the upper (lower) multiplication in the lattice Ω in the manner that, to every two-membered sequence of subgroups $\mathfrak{A}$, $\mathfrak{B} \in \Omega$, there corresponds their product $\mathfrak{A}\mathfrak{B}$ (intersection $\mathfrak{A} \cap \mathfrak{B}$):

$$\mathfrak{A} \cup \mathfrak{B} = \mathfrak{A}\mathfrak{B}, \quad \mathfrak{A} \cap \mathfrak{B} = \mathfrak{A} \cap \mathfrak{B}.$$

Then we obtain the upper (lower) partial ordering $\boldsymbol{u}$ ($\boldsymbol{l}$) of Ω by associating, with each $\mathfrak{A} \in \Omega$, all its supergroups (subgroups) $\mathfrak{B} \in \Omega$. From 22.2.1 it is evident that every three-membered sequence of subgroups $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C} \in \Omega$ for which $\mathfrak{A} \leqq \mathfrak{C}$ ($\boldsymbol{u}$), satisfies the upper modular relation. Consequently, Ω is modular.

Let us, furthermore, associate with every subgroup $\mathfrak{A} \in \Omega$ the decomposition $\mathfrak{G}/_l\mathfrak{A}$ and denote the corresponding system of the left decompositions of $\mathfrak{G}$ by the symbol O^*. Considering 21.4, 21.5, we realize that the system O^* is closed with respect to the operations ( ), [ ] and therefore includes, with every pair of left decompositions $\mathfrak{G}/_l\mathfrak{A}$, $\mathfrak{G}/_l\mathfrak{B} \in O^*$, even their greatest common refinement and their least common covering:

$$(\mathfrak{G}/_l\mathfrak{A}, \mathfrak{G}/_l\mathfrak{B}), [\mathfrak{G}/_l\mathfrak{A}, \mathfrak{G}/_l\mathfrak{B}] \in O^*.$$

Two multiplications in O^* may be defined by associating, with each two-membered sequence of the left decompositions $\mathfrak{G}/_l\mathfrak{A}$, $\mathfrak{G}/_l\mathfrak{B} \in O^*$, first, the greatest common refinement and, next, the least common covering of these decompositions. Thus we obtain, on O^*, a pair of groupoids Ω^* which, as it can again be verified, is a lattice.

The function $\boldsymbol{i}$, associating with each subgroup $\mathfrak{A} \in \Omega$ the left decomposition $\mathfrak{G}/_l\mathfrak{A} \in \Omega^*$, is clearly a simple mapping of Ω onto Ω^* such that for every $\mathfrak{A}$, $\mathfrak{B} \in \Omega$ there holds

$$\boldsymbol{i}(\mathfrak{A} \cap \mathfrak{B}) = \mathfrak{G}/_l(\mathfrak{A} \cap \mathfrak{B}), \quad \boldsymbol{i}\mathfrak{A}\mathfrak{B} = \mathfrak{G}/_l\mathfrak{A}\mathfrak{B},$$

i.e.,

$$\boldsymbol{i}(\mathfrak{A} \cap \mathfrak{B}) = \boldsymbol{i}\mathfrak{A} \cap \boldsymbol{i}\mathfrak{B}, \quad \boldsymbol{i}(\mathfrak{A} \cup \mathfrak{B}) = \boldsymbol{i}\mathfrak{A} \cup \boldsymbol{i}\mathfrak{B}.$$

The mapping $\boldsymbol{i}$ is therefore an isomorphism of Ω onto Ω^*. Since Ω is modular, Ω^* is modular as well (18.7.14).

The result:

A nonempty system of subgroups of $\mathfrak{G}$ any two elements of which are interchangeable and which is closed with respect to the intersections and the products of any two subgroups forms — together with the multiplications defined by the forming of the intersections and the products — a modular lattice. The system of the left (right) decompositions of $\mathfrak{G}$, generated by the individual elements of this lattice is, with respect to the operations ( ), [ ], closed and forms — with the multiplications defined by these operations — also a modular lattice which is isomorphic with the former.

22.4. Exercises

1. The order of any group consisting of permutations of a finite set of order n is a divisor of $n!$.
2. In every finite Abelian group of order N the number of elements inverse of themselves is a divisor of N.

23. Special decompositions of groups, generated by subgroups

23.1. Semi-coupled and coupled left decompositions

Consider the subgroups $\mathfrak{A} \supset \mathfrak{B}, \mathfrak{C} \supset \mathfrak{D}$ of $\mathfrak{G}$. Their fields are denoted by A, B, C, D.

We first ask under what conditions the left decompositions $\mathfrak{A}/_l\mathfrak{B}, \mathfrak{C}/_l\mathfrak{D}$ are semi-coupled or coupled.

Since the intersection $A \cap B$ contains the unit of $\mathfrak{G}$ and therefore is not empty, it is obvious, with respect to 4.1, that the mentioned decompositions are semi-coupled if and only if

$$\mathfrak{A}/_l\mathfrak{B} \sqcap \mathfrak{C} = \mathfrak{C}/_l\mathfrak{D} \sqcap \mathfrak{A}.$$

In accordance with 21.2.1, this may be written

$$(\mathfrak{A} \cap \mathfrak{C})/_l\,(\mathfrak{C} \cap \mathfrak{B}) = (\mathfrak{A} \cap \mathfrak{C})/_l\,(\mathfrak{A} \cap \mathfrak{D}).$$

This equality is evidently true if and only if

$$\mathfrak{A} \cap \mathfrak{D} = \mathfrak{C} \cap \mathfrak{B}. \tag{1}$$

Thus we have verified that *the left decompositions* $\mathfrak{A}/_l\mathfrak{B}, \mathfrak{C}/_l\mathfrak{D}$ *are semi-coupled if and only if the subgroups* $\mathfrak{A} \cap \mathfrak{D}, \mathfrak{C} \cap \mathfrak{B}$ *coincide, i.e., if* $\mathfrak{A} \cap \mathfrak{D} = \mathfrak{C} \cap \mathfrak{B}$.

Now suppose the left decompositions $\mathfrak{A}/_l\mathfrak{B}$, $\mathfrak{C}/_l\mathfrak{D}$ are coupled. Then (by 4.1; 20.3.2) we have, besides (1), even:

$$A = (A \cap C)B, \quad C = (C \cap A)D,$$

from which it follows (19.7.8) that $\mathfrak{A} \cap \mathfrak{C}$ is interchangeable with both $\mathfrak{B}$ and $\mathfrak{D}$ and so:

$$\mathfrak{A} = (\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}, \quad \mathfrak{C} = (\mathfrak{C} \cap \mathfrak{A})\,\mathfrak{D}. \tag{2}$$

Conversely, if (1) and (2) simultaneously apply, then with respect to 4.1 and 21.2.1, the left decompositions $\mathfrak{A}/_l\mathfrak{B}$, $\mathfrak{C}/_l\mathfrak{D}$ are coupled.