15. Factoroids

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ciative; b) the decomposition of $\mathfrak{G}$, the elements of which are the sets of all the numbers in $\mathfrak{G}$ expressed, in the decimal system, by symbols containing the same number of digits, is generating.

3. The groupoid $\mathfrak{G}$, whose field is an arbitrary set and the multiplication given by $ab = a$ ($ab = b$) for $a, b \in \mathfrak{G}$, is associative and all its decompositions are generating.

15. Factoroids

The notion of a factoroid we shall now be concerned with plays an important part throughout the following theory.

15.1. Basic concepts

Let again $\bar{A}$ denote an arbitrary generating decomposition in $\mathfrak{G}$. With $\bar{A}$ we can uniquely associate a groupoid denoted $\bar{\mathfrak{A}}$ and defined as follows: The field of $\bar{\mathfrak{A}}$ is the decomposition $\bar{A}$ and the multiplication is defined in the following way: the product of any element $\bar{a} \in \bar{A}$ and any element $\bar{b} \in \bar{A}$ is the element $\bar{c} \in \bar{A}$ for which $\bar{a}\bar{b} \subset \bar{c}$. Then we generally write

$$\bar{a} \circ \bar{b} = \bar{c},$$

and we have $\bar{a}\bar{b} \subset \bar{a} \circ \bar{b} \in \bar{\mathfrak{A}}$. We employ the symbol $\circ$ to denote the products in $\bar{\mathfrak{A}}$ in the same way as we use the symbol . to denote the products in $\mathfrak{G}$.

$\bar{\mathfrak{A}}$ is called a *factoroid in* $\mathfrak{G}$; if $\bar{A}$ is *on* $\mathfrak{G}$, then it is a *factoroid on* $\mathfrak{G}$ or a *factoroid of* $\mathfrak{G}$. Every generating decomposition in $\mathfrak{G}$ uniquely determines a certain factoroid in $\mathfrak{G}$, namely the one whose field it is; we say that to every generating decomposition in $\mathfrak{G}$ there *corresponds* or *belongs* a certain factoroid in $\mathfrak{G}$.

Note that on $\mathfrak{G}$ there exist at least two factoroids, namely the so-called *greatest factoroid*, $\bar{\mathfrak{G}}_{\max}$, belonging to the greatest generating decomposition $\bar{G}_{\max}$ and the *least factoroid*, $\bar{\mathfrak{G}}_{\min}$, belonging to the least generating decomposition $\bar{G}_{\min}$ of the groupoid $\mathfrak{G}$. These extreme factoroids on $\mathfrak{G}$ are either different from each other or coincide according as $\mathfrak{G}$ contains more than one or precisely one element.

15.2. Example of a factoroid

Consider, for example, the groupoid $\mathfrak{Z}$. Let n be an arbitrary positive integer and $\bar{a}_i$, where i runs over the numbers $0, \dots, n-1$, stand for the set of all the elements of $\mathfrak{Z}$ that, in the division by n, leave the remainder i. The sets $\bar{a}_0, \dots, \bar{a}_{n-1}$ are:

$$\begin{array}{llllllll}
\bar{a}_0 & = \{\dots, -2n, & -n, & 0, & n, & 2n, & \dots\}, \\
\bar{a}_1 & = \{\dots, -2n+1, & -n+1, & 1, & n+1, & 2n+1, & \dots\}, \\
\bar{a}_2 & = \{\dots, -2n+2, & -n+2, & 2, & n+2, & 2n+2, & \dots\}, \\
\hline
\bar{a}_{n-1} & = \{\dots, -2n+(n-1), & -n+(n-1), & n-1, & n+(n-1), & 2n+(n-1), & \dots\}.
\end{array}$$

We see that the system $\{\bar{a}_0, \dots, \bar{a}_{n-1}\}$ is a decomposition of $\mathfrak{Z}$; let us denote it $\bar{Z}_n$ and show that it is generating. To that purpose we shall verify that the product $\bar{a}_i\bar{a}_j$ of an element $\bar{a}_i \in \bar{Z}_n$ and an element $\bar{a}_j \in \bar{Z}_n$ is a part of an element $\bar{a}_k \in \bar{Z}_n$. By its definition, the set $\bar{a}_i \,.\, \bar{a}_j$ consists of the products $a \,.\, b$ where a and b run over all the elements of $\bar{a}_i$ and $\bar{a}_j$, respectively. Now let a be an element of $\bar{a}_i$ so that the remainder in the division of a by n is i, and let b denote an element of $\bar{a}_j$ so that the remainder in the division of b by n is j. By the definition of the multiplication in $\mathfrak{Z}$, we have $a \,.\, b = a + b \in \bar{a}_k$ where k is the remainder in the division of $i + j$ by n because both $a + b$ and $i + j$ leave, in the division by n, the same remainder. So we have $\bar{a}_i\bar{a}_j \subset \bar{a}_k$, hence $\bar{Z}_n$ is generating. The corresponding factoroid $\bar{\mathfrak{Z}}_n$ therefore consists of n elements: $\bar{a}_0, \dots, \bar{a}_{n-1}$ and its multiplication is defined by the rule that the product $\bar{a}_i \,.\, \bar{a}_j$ is the element $\bar{a}_k$ where k is the remainder in the division of $i + j$ by n. Obviously $\bar{\mathfrak{Z}}_1$ is the greatest factoroid on $\mathfrak{Z}$.

15.3. Factoroids in groupoids

Before proceeding with our study, let us remember that we apply, to groupoids, all the concepts, symbols and results defined for their fields and multiplication. The same holds for factoroids. The most important concepts, symbols and results arrived at in this way are:

1. *Coverings and refinements.* Let $\bar{\mathfrak{A}}, \bar{\mathfrak{B}}$ stand for factoroids in $\mathfrak{G}$.

$\bar{\mathfrak{A}}$ ($\bar{\mathfrak{B}}$) is called a *covering* (*refinement*) *of* $\bar{\mathfrak{B}}$ ($\bar{\mathfrak{A}}$) if, for the fields $\bar{A}$, $\bar{B}$ of $\bar{\mathfrak{A}}, \bar{\mathfrak{B}}$, there holds $\bar{A} \geqq \bar{B}$. We write $\bar{\mathfrak{A}} \geqq \bar{\mathfrak{B}}$ or $\bar{\mathfrak{B}} \leqq \bar{\mathfrak{A}}$. The meaning of a *normal* and a *pure covering* (*refinement*) *of* $\bar{\mathfrak{B}}$ ($\bar{\mathfrak{A}}$) is obvious (2.4). The relation $\bar{\mathfrak{A}} \geqq \bar{\mathfrak{B}}$ yields $\mathbf{s}\bar{\mathfrak{A}} \supset \mathbf{s}\bar{\mathfrak{B}}$ and, in case of a pure covering (refinement): $\mathbf{s}\bar{\mathfrak{A}} = \mathbf{s}\bar{\mathfrak{B}}$. If $\bar{\mathfrak{A}} \geqq \bar{\mathfrak{B}}$ and, at the same time, $\bar{\mathfrak{A}} \neq \bar{\mathfrak{B}}$, then $\bar{\mathfrak{A}}$ ($\bar{\mathfrak{B}}$) is a *proper covering* (*proper refinement*) *of* $\bar{\mathfrak{B}}$ ($\bar{\mathfrak{A}}$); then we sometimes write $\bar{\mathfrak{A}} > \bar{\mathfrak{B}}$ or $\bar{\mathfrak{B}} < \bar{\mathfrak{A}}$.

2. *Closures and intersections.* Let $\mathfrak{B} \subset \mathfrak{G}$ stand for a subgroupoid and $\bar{\mathfrak{A}}, \bar{\mathfrak{C}}$ for factoroids in $\mathfrak{G}$.

If $B \cap s\bar{C} \neq \emptyset$, then (14.3.2) $B \sqsubset \bar{C}$ and $B \sqcap \bar{C}$ are generating decompositions in $\mathfrak{G}$. The corresponding factoroids in $\mathfrak{G}$ are called the *closure of the subgroupoid* $\mathfrak{B}$ *in the factoroid* $\overline{\mathfrak{C}}$ and the *intersection of the subgroupoid* $\mathfrak{B}$ *(factoroid* $\overline{\mathfrak{C}}$*) and the factoroid* $\overline{\mathfrak{C}}$ *(subgroupoid* $\mathfrak{B}$*)*; notation for closure: $\mathfrak{B} \sqsubset \overline{\mathfrak{C}}$ or $\overline{\mathfrak{C}} \sqsupset \mathfrak{B}$, for intersection: $\mathfrak{B} \sqcap \overline{\mathfrak{C}}$ or $\overline{\mathfrak{C}} \sqcap \mathfrak{B}$.

The meaning of the concepts defined for $s\bar{A} \cap s\bar{C} \neq \emptyset$ and denoted $\overline{\mathfrak{A}} \sqsubset \overline{\mathfrak{C}}$ or $\overline{\mathfrak{C}} \sqsupset \overline{\mathfrak{A}}$ and $\overline{\mathfrak{A}} \sqcap \overline{\mathfrak{C}}$ is obvious as well; the former is called the *closure of* $\overline{\mathfrak{A}}$ *in* $\overline{\mathfrak{C}}$, the latter is the *intersection of* $\overline{\mathfrak{A}}$ *and* $\overline{\mathfrak{C}}$. Evidently: $\overline{\mathfrak{A}} \sqcap \overline{\mathfrak{C}} = \overline{\mathfrak{C}} \sqcap \overline{\mathfrak{A}}$.

Note that $\mathfrak{B} \sqsubset \overline{\mathfrak{C}}$ is a subgroupoid in $\overline{\mathfrak{C}}$ and $\mathfrak{B} \sqcap \overline{\mathfrak{C}}$ a factoroid in $\mathfrak{B}$.

If, in particular, $\overline{\mathfrak{C}}$ lies on $\mathfrak{G}$, then the above assumption $B \cap s\bar{C} \neq \emptyset$ is satisfied and $\mathfrak{B} \sqcap \overline{\mathfrak{C}}$ is a factoroid on $\mathfrak{B}$. Every factoroid $\overline{\mathfrak{C}}$ on $\mathfrak{G}$ and a subgroupoid $\mathfrak{B}$ of $\mathfrak{G}$ thus uniquely determine a subgroupoid $\mathfrak{B} \sqsubset \overline{\mathfrak{C}}$ in $\overline{\mathfrak{C}}$ and a factoroid $\mathfrak{B} \sqcap \overline{\mathfrak{C}}$ on $\mathfrak{B}$.

Similarly, a factoroid $\overline{\mathfrak{A}}$ in $\mathfrak{G}$ and a factoroid $\overline{\mathfrak{C}}$ on $\mathfrak{G}$ determine a factoroid $\overline{\mathfrak{A}} \sqsubset \overline{\mathfrak{C}}$ and a factoroid $\overline{\mathfrak{A}} \sqcap \overline{\mathfrak{C}}$; the former is a subgroupoid of $\overline{\mathfrak{C}}$ and the latter a factoroid on $s\overline{\mathfrak{A}}$.

Finally, let us remark that *if* $\overline{\mathfrak{A}}$ *and* $\overline{\mathfrak{C}}$ *cover* $\mathfrak{G}$, *then their intersection coincides with the greatest common refinement* $(\overline{\mathfrak{A}}, \overline{\mathfrak{C}})$ *of* $\overline{\mathfrak{A}}, \overline{\mathfrak{C}}$ and so $\overline{\mathfrak{A}} \sqcap \overline{\mathfrak{C}} = (\overline{\mathfrak{A}}, \overline{\mathfrak{C}})$ (15.4.5).

Example. In order to illustrate the above notions by an example, let us again consider the factoroid $\overline{\mathfrak{Z}}_n$ on the groupoid $\mathfrak{Z}$ $(n \geqq 1)$. Let $\mathfrak{A}_m$ denote the subgroupoid of $\mathfrak{Z}$, with the field consisting of all multiples of a given positive integer m and suppose (to simplify our example) that the greatest common divisor of m and n is 1.

Which elements do the factoroids $\mathfrak{A}_m \sqsubset \overline{\mathfrak{Z}}_n$, $\overline{\mathfrak{Z}}_n \sqcap \mathfrak{A}_m$ consist of?

Consider which of the elements $\bar{a}_0, \ldots, \bar{a}_{n-1} \in \overline{\mathfrak{Z}}_n$ are incident with the subgroupoid $\mathfrak{A}_m$. Any element $\bar{a}_i \in \overline{\mathfrak{Z}}_n$ is incident with $\mathfrak{A}_m$ if and only if it comprises a multiple xm of m (x integer). Since each element of $\bar{a}_i$ is of the form $yn + i$ where y also denotes an integer, we see that $\bar{a}_i$ is incident with $\mathfrak{A}_m$ if and only if the equation $xm = yn + i$ and therefore even $xm - yn = i$ has an integral solution. Since the greatest common divisor of m and n is 1, there exist integers a, b satisfying $am - bn = 1$. Consequently, $xm - yn = i$ has, for every number $i = 0, \ldots, n - 1$, an integral solution, namely $x = ai$, $y = bi$, hence every element $\bar{a}_i \in \overline{\mathfrak{Z}}_n$ is incident with $\mathfrak{A}_m$. Thus the factoroid $\mathfrak{A}_m \sqsubset \overline{\mathfrak{Z}}_n$ is identical with $\overline{\mathfrak{Z}}_n$ and the elements of $\overline{\mathfrak{Z}}_n \sqcap \mathfrak{A}_m$ are sets consisting of all the multiples of m contained in the individual elements $\bar{a}_0, \ldots, \bar{a}_{n-1}$ of the factoroid $\overline{\mathfrak{Z}}_n$.

3. *Semi-coupled or loosely coupled and coupled factoroids.* Let $\overline{\mathfrak{A}}, \overline{\mathfrak{C}}$ be factoroids in $\mathfrak{G}$. The factoroids $\overline{\mathfrak{A}}, \overline{\mathfrak{C}}$ are said to be *semi-coupled* or *loosely coupled* (*coupled*) if their fields $\bar{A}, \bar{C}$ have the same property (4.1).

For example, the closure $\mathfrak{X} \sqsubset \overline{\mathfrak{Y}}$ of an arbitrary subgroupoid $\mathfrak{X} \subset \mathfrak{G}$ in the factoroid $\overline{\mathfrak{Y}}$ in $\mathfrak{G}$ and the intersection $\overline{\mathfrak{Y}} \sqcap \mathfrak{X}$ $(X \cap s\bar{Y} \neq \emptyset)$ are coupled factoroids.

In what follows we shall assume that $\overline{\mathfrak{A}} = \overline{\mathfrak{C}} \sqsubset \overline{\mathfrak{A}}, \overline{\mathfrak{C}} = \overline{\mathfrak{A}} \sqsubset \overline{\mathfrak{C}}$.

In that case there lies, in $\mathfrak{G}$, the subgroupoid $s\overline{\mathfrak{A}} \cap s\overline{\mathfrak{C}}$ and, on the latter, the factoroids $\overline{\mathfrak{A}} \sqcap s\overline{\mathfrak{C}}, \overline{\mathfrak{C}} \sqcap s\overline{\mathfrak{A}}$. From the theorem in 14.3.3 we conclude that *every* *common covering* $\overline{\mathfrak{B}}$ *of* $\overline{\mathfrak{A}} \sqcap \mathbf{s}\overline{\mathfrak{C}}$ *and* $\overline{\mathfrak{C}} \sqcap \mathbf{s}\overline{\mathfrak{A}}$ *enforces coupled coverings* $\mathring{\mathfrak{A}} \geqq \overline{\mathfrak{A}}$, $\mathring{\mathfrak{C}} \geqq \overline{\mathfrak{C}}$ *of* $\overline{\mathfrak{A}}, \overline{\mathfrak{C}}$ *intersecting each other in the factoroid* $\overline{\mathfrak{B}}$: $\mathring{\mathfrak{A}} \sqcap \mathring{\mathfrak{C}} = \overline{\mathfrak{B}}$.

4. *Adjoint factoroids.* Let $\overline{\mathfrak{A}}, \overline{\mathfrak{C}}$ be factoroids and $\mathfrak{B}, \mathfrak{D}$ subgroupoids of $\mathfrak{G}$. Denote $\mathfrak{A} = \mathbf{s}\overline{\mathfrak{A}}$, $\mathfrak{C} = \mathbf{s}\overline{\mathfrak{C}}$ and let B, D stand for the fields of the subgroupoids $\mathfrak{B}, \mathfrak{D}$.

Suppose there holds: $\mathfrak{B} \in \overline{\mathfrak{A}}$, $\mathfrak{D} \in \overline{\mathfrak{C}}$; $\mathfrak{B} \cap \mathfrak{D} \neq \emptyset$, the first formulae expressing the relations $B \in \overline{\mathfrak{A}}$, $D \in \overline{\mathfrak{C}}$.

The factoroids $\overline{\mathfrak{A}}, \overline{\mathfrak{C}}$ are said to be *adjoint with regard to* $\mathfrak{B}, \mathfrak{D}$ if the decompositions $\bar{A}, \bar{C}$ have the same property with regard to B, D (4.2). This may be expressed by the formula:

$$\mathbf{s}(\mathfrak{D} \sqsubset \overline{\mathfrak{A}} \sqcap \mathfrak{C}) = \mathbf{s}(\mathfrak{B} \sqsubset \overline{\mathfrak{C}} \sqcap \mathfrak{A}).$$

Suppose $\overline{\mathfrak{A}}, \overline{\mathfrak{C}}$ are adjoint with regard to $\mathfrak{B}, \mathfrak{D}$. Then

$$\overline{\mathfrak{A}}_1 = \mathfrak{C} \sqsubset \overline{\mathfrak{A}}, \quad \overline{\mathfrak{A}}_2 = \mathfrak{D} \sqsubset \overline{\mathfrak{A}},$$
$$\overline{\mathfrak{C}}_1 = \mathfrak{A} \sqsubset \overline{\mathfrak{C}}, \quad \overline{\mathfrak{C}}_2 = \mathfrak{B} \sqsubset \overline{\mathfrak{C}}$$

are factoroids in $\mathfrak{G}$. Denote $\mathfrak{A}_1 = \mathbf{s}\overline{\mathfrak{A}}_1$, $\mathfrak{A}_2 = \mathbf{s}\overline{\mathfrak{A}}_2$; $\mathfrak{C}_1 = \mathbf{s}\overline{\mathfrak{C}}_1$, $\mathfrak{C}_2 = \mathbf{s}\overline{\mathfrak{C}}_2$. From the result in 4.2 there follows, with respect to 14.4.2 and 14.3.3, the following theorem:

The factoroids $\overline{\mathfrak{A}}_1, \overline{\mathfrak{C}}_1$ *have coupled coverings* $\mathring{\mathfrak{A}}, \mathring{\mathfrak{C}}$ *such that* $\mathfrak{A}_2 \in \mathring{\mathfrak{A}}$, $\mathfrak{C}_2 \in \mathring{\mathfrak{C}}$; the coverings $\mathring{\mathfrak{A}}, \mathring{\mathfrak{C}}$ are given by the construction described in 4.2a. *The subgroupoids* $\mathfrak{A}_2, \mathfrak{C}_2$ *are incident.*

5. *Chains of factoroids.* Let $\mathfrak{A} \supset \mathfrak{B}$ denote subgroupoids of $\mathfrak{G}$.

A *chain of factoroids from* $\mathfrak{A}$ *to* $\mathfrak{B}$, briefly, a *chain from* $\mathfrak{A}$ *to* $\mathfrak{B}$, is a finite sequence consisting of α $(\geqq 1)$ factoroids $\overline{\mathfrak{K}}_1, \ldots, \overline{\mathfrak{K}}_\alpha$ in $\mathfrak{G}$ with the following properties: a) the factoroid $\overline{\mathfrak{K}}_1$ lies on $\mathfrak{A}$; b) for $1 \leqq \gamma \leqq \alpha - 1$ the factoroid $\overline{\mathfrak{K}}_{\gamma+1}$ lies on an element of $\overline{\mathfrak{K}}_\gamma$; c) $\mathfrak{B} \in \overline{\mathfrak{K}}_\alpha$. Such a chain is denoted

$$\overline{\mathfrak{K}}_1 \to \cdots \to \overline{\mathfrak{K}}_\alpha,$$

briefly: $[\overline{\mathfrak{K}}]$.

The notions relative to chains of decompositions, defined in 2.5 and 4.2, can be directly applied to chains of factoroids. In particular, the concept of adjoint chains of factoroids is defined as follows:

Let $\mathfrak{A} \supset \mathfrak{B}$, $\mathfrak{C} \supset \mathfrak{D}$ stand for subgroupoids of $\mathfrak{G}$ and let

$$([\overline{\mathfrak{K}}] =) \ \overline{\mathfrak{K}}_1 \to \cdots \to \overline{\mathfrak{K}}_\alpha,$$
$$([\overline{\mathfrak{L}}] =) \ \overline{\mathfrak{L}}_1 \to \cdots \to \overline{\mathfrak{L}}_\beta$$

be chains of factoroids from $\mathfrak{A}$ to $\mathfrak{B}$ and from $\mathfrak{C}$ to $\mathfrak{D}$.

The chains $[\overline{\mathfrak{K}}]$, $[\overline{\mathfrak{L}}]$ are called *adjoint* if a) their ends coincide, i.e., $\mathfrak{A} = \mathfrak{C}$, $\mathfrak{B} = \mathfrak{D}$; b) every two members $\overline{\mathfrak{K}}_\gamma$, $\overline{\mathfrak{L}}_\delta$ are adjoint with regard to $\mathbf{s}\overline{\mathfrak{K}}_{\gamma+1}$, $\mathbf{s}\overline{\mathfrak{L}}_{\delta+1}$ for $\gamma = 1, \ldots, \alpha$; $\delta = 1, \ldots, \beta$ while $\mathbf{s}\overline{\mathfrak{K}}_{\alpha+1} = \mathfrak{B}$, $\mathbf{s}\overline{\mathfrak{L}}_{\beta+1} = \mathfrak{D}$.

15.4. Factoroids on groupoids

Let us now deal with factoroids on groupoids. The results can often be applied even to factoroids in groupoids, since every factoroid $\overline{\mathfrak{A}}$ in $\mathfrak{G}$ lies on the subgroupoid $s\overline{\mathfrak{A}}$.

1. *Coverings and refinements.* We shall start from the notions of a covering and a refinement of a factoroid in $\mathfrak{G}$, described in 15.3.1 and proceed to the case of factoroids on $\mathfrak{G}$.

Let $\overline{\mathfrak{A}}$, $\overline{\mathfrak{B}}$ denote factoroids on $\mathfrak{G}$.

We know that $\overline{\mathfrak{A}}$ ($\overline{\mathfrak{B}}$) is called a covering (refinement) of $\overline{\mathfrak{B}}$ ($\overline{\mathfrak{A}}$) and that we write $\overline{\mathfrak{A}} \geqq \overline{\mathfrak{B}}$ or $\overline{\mathfrak{B}} \leqq \overline{\mathfrak{A}}$ if, for the fields $\bar{A}$ and $\bar{B}$ of $\overline{\mathfrak{A}}$ and $\overline{\mathfrak{B}}$, respectively, there holds $\bar{A} \geqq \bar{B}$.

For example, $\overline{\mathfrak{G}}_{\max}$ ($\overline{\mathfrak{B}}$) is the greatest (least) covering of $\overline{\mathfrak{B}}$ in the sense that every covering of $\overline{\mathfrak{B}}$ is a refinement of $\overline{\mathfrak{G}}_{\max}$ and, of course, a covering of $\overline{\mathfrak{B}}$; analogously, $\overline{\mathfrak{A}}$ ($\overline{\mathfrak{G}}_{\min}$) is the greatest (least) refinement of $\overline{\mathfrak{A}}$.

If $\overline{\mathfrak{A}} \geqq \overline{\mathfrak{B}}$, then $\bar{A}$ is a covering of $\bar{B}$ so that $\bar{A}$ is enforced by a certain decomposition $\bar{\bar{B}}$ lying on $\bar{B}$ and, naturally, also on $\overline{\mathfrak{B}}$ (2.4). Note that every element $\bar{\bar{b}} \in \bar{\bar{B}}$ is a system of subsets in $\mathfrak{G}$ which are elements of $\overline{\mathfrak{B}}$ and that $\bar{A}$ is obtained by summing all the elements of $\overline{\mathfrak{B}}$ lying in the individual elements $\bar{\bar{b}}$.

Conversely, every decomposition $\bar{\bar{B}}$ on the factoroid $\overline{\mathfrak{B}}$ enforces a certain covering of $\bar{B}$ which, however, is not necessarily generating. We observe that the covering enforced by $\bar{\bar{B}}$ need not be the field of a factoroid.

Let us now prove the following theorem:

Let $\overline{\mathfrak{B}}$ stand for a factoroid on $\mathfrak{G}$, $\bar{\bar{B}}$ for a decomposition of $\overline{\mathfrak{B}}$ and $\bar{A}$ for the covering of the field $\bar{B}$ of $\overline{\mathfrak{B}}$ enforced by $\bar{\bar{B}}$. The decomposition $\bar{A}$ is generating if and only if $\bar{\bar{B}}$ is generating.

Proof. a) Suppose $\bar{\bar{B}}$ is generating. Consider arbitrary elements $\bar{a}_1, \bar{a}_2 \in \bar{A}$. We are to show that there exists an element $\bar{a}_3 \in \bar{A}$ such that $\bar{a}_1\bar{a}_2 \subset \bar{a}_3$. Now, with regard to the definition of $\bar{A}$, there holds $\bar{a}_1 = \bigcup_1 \bar{b}_1$, $\bar{a}_2 = \bigcup_2 \bar{b}_2$, the symbol $\bigcup_1$ ($\bigcup_2$) relating to all the elements of the factoroid $\overline{\mathfrak{B}}$ contained in a certain element $\bar{\bar{b}}_1$ ($\bar{\bar{b}}_2$) of $\bar{\bar{B}}$. Since $\bar{\bar{B}}$ is generating, there exists an element $\bar{\bar{b}}_3 \in \bar{\bar{B}}$ such that $\bar{\bar{b}}_1 \circ \bar{\bar{b}}_2 \subset \bar{\bar{b}}_3$. Let $\bar{a}_3$ be the sum of all the elements of $\overline{\mathfrak{B}}$ contained in $\bar{\bar{b}}_3$ so that $\bar{a}_3 \in \bar{A}$. For every element $\bar{b}_1$ ($\bar{b}_2$) to which the symbol $\bigcup_1$ ($\bigcup_2$) applies we evidently have $\bar{b}_1 \circ \bar{b}_2 \in \bar{\bar{b}}_1 \circ \bar{\bar{b}}_2 \subset \bar{\bar{b}}_3$. Hence the relations:

$$\bar{a}_1\bar{a}_2 = \bigcup_1\bigcup_2 \bar{b}_1\bar{b}_2 \subset \bigcup_1\bigcup_2 \bar{b}_1 \circ \bar{b}_2 \subset \bar{a}_3$$

which prove the first part of the theorem.

b) Suppose $\bar{A}$ is generating. Consider arbitrary elements $\bar{\bar{b}}_1, \bar{\bar{b}}_2 \in \bar{\bar{B}}$ and let $\bar{a}_1, \bar{a}_2$, $\bar{b}_1, \bar{b}_2$ have the above meaning. Since $\bar{A}$ is generating, there exists an element $\bar{a}_3 \in \bar{A}$ such that $\bar{a}_1\bar{a}_2 \subset \bar{a}_3$. By the definition of $\bar{A}$ there exist elements $\bar{b}_3 \in \overline{\mathfrak{B}}$ such that

$\bar{a}_3 = \bigcup \bar{\bar{b}}_3$ and the set of these elements is an element $\bar{\bar{b}}_3 \in \bar{\bar{B}}$. For any elements $\bar{b}_1 \in \bar{\bar{b}}_1$, $\bar{b}_2 \in \bar{\bar{b}}_2$ there holds $\bar{b}_1\bar{b}_2 \subset \bar{a}_3$ and we see that there exists an element $\bar{b}_3 \in \bar{\bar{b}}_3$ such that $\bar{b}_1\bar{b}_2 \subset \bar{b}_3$, Hence $\bar{b}_1 \circ \bar{b}_2 = \bar{b}_3 \in \bar{\bar{b}}_3$ so that $\bar{\bar{b}}_1 \circ \bar{\bar{b}}_2 \subset \bar{\bar{b}}_3$ and the proof is complete.

Thus *both decompositions* $\bar{A}$ *and* $\bar{\bar{B}}$ *are simultaneously generating,* i.e., if one is generating, then the other is generating as well. If they are generating, then there corresponds to $\bar{A}$ a certain factoroid $\bar{\mathfrak{A}}$ on $\mathfrak{G}$ and there holds $\bar{\mathfrak{A}} \geqq \bar{\mathfrak{B}}$; similarly, there corresponds to $\bar{\bar{B}}$ a certain factoroid $\bar{\bar{\bar{\mathfrak{B}}}}$ on $\bar{\mathfrak{B}}$. $\bar{\mathfrak{A}}$ is called the *covering of* $\bar{\mathfrak{B}}$ *enforced by* $\bar{\bar{\bar{\mathfrak{B}}}}$. Every factoroid on an arbitrary factoroid $\bar{\mathfrak{B}}$ of $\mathfrak{G}$ therefore enforces a certain covering of $\bar{\mathfrak{B}}$ and, conversely, every covering of $\bar{\mathfrak{B}}$ is enforced by a factoroid on $\bar{\mathfrak{B}}$.

Example. To illustrate the above notions by an example, let us again consider the factoroid $\bar{\mathfrak{Z}}_n$ on the groupoid $\mathfrak{Z}$ (15.2). Suppose the number n is greater than 1 and is not a prime number. Then there exists a divisor $(1 <)\, d\, (< n)$ of the number n and we have $n = qd$ where q is a positive integer $1 < q < n$. We shall now be concerned with the decomposition $\bar{\bar{Z}}_d$ of $\bar{\mathfrak{Z}}_n$ whose elements are:

$$\begin{aligned}
\bar{\bar{a}}_0 &= \{\bar{a}_0,\ \bar{a}_d,\ \bar{a}_{2d},\ \ldots,\ \bar{a}_{(q-1)d}\},\\
\bar{\bar{a}}_1 &= \{\bar{a}_1,\ \bar{a}_{d+1},\ \bar{a}_{2d+1},\ \ldots,\ \bar{a}_{(q-1)d+1}\},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\bar{\bar{a}}_{d-1} &= \{\bar{a}_{d-1},\ \bar{a}_{d+d-1},\ \bar{a}_{2d+d-1},\ \ldots,\ \bar{a}_{(q-1)d+d-1}\},
\end{aligned}$$

and so any element $\bar{\bar{a}}_i$ $(i = 0, \ldots, d-1)$ of $\bar{\bar{Z}}_d$ consists of those elements of $\bar{\mathfrak{Z}}_n$ whose indices are congruent to i modulo d. Let us prove that $\bar{\bar{Z}}_d$ is generating. Consider arbitrary elements $\bar{\bar{a}}_i$, $\bar{\bar{a}}_j$ of the decomposition $\bar{\bar{Z}}_d$. We shall show that there holds $\bar{\bar{a}}_i \,.\, \bar{\bar{a}}_j \subset \bar{\bar{a}}_k$ where k is the remainder of $i + j$ divided by d. Let $\bar{a}_\alpha$ and $\bar{a}_\beta$ be arbitrary elements of $\bar{\bar{a}}_i$ and $\bar{\bar{a}}_j$, respectively, so that divided by d, α leaves the remainder i and β the remainder j; consequently, $\alpha + \beta$, $i + j$ differ only by an integer multiple of d. In accordance with the definition of the multiplication in $\bar{\mathfrak{Z}}_n$, there holds $\bar{a}_\alpha \circ \bar{a}_\beta = \bar{a}_\gamma$ where γ is the remainder of $\alpha + \beta$ divided by n. Since d is a divisor of n, the numbers $\alpha + \beta$, γ and hence even $i + j$, γ differ by an integer multiple of d; consequently, γ divided by d leaves the remainder k. So we have $\bar{a}_\alpha \circ \bar{a}_\beta = \bar{a}_\gamma \in \bar{\bar{a}}_k$ which yields $\bar{\bar{a}}_i \,.\, \bar{\bar{a}}_j \subset \bar{\bar{a}}_k$. The covering of $\bar{\mathfrak{Z}}_n$, enforced by the factoroid $\bar{\bar{\bar{\mathfrak{Z}}}}_d$ belonging to the generating decomposition $\bar{\bar{Z}}_d$, consists of d elements

$$\{\ldots, -n + i, -n + d + i, \ldots, -n + (q-1)d + i, i, d + i, \ldots,$$
$$(q-1)d + i, n + i, n + d + i, \ldots, n + (q-1)d + i, \ldots\},$$

where i denotes one of the numbers $0, \ldots, d-1$.

2. *Local properties of coverings and refinements.* Let $\bar{\mathfrak{A}} \geqq \bar{\mathfrak{B}}$ stand for arbitrary factoroids on $\mathfrak{G}$.

Consider arbitrary elements $\bar{a}_1, \bar{a}_2 \in \overline{\mathfrak{A}}$ and $\bar{b}_1, \bar{b}_2 \in \overline{\mathfrak{B}}$ such that $\bar{a}_1 \supset \bar{b}_1$, $\bar{a}_2 \supset \bar{b}_2$ and, furthermore, the following decompositions in $\mathfrak{G}$: $\bar{a}_1 \sqcap \overline{\mathfrak{B}}$, $\bar{a}_2 \sqcap \overline{\mathfrak{B}}$. With regard to the relation $\overline{\mathfrak{A}} \geqq \overline{\mathfrak{B}}$, the mentioned decompositions are complexes in $\overline{\mathfrak{B}}$.

We shall show that *there holds*:

$$\bar{a}_1 \circ \bar{a}_2 \supset \bar{b}_1 \circ \bar{b}_2, \tag{1}$$

$$(\bar{a}_1 \sqcap \overline{\mathfrak{B}}) \circ (\bar{a}_2 \sqcap \overline{\mathfrak{B}}) \subset \bar{a}_1 \circ \bar{a}_2 \sqcap \overline{\mathfrak{B}}. \tag{2}$$

Proof. a) From $\bar{b}_1\bar{b}_2 \subset \bar{b}_1 \circ \bar{b}_2 \cap \bar{a}_1\bar{a}_2 \subset \bar{b}_1 \circ \bar{b}_2 \cap \bar{a}_1 \circ \bar{a}_2$ there follows that the elements $\bar{b}_1 \circ \bar{b}_2 \in \overline{\mathfrak{B}}$, $\bar{a}_1 \circ \bar{a}_2 \in \overline{\mathfrak{A}}$ are incident. Hence, with regard to $\overline{\mathfrak{A}} \geqq \overline{\mathfrak{B}}$ (3.2), we have the formula (1).

b) The product $\bar{x} \circ \bar{y}$ with arbitrary factors $\bar{x} \in \bar{a}_1 \sqcap \overline{\mathfrak{B}}$, $\bar{y} \in \bar{a}_2 \sqcap \overline{\mathfrak{B}}$ is the element $\bar{z} \in \overline{\mathfrak{B}}$ for which $\bar{x}\bar{y} \subset \bar{z}$; $\bar{z}$ is an element of the decomposition $\bar{a}_1 \circ \bar{a}_2 \sqcap \overline{\mathfrak{B}}$ (14.4.1).

We observe, in particular, that if any element $\bar{a} \in \overline{\mathfrak{A}}$ is a groupoidal subset of $\mathfrak{G}$ and so $\bar{a} \circ \bar{a} = \bar{a}$, then the formula (2) yields (for $\bar{a}_1 = \bar{a}_2 = \bar{a}$): $(\bar{a} \sqcap \overline{\mathfrak{B}}) \circ (\bar{a} \sqcap \overline{\mathfrak{B}}) \subset \bar{a} \sqcap \overline{\mathfrak{B}}$. In that case the decomposition $\bar{a} \sqcap \overline{\mathfrak{B}}$ is a groupoidal complex in the factoroid $\overline{\mathfrak{B}}$.

If any element $\bar{a} \in \overline{\mathfrak{A}}$ is a groupoidal subset of $\mathfrak{G}$, then the decomposition $\bar{a} \sqcap \overline{\mathfrak{B}}$ generates, on the corresponding subgroupoid $\bar{\mathfrak{a}} \subset \mathfrak{G}$, the factoroid $\bar{\mathfrak{a}} \sqcap \overline{\mathfrak{B}}$.

In particular, every element $\bar{a} \in \overline{\mathfrak{A}}$ comprising an idempotent point $a \in \bar{a}$ (i.e., such that $aa = a \in \bar{a}$) is a groupoidal subset of $\mathfrak{G}$ (15.6.4).

It is easy to see that, if $a \in \mathfrak{G}$ is idempotent, then the element $\bar{a} \in \overline{\mathfrak{A}}$ containing it is a groupoidal subset of $\mathfrak{G}$ and that the decomposition $\bar{a} \sqcap \overline{\mathfrak{B}}$ generates, on the corresponding subgroupoid $\bar{\mathfrak{a}} \subset \mathfrak{G}$, the factoroid $\bar{\mathfrak{a}} \sqcap \overline{\mathfrak{B}}$.

3. *Common covering and common refinement of two factoroids.* Let $\overline{\mathfrak{A}}, \overline{\mathfrak{B}}$ denote arbitrary factoroids on $\mathfrak{G}$.

A *common covering*, briefly, a *covering of* $\overline{\mathfrak{A}}, \overline{\mathfrak{B}}$ is any factoroid on $\mathfrak{G}$ that is a covering of either of the factoroids $\overline{\mathfrak{A}}, \overline{\mathfrak{B}}$.

Analogously, by a *common refinement*, briefly, a *refinement of the factoroids* $\overline{\mathfrak{A}}, \overline{\mathfrak{B}}$ we mean any factoroid on $\mathfrak{G}$ that is a refinement of either of the factoroids $\overline{\mathfrak{A}}, \overline{\mathfrak{B}}$.

For example, the greatest factoroid $\overline{\mathfrak{G}}_{\max}$ is a common covering and the least factoroid $\overline{\mathfrak{G}}_{\min}$ a common refinement of the factoroids $\overline{\mathfrak{A}}, \overline{\mathfrak{B}}$.

It is obvious that *every covering of any common covering of* $\overline{\mathfrak{A}}, \overline{\mathfrak{B}}$ *is again a covering of* $\overline{\mathfrak{A}}, \overline{\mathfrak{B}}$; analogously, *every refinement of any common refinement of* $\overline{\mathfrak{A}}, \overline{\mathfrak{B}}$ *is again their refinement.*

4. *The least common covering of two factoroids.* From 14.4.2 we know that the least common covering of the fields of $\overline{\mathfrak{A}}, \overline{\mathfrak{B}}$ is a generating decomposition of $\mathfrak{G}$. The factoroid corresponding to the least common covering of the fields of $\overline{\mathfrak{A}}, \overline{\mathfrak{B}}$ is called the *least common covering*, briefly, the *least covering of* $\overline{\mathfrak{A}}, \overline{\mathfrak{B}}$ and is denoted by $[\overline{\mathfrak{A}}, \overline{\mathfrak{B}}]$ or $[\overline{\mathfrak{B}}, \overline{\mathfrak{A}}]$.

From the definition of the factoroid $[\bar{\mathfrak{A}}, \bar{\mathfrak{B}}]$ it follows that its field is a refinement of any common covering of the fields of $\bar{\mathfrak{A}}, \bar{\mathfrak{B}}$ and therefore also of any generating common covering of the fields of $\bar{\mathfrak{A}}, \bar{\mathfrak{B}}$. Hence the factoroid $[\bar{\mathfrak{A}}, \bar{\mathfrak{B}}]$ is the least common covering of $\bar{\mathfrak{A}}, \bar{\mathfrak{B}}$, least in the sense that any common covering of both factoroids is a covering of $[\bar{\mathfrak{A}}, \bar{\mathfrak{B}}]$.

5. *The greatest common refinement of two factoroids.* From 14.4.3 we know that the greatest common refinement of the fields of $\bar{\mathfrak{A}}, \bar{\mathfrak{B}}$ is a generating decomposition of $\mathfrak{G}$. The factoroid corresponding to the greatest common refinement of the fields of $\bar{\mathfrak{A}}, \bar{\mathfrak{B}}$ is called the *greatest common refinement*, briefly, the *greatest refinement of* $\bar{\mathfrak{A}}, \bar{\mathfrak{B}}$ and is denoted by $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ or $(\bar{\mathfrak{B}}, \bar{\mathfrak{A}})$.

From the definition of the factoroid $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ it follows that its field is a covering of any common refinement of the fields of $\bar{\mathfrak{A}}, \bar{\mathfrak{B}}$ and therefore also of any generating common refinement of the fields of $\bar{\mathfrak{A}}, \bar{\mathfrak{B}}$. Hence $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ is the greatest common refinement of $\bar{\mathfrak{A}}, \bar{\mathfrak{B}}$, greatest in the sense that any common refinement of both factoroids is a refinement of $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$.

On this occasion, let us note the formula: $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}}) = \bar{\mathfrak{A}} \sqcap \bar{\mathfrak{B}}$ (15.3.2).

6. *Modular factoroids.* Let $\bar{\mathfrak{X}}, \bar{\mathfrak{A}}, \bar{\mathfrak{B}}$ be factoroids on $\mathfrak{G}$ such that $\bar{\mathfrak{X}} \geqq \bar{\mathfrak{A}}$.

The factoroid $\bar{\mathfrak{B}}$ is said to be *modular with regard to* $\bar{\mathfrak{X}}, \bar{\mathfrak{A}}$ (in this order) if there holds:

$$[\bar{\mathfrak{A}}, (\bar{\mathfrak{X}}, \bar{\mathfrak{B}})] = (\bar{\mathfrak{X}}, [\bar{\mathfrak{A}}, \bar{\mathfrak{B}}]).$$

If, for example, $\bar{\mathfrak{X}} = \bar{\mathfrak{A}}$ or $\bar{\mathfrak{A}} = \bar{\mathfrak{G}}_{\max}$, then $\bar{\mathfrak{B}}$ is modular with regard to $\bar{\mathfrak{X}}, \bar{\mathfrak{A}}$.

Let $\bar{\mathfrak{X}}, \bar{\mathfrak{Y}}$ and $\bar{\mathfrak{A}}, \bar{\mathfrak{B}}$ denote arbitrary factoroids on $\mathfrak{G}$ such that $\bar{\mathfrak{X}} \geqq \bar{\mathfrak{A}}, \bar{\mathfrak{Y}} \geqq \bar{\mathfrak{B}}$ and suppose that $\bar{\mathfrak{B}}$ is modular with regard to $\bar{\mathfrak{X}}, \bar{\mathfrak{A}}$ and $\bar{\mathfrak{A}}$ is modular with regard to $\bar{\mathfrak{Y}}, \bar{\mathfrak{B}}$.

Then we have:

$$(\mathring{\mathfrak{A}} =)\ [\bar{\mathfrak{A}}, (\bar{\mathfrak{X}}, \bar{\mathfrak{B}})] = (\bar{\mathfrak{X}}, [\bar{\mathfrak{A}}, \bar{\mathfrak{B}}]),$$

$$(\mathring{\mathfrak{B}} =)\ [\bar{\mathfrak{B}}, (\bar{\mathfrak{Y}}, \bar{\mathfrak{A}})] = (\bar{\mathfrak{Y}}, [\bar{\mathfrak{B}}, \bar{\mathfrak{A}}]),$$

$\mathring{\mathfrak{A}}$ and $\mathring{\mathfrak{B}}$ denoting factoroids defined by the first and the second formula, respectively.

In this situation there hold the interpolation formulae

$$\bar{\mathfrak{X}} \geqq \mathring{\mathfrak{A}} \geqq \bar{\mathfrak{A}}, \quad \bar{\mathfrak{Y}} \geqq \mathring{\mathfrak{B}} \geqq \bar{\mathfrak{B}}$$

and, furthermore, the equalities (4.3)

$$[\mathring{\mathfrak{A}}, \mathring{\mathfrak{B}}] = [\bar{\mathfrak{A}}, \bar{\mathfrak{B}}], \quad [\bar{\mathfrak{X}}, \mathring{\mathfrak{B}}] = [\bar{\mathfrak{X}}, \bar{\mathfrak{B}}], \quad [\bar{\mathfrak{Y}}, \mathring{\mathfrak{A}}] = [\bar{\mathfrak{Y}}, \bar{\mathfrak{A}}], \tag{1}$$

$$(\mathring{\mathfrak{A}}, \mathring{\mathfrak{B}}) = (\bar{\mathfrak{X}}, \mathring{\mathfrak{B}}) = (\bar{\mathfrak{Y}}, \mathring{\mathfrak{A}}) = \big((\bar{\mathfrak{X}}, \bar{\mathfrak{Y}}), [\bar{\mathfrak{A}}, \bar{\mathfrak{B}}]\big). \tag{2}$$

7. *Complementary* (*commuting*) *factoroids*. Let $\overline{\mathfrak{A}}$, $\overline{\mathfrak{B}}$ stand for arbitrary factoroids on $\mathfrak{G}$. $\overline{\mathfrak{A}}, \overline{\mathfrak{B}}$ are called *complementary* (*commuting*) if their fields are complementary, i.e., if any two elements $\bar{a} \in \overline{\mathfrak{A}}$, $b \in \overline{\mathfrak{B}}$ lying in the same element $\bar{u} \in [\overline{\mathfrak{A}}, \overline{\mathfrak{B}}]$ are incident.

If, for example, one of the two factoroids is a covering of the other, then both factoroids are complementary.

If there holds, for a certain factoroid $\overline{\mathfrak{X}}$ on $\mathfrak{G}$, the relation $\overline{\mathfrak{X}} \geqq \overline{\mathfrak{A}}$ and $\overline{\mathfrak{A}}, \overline{\mathfrak{B}}$ are complementary, then $\overline{\mathfrak{B}}$ is modular with respect to $\overline{\mathfrak{X}}$, $\overline{\mathfrak{A}}$ (5.4).

Later (25.3) we shall see that there exist groupoids on which any two factoroids are complementary. Generally, however, two factoroids of a given groupoid are not complementary. For example, on the groupoid whose field consists of four elements a, b, c, d and the multiplication is given by $xy = y$, all the decompositions are generating (14.5.3); factoroids whose fields are, e.g., the two decompositions $\{a, b\}$, $\{c, d\}$ and $\{a\}$, $\{b, c, d\}$ are not complementary (5.6.2).

15.5. α-grade groupoidal structures

Let us now proceed to the definition of a more complicated notion based on the concept of an α-grade set structure, which plays an important part in the following considerations.

Let $\alpha(\geqq 1)$ be an arbitrary natural number, $([\mathfrak{A}] =)$ $(\mathfrak{A}_1, \ldots, \mathfrak{A}_\alpha)$ be an α-membered sequence of groupoids and the symbol A_γ denote the field of $\mathfrak{A}_\gamma$, $\gamma = 1, 2, \ldots, \alpha$.

By an *α-grade groupoidal structure*, briefly, a *groupoidal structure* or a *structure with regard to the sequence* $[\mathfrak{A}]$ we mean a groupoid $\tilde{\mathfrak{A}}$ of the following form:

The field of the groupoid $\tilde{\mathfrak{A}}$ is an α-grade set structure with regard to the sequence $(A_1, \ldots, A_\alpha)$; each element

$$\bar{\bar{a}} = (\bar{a}_1, \ldots, \bar{a}_\alpha) \in \tilde{\mathfrak{A}}$$

is, consequently, an α-membered sequence every member $\bar{a}_\gamma$, $(\gamma = 1, \ldots, \alpha)$ of which is a complex in $\mathfrak{A}_\gamma$. The multiplication in $\tilde{\mathfrak{A}}$ is such that for any two elements

$$\bar{\bar{a}} = (\bar{a}_1, \ldots, \bar{a}_\alpha), \quad \bar{\bar{b}} = (\bar{b}_1, \ldots, \bar{b}_\alpha)$$

and their product

$$\bar{\bar{a}}\bar{\bar{b}} = \bar{\bar{c}} = (\bar{c}_1, \ldots, \bar{c}_\alpha) \in \tilde{\mathfrak{A}}$$

there holds

$$\bar{a}_1\bar{b}_1 \subset \bar{c}_1, \ldots, \bar{a}_\alpha\bar{b}_\alpha \subset \bar{c}_\alpha.$$

In what follows we shall be particularly concerned with the case when the groupoids $\mathfrak{A}_1, \ldots, \mathfrak{A}_\alpha$ are factoroids $\overline{\mathfrak{A}}_1, \ldots, \overline{\mathfrak{A}}_\alpha$ on $\mathfrak{G}$. Such α-grade groupoidal structures are, consequently, formed in the following way: Every element $\bar{\bar{a}} = (\bar{a}_1, \ldots, \bar{a}_\alpha) \in \tilde{\mathfrak{A}}$ is an α-membered sequence each member $\bar{a}_\gamma$, $(\gamma = 1, \ldots, \alpha)$ of which is a decomposition in $\mathfrak{G}$ and, in fact, a complex in $\overline{\mathfrak{A}}_\gamma$. The multiplication in $\tilde{\mathfrak{A}}$ is such that, for any two elements

$$\bar{\bar{a}} = (\bar{a}_1, \cdots, \bar{a}_\alpha), \bar{\bar{b}} = (\bar{b}_1, \ldots, \bar{b}_\alpha) \in \tilde{\mathfrak{A}}$$

and their product

$$\bar{\bar{a}}\bar{\bar{b}} = \bar{\bar{c}} = (\bar{c}_1, \ldots, \bar{c}_\alpha) \in \tilde{\mathfrak{A}},$$

there holds:

$$\bar{a}_1 \circ \bar{b}_1 \subset \bar{c}_1, \ldots, \bar{a}_\alpha \circ \bar{b}_\alpha \subset \bar{c}_\alpha.$$

15.6. Exercises

1. Show that the groupoids $\mathfrak{Z}_n$, $\overline{\mathfrak{Z}}_n$ $(n \geqq 1)$ are isomorphic.
2. Let $\mathfrak{A}_m$ stand for the subgroupoid of $\mathfrak{Z}$ whose field consists of all the integer multiples of a certain natural number $m > 1$. Of which elements do the factoroids $\mathfrak{A}_m \sqsubset \overline{\mathfrak{Z}}_n$ and $\overline{\mathfrak{Z}}_n \sqcap \mathfrak{A}_m$ $(n > 1)$ consist if m, n are not relatively prime?
3. Every factoroid on an Abelian (associative) groupoid is Abelian (associative).
4. If a groupoid $\mathfrak{G}$ contains an element a such that $aa = a$, i.e., a so-called *idempotent element* (15.4.2), then the element of any factoroid in $\mathfrak{G}$ comprising a is idempotent as well.

16. Deformations of factoroids

16.1. The isomorphism theorems for groupoids

Let us now proceed to the isomorphism theorems for groupoids. These theorems describe situations occurring under homomorphic mappings of groupoids or factoroids and connected with the concept of isomorphism. The set structure of these theorems is expressed by the equivalence theorems dealt with in 6.8.

1. *The first theorem.* Let $\mathfrak{G}$, $\mathfrak{G}^*$ be groupoids and suppose there exists a deformation $\boldsymbol{d}$ of $\mathfrak{G}$ onto $\mathfrak{G}^*$. In 14.2 we have shown that the decomposition $\bar{D}$ of $\mathfrak{G}$ corresponding to d is generating. Let $\overline{\mathfrak{D}}$ stand for the factoroid corresponding to $\bar{D}$. Associating with each element $\bar{a} \in \overline{\mathfrak{D}}$ that element $a^* \in \mathfrak{G}^*$ of whose $\boldsymbol{d}$-inverse